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Fixed Points of Almost Generalized (α, ψ)-Contractive Type Maps in G-Metric Spaces

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Abstract : In this paper, we introduce almost generalized (α, ψ)-contractive type mappings and prove the existence of fixed points for such mappings in G-metric spaces. Further, we define 'Condition (H)' and under this additional assumption, we prove uniqueness of fixed point. Our results generalize the results of Alghamdi and Karapinar [1] and Mustafa and Sims [2].

Keywords : G-metric space, G-convergent, G-cauchy, α-admissible maps, almost generalized (α, ψ)-contractive type maps.
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1 Introduction

Fixed points and fixed point theorems have always been a basic tool to find the existence of solutions of problems that arise in theoretical mathematics. Recently Samet, Vetro and Vetro [3] introduced a new concept namely '(α, ψ)-contractive' mappings and proved the existence of fixed points of such mappings in metric space setting which generalizes Banach contraction principle. After that Karapinar and Samet [4] introduced 'generalized (α, ψ)-contractive' mappings and proved fixed point results.

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In the context of generalization of contraction condition, in 2004, Berinde [5] introduced 'weak contraction maps' which are named as 'almost contraction maps' in his later work and proved fixed point results in complete metric spaces. In this paper, we use the terminology almost contractions for weak contractions. In 2008, Babu, Sandhya and Kameswari [6] modified the above definition by introducing 'condition (B)' and proved that every selfmap T of a complete metric space satisfying 'condition (B)' has a unique fixed point.

In the direction of generalization of ambient spaces, in 2005, Mustafa and Sims [2] introduced a new notion namely generalized metric space called G-metric space and studied the existence of fixed points of various types of contraction mappings in G-metric spaces. For more works on the existence of fixed points and coupled fixed points in G-metric spaces, we refer [[7], [1], [8], [9]].

In this paper, we introduce almost generalized (α, ψ)-contractive mappings in G-metric spaces. Further, we define 'Condition (H)' and under this additional assumption, we prove uniqueness of fixed point. Our results generalize the results of Alghamdi and Karapinar [1] and Mustafa and Sims [2].

In the following Section, we present some preliminaries that are needed to prove our main results of Section 3.

2 Preliminaries

Throughout this paper we denote by Ψ the family of nondecreasing functions $\psi : [0,\ \infty) \to [0,\ \infty)$ which satisfies $\sum\limits_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$ where ψ^n is the n^{th} iterate of ψ.

Remark 2.1. *Any function $\psi \in \Psi$ satisfies $\lim\limits_{n\to\infty} \psi^n(t) = 0$, $\psi(t) < t$ for any $t > 0$ and ψ is continuous at 0.*

Definition 2.2. (Samet, Vetro and Vetro [3, Definition 2.1]) Let (X, d) be a metric space and $T : X \to X$. We say that T is (α, ψ)-contractive mapping if there exist two functions $\alpha : X \times X \to [0,\ \infty)$
and $\psi \in \Psi$ such that
$$\alpha(x,\ y)d(Tx,\ Ty) \leq \psi(d(x,\ y)) \text{ for all } x,\ y \in X. \tag{2.2.1}$$

Definition 2.3. (Samet, Vetro and Vetro [3, Definition 2.1]) Let (X, d) be a metric space, $T : X \to X$ and
$\alpha : X \times X \to [0,\ \infty)$. We say that T is α-admissible if $x,\ y \in X$
$$\alpha(x,\ y) \geq 1 \Rightarrow \alpha(Tx,\ Ty) \geq 1. \tag{2.3.1}$$

For examples on α-admissible functions, we refer [9] and for more works on α-admissible functions, we refer [[4], [10], [11], [3]].

Theorem 2.4. (Samet, Vetro and Vetro [3, Theorem 2.2]) *Let $(X,\ d)$ be a complete metric space and $T : X \to X$. Suppose that there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that*
T is (α, ψ)-contractive map. Also, assume that

- (i) *T is α-admissible;*
- (ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$; either*
- (iii) *T is continuous; (or)*
- (iv) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all n.*

Then T has a fixed point. i.e., there exists $u \in X$ such that $Tu = u$.

Definition 2.5. (Karapinar and Samet [4, Definition 2.1]) Let (X, d) be a metric space and $T : X \to X$ be a given mapping. We say that T is a generalized (α, ψ)-contractive mapping if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that for all $x,\ y \in X$
$\alpha(x, y)d(Tx, Ty) \leq \psi(M(x, y))$, where (2.5.1)
$M(x, y) = \max\{d(x, y), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Tx)}{2}\}$.

Theorem 2.6. (Karapinar and Samet [4, Theorem 2.3]) *Let (X, d) be a complete metric space and $T : X \to X$. Suppose that there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that T is a generalized (α, ψ)-contractive map*
Also, assume that the following conditions are satisfied:

- (i) *T is α- admissible;*
- (ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$; either*
- (iii) *T is continuous; (or)*
- (iv) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,\ x_{n+1}) \geq 1$ for all n and $x_n \to x$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k},\ x) \geq 1$ for all k.*

Then there exists $u \in X$ such that $Tu = u$.

Definition 2.7. (Berinde [5, Definition 1]) Let (X, d) be a metric space. A map $T : X \to X$ is called an 'almost contraction' if there exist a constant $\delta \in (0, 1)$ and $L \geq 0$ such that
$d(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx)$ for all $x, y \in X$. (2.7.1)

Definition 2.8. (Babu, Sandhya and Kameswari [6, Definition 2.1]) Let (X, d) be a metric space. A map $T : X \to X$ is said to satisfy 'condition (B)' if there exist a $0 < \delta < 1$ and $L \geq 0$ such that for all $x, y \in X$
$d(Tx, Ty) \leq \delta d(x, y) + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$. (2.8.1)

Mustafa and Sims [2] introduced the concept of G-metric space as follows:

Definition 2.9. (Mustafa and Sims [2]) Let X be a non empty set and let $G : X^3 \to \mathbb{R}_+, \mathbb{R}_+ = [0, \infty)$, be a function satisfying:

(G1) $G(x, y, z) = 0$ if $x = y = z$,

(G2) $0 \leq G(x, x, y)$ for all $x, y \in X$, with $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y \in X$ with $z \neq y$,

(G4) $G(x, y, z) = G(y, x, z) = G(y, z, x) = ...$ (symmetry in all the three variables) and,

(G5) $G(x, y, z) \leq G(x, a, a)+G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then the function G is called a generalized metric (or) more specifically a G-metric on X, and the pair (X, G) is called a *G-metric space*.

Example 2.10. (Mustafa and Sims [2]) Let (X, d) be a metric space. The mapping $G_s : X^3 \to \mathbb{R}_+$ defined by

$$G_s(x, y, z) = d(x, y) + d(y, z) + d(z, x)$$

for all $x, y, z \in X$ is a G-metric and so (X, G_s) is a G-metric space.

Example 2.11. (Mustafa and Sims [2]) Let (X, d) be a metric space. The mapping $G_m : X^3 \to \mathbb{R}_+$ defined by

$$G_m(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$$

for all $x, y, z \in X$ is a G-metric and so (X, G_m) is a G-metric space.

Definition 2.12. (Mustafa and Sims [2])A G-metric space (X, G) is called *symmetric* if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

Definition 2.13. (Mustafa and Sims [2])Let (X, G) be a G-metric space and let $\{x_n\}$ be a sequence of points of X. We say that $\{x_n\}$ is *G-Convergent* to x if $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$; that is, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \epsilon$ for all $n, m \geq \mathbb{N}$. We refer to x as the limit of the sequence $\{x_n\}$. We denote it by $\lim_{n\to\infty} x_n =^G x$.

Definition 2.14. (Mustafa and Sims [2])Let (X, G) be a G-metric space and let $T : X \to X$. We say that $\{x_n\}$ is *G-Continuous* to $x \in X$ if for any sequence $\{x_n\} \subset X$ with $\lim_{n\to\infty} G(x_n, u, u) = 0$ implies $\lim_{n\to\infty} G(Tx_n, Tu, Tu) = 0$. *i.e.*, $\lim_{n\to\infty} x_n =^G u$ implies $\lim_{n\to\infty} Tx_n =^G Tu$.

Proposition 2.15. (Mustafa and Sims [2]) *Let (X, G) be a G-metric space. Then for any $x, y, z, a \in X$ we have that:*

(*1*) *if $G(x, y, z) = 0$ then $x = y = z$.*

(2) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$.

(3) $G(x, y, y) \leq 2G(y, x, x)$.

(4) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$.

(5) $G(x, y, z) \leq \frac{2}{3}(G(x, y, a) + G(x, a, z) + G(a, y, z))$.

Proposition 2.16. (Mustafa and Sims [2]) *Let (X, G) be a G-metric space. Then the following statements are equivalent:*

(1) *$\{x_n\}$ is G-convergent to x.*

(2) *$G(x_n, x_n, x) \to 0$ as $n \to \infty$.*

(3) *$G(x_n, x, x) \to 0$ as $n \to \infty$.*

Definition 2.17. (Mustafa and Sims [2]) Let (X, G) be a G-metric space. A sequence $\{x_n\}$ is *G-Cauchy* in X if given $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geq \mathbb{N}$; that is, if $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

Proposition 2.18. (Mustafa and Sims [2]) *In a G-metric space X, the following two statements are equivalent:*

(1) *The sequence $\{x_n\}$ is G-Cauchy.*

(2) *For every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \epsilon$ for all $n, m \geq \mathbb{N}$.*

Definition 2.19. (Mustafa and Sims [2])A G-metric space X is said to be *G-Complete*(or a complete G-metric space) if every G-Cauchy sequence in X is G-Convergent in X.

Proposition 2.20. (Mustafa and Sims [2]) *Let X be a G-metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Proposition 2.21. (Mustafa and Sims [2]) *Every G-metric space X defines a metric space (X, d_G) by*
$d_G(x, y) = G(x, y, y) + G(y, x, x)$ for all $x, y \in X$.

In 2005, Mustafa [2] proved the following results.

Theorem 2.22. (Mustafa and Sims [2]) *Let (X, G) be a complete G-metric space and let $T : X \to X$ be a mapping satisfying the following condition for all $x, y, z \in X$*

$$G(Tx, Ty, Tz) \leq kG(x, y, z), \tag{2.22.1}$$

where $k \in [0, 1)$. Then T has a unique fixed point.

Remark: A map T that satisfies the inequality (2.22.1) is called a G-contraction map on X. Sometimes we also call it G-contraction map of type I.

Theorem 2.23. (Mustafa and Sims [2]) *Let (X, G) be a complete G-metric space and let $T : X \to X$ be a mapping satisfying the following condition for all $x, y \in X$*

$$G(Tx, Ty, Ty) \leq kG(x, y, y), \tag{2.23.1}$$

where $k \in [0, 1)$. Then T has a unique fixed point.

A map that satisfies the inequality (2.23.1) is called a 'G-contraction map of type II.

Remark 2.24. *The condition* (2.22.1) *implies the condition* (2.23.1). *The converse is true only if $k \in [0, \frac{1}{2})$.*

Alghamdi and Karapinar [1] proved the following result.

Definition 2.25. (Alghamdi and Karapinar [1])Let (X, G) be a G-metric space and let $T : X \to X$ be a given mapping. We say that T is a (G, α, ψ)-*contractive mapping of type I* if there exist two functions $\alpha : X \times X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that for all $x, y, z \in X$ we have

$$\alpha(x, y, z)G(Tx, Ty, Tz) \leq \psi(G(x, y, z)). \tag{2.25.1}$$

Definition 2.26. (Alghamdi and Karapinar [1]) Let (X, G) be a G-metric space and let $T : X \to X$ be a given mapping. We say that T is a (G, α, ψ)-*contractive mapping of type II* if there exist two functions $\alpha : X \times X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that for all $x, y \in X$ we have

$$\alpha(x, y, y)G(Tx, Ty, Ty) \leq \psi(G(x, y, y)). \tag{2.26.1}$$

Definition 2.27. (Alghamdi and Karapinar [1])Let (X, G) be a G-metric space and let $T : X \to X$ be a given mapping. We say that T is a (G, α, ψ)-*contractive mapping of type A* if there exist two functions $\alpha : X \times X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that for all $x, y \in X$ we have

$$\alpha(x, y, Tx)G(Tx, Ty, T^2x) \leq \psi(G(x, y, T^2x)). \tag{2.27.1}$$

Theorem 2.28. (Alghamdi and Karapinar [1]) *Let (X, G) be a complete G-metric space. Suppose that $T : X \to X$ is a (G, α, ψ)-contractive mapping of type A that satisfies the following conditions:*

- (*i*) *T is α-admissible;*
- (*ii*) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, Tx_0) \geq 1$; either*
- (*iii*) *T is G-continuous;* (*or*)
- (*iv*) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, x_{n+1}) \geq 1$ for all n and $\{x_n\}$ is G-convergent to $x \in X$, then $\alpha(x_n, x, x_{n+1}) \geq 1$ for all n.*

Then there exists $u \in X$ such that $Tu = u$.

Theorem 2.29. (Alghamdi and Karapinar [1]) *Let (X, G) be a complete G-metric space. Suppose that $T : X \to X$ is a (G, α, ψ)-contractive mapping of type II that satisfies the following conditions:*

(i) *T is α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, Tx_0) \geq 1$; either*

(iii) *T is G-continuous; (or)*

(iv) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, x_{n+1}) \geq 1$ for all n and $\{x_n\}$ is G-convergent to $x \in X$, then $\alpha(x_n, x, x_{n+1}) \geq 1$ for all n.*

Then there exists $u \in X$ such that $Tu = u$.

In the following, we introduce almost generalized (α, ψ)-contractive type map.

Definition 2.30. Let (X, G) be a G-metric space and $T : X \to X$ be a given mapping. We say that T is an *almost generalized (α, ψ)-contractive type mapping* if there exist two functions $\alpha : X \times X \times X \to [0, \infty)$, $L \geq 0$, $\psi \in \Psi$ and a constant $L \geq 0$ such that for all $x, y \in X$

$$\alpha(x, y, y)G(Tx, Ty, Ty) \leq \psi(M(x, y, y)) + LN(x, y, y), \text{ where} \quad (2.30.1)$$

$$M(x, y, y) = \max\{G(x, y, y), G(x, Tx, Tx), G(y, Ty, Ty), \tfrac{G(x,Ty,Ty)+G(y,Tx,Tx)}{2}\}$$

and $N(x, y, y) = \min\{G(x, Tx, Tx), G(y, Ty, Ty), G(x, Ty, Ty), G(y, Tx, Tx)\}$.

Note: Every G-contraction map of type II is an almost generalized (α, ψ)-contractive type map with $\alpha(x, y, y) = 1 \ \forall \ x, y \in X$, $L = 0$ and $\psi(t) = kt$, $t \geq 0$ where $k \in [0, 1)$.

Note: Clearly, a map T that satisfies (2.26.1) then it satisfies the inequality (2.30.1) with $L = 0$ so that T is an almost generalized (α, ψ)-contractive type map. But its converse need not be true.

Example 2.31. Let $X = [0, 3]$. We define $G : X^3 \to \mathbb{R}_+$ by $G(x, y, z) = d(x, y) + d(y, z) + d(z, x)$. Let $A = \Delta \cup \{(1, 3, 3), (2, 3, 3), (0, 1, 1), (0, 1, 2)\}$. We define mappings $T : X \to X$ by

$$T(x) = \begin{cases} 2 + x^2 & if\ x \in [0, 1] \\ 3 & if\ x \in (1, 3]. \end{cases}$$

We define $\alpha : X^3 \to [0, +\infty)$ and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\alpha(x, y, z) = \begin{cases} 3 & if\ x, y, z \in A \\ 0 & otherwise \end{cases}$$

and $\psi(t) = \frac{3t}{4}$ for all $t > 0$.

The following three cases arise to verify the inequality (2.30.1).

<u>Case (i):</u> $x = 1$ and $y = 3$.

In this case, the inequality (2.30.1) holds trivially.

<u>Case (ii):</u> $(x, y, y) = (2, 3, 3)$ or $(0, 1, 2)$.

In this case, the inequality (2.30.1) holds trivially.

<u>Case (iii):</u> $x = 0$ and $y = 1$.

In this case, $G(T0, T1, T1) = 2$, $M(0, 1, 1) = 4$ and $N(0, 1, 1) = 2$.

$\alpha(x, y, y)G(Tx, Ty, Ty) = \alpha(0, 1, 1)G(T0, T1, T1)$

$= 6 \leq \psi(4) + L.2 = \psi(M(0,1,1)) + L.N(0,1,1)$
$= \psi(M(x,y,y)) + L.N(x,y,y)$ holds with $L = 2$.
Hence, from the above cases, we choose $L = 2$, so that T is an almost generalized (α,ψ)-contractive type map.

Here we note that if $L = 0$ in the inequality (2.30.1), then for $x = 0$ and $y = 1$ we have
$\alpha(0,1,1)G(T0,T1,T1) = 6 \nleq \psi(4) = \psi(M(0,1,1))$, for any $\psi \in \Psi$, so that the inequality (2.30.1) fails to hold, which shows the importance of L.

Further, we observe that the inequality (2.25.1) fails to hold.
For, by choosing $(x,y,z) = (0,1,2)$ we have
$\alpha(0,1,2)G(T0,T1,T2) = 6 \nleq \psi(4) = \psi(G(0,1,2))$ for this α.
This shows that the inequality (2.25.1) fails to hold so that T is not (G,α,ψ)-contractive mapping of type I.

Here, we observe that the inequality (2.26.1) also fails to hold.
For, by choosing $(x,y,y) = (0,1,1)$ we have
$\alpha(0,1,1)G(T0,T1,T1) = 6 \nleq \psi(2) = \psi(G(0,1,1))$ for this α.
This shows that the inequality (2.26.1) fails to hold so that T is not (G,α,ψ)-contractive mapping of type II.

Also, we observe that the inequality (2.27.1) fails to hold.
For, by choosing $(x,y,y) = (0,1,1)$ we have
$\alpha(0,1,T0)G(T0,T1,T^20) = 6 \nleq \psi(6) = \psi(G(0,1,1))$ for this α.
This shows that the inequality (2.27.1) fails to hold so that T is not (G,α,ψ)-contractive mapping of type A.
If $\alpha = 1$, we note that the inequality (2.23.1) fails to hold.
For, by choosing $(x,y) = (0,1)$ we have

$$G(T0,T1,T1) = 2 \nleq k \cdot 2 = kG(0,1,1).$$

This shows that the inequality (2.23.1) fails to hold, for any α and $k \in [0,1)$.

In conclusion, we have the following remark.

Remark 2.32. *From Example 2.31, we conclude that the class of almost generalized (α,ψ)-contractive type maps is more general than the class of (G,α,ψ)-contractive mappings of type I, the class of (G,α,ψ)-contractive mappings of type II and the class of (G,α,ψ)-contractive mappings of type A.*

In Section 3, we prove the existence of fixed points of almost generalized (α,ψ)-contractive type mappings. Further, we define 'Condition (H)' and under this additional assumption, we prove uniqueness of fixed point. In Section 4, we provide examples in support of our main results.

3 Main Results

Theorem 3.1. *Let (X,G) be a complete G-metric space and $T : X \to X$ be an almost generalized (α,ψ)-contractive type mapping. Suppose that* (*i*) *T is α-admissible;*

(*ii*) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, Tx_0) \geq 1$;*

(*iii*) *T is G-continuous.*

Then there exists $u \in X$ such that $Tu = u$.

Proof. Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0, Tx_0) \geq 1$. We define the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \geq 0$. (3.1.1)
If $x_n = x_{n+1}$ for some n, then $x_n = x_{n+1} = Tx_n$ so that x_n is a fixed point of T.
Hence w.l.g. we assume that $x_n \neq x_{n+1}$ for all n.
Since $\alpha(x_0, Tx_0, Tx_0) \geq 1$. *i.e.,* $\alpha(x_0, x_1, x_1) \geq 1$ and since T is α-admissible, we have
$\alpha(Tx_0, Tx_1, Tx_1) \geq 1$. *i.e.,* $\alpha(x_1, x_2, x_2) \geq 1$.
By repeating this process, we get
$\alpha(x_n, x_{n+1}, x_{n+1}) \geq 1$ for all $n = 0, 1, 2, \ldots$. (3.1.2)
Now, from (2.30.1) and (3.1.2), we have

$$\begin{aligned} G(x_{n+1}, x_n, x_n) &= G(x_n, x_{n+1}, x_{n+1}) \\ &= G(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq \alpha(x_{n-1}, x_n, x_n) G(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq \psi(M(x_{n-1}, x_n, x_n)) + LN(x_{n-1}, x_n, x_n), \end{aligned} \quad (3.1.3)$$

where

$$\begin{aligned} M(x_{n-1}, x_n, x_n) &= \max\{G(x_{n-1}, x_n, x_n), G(x_{n-1}, Tx_{n-1}, Tx_{n-1}), G(x_n, Tx_n, Tx_n), \\ &\qquad \tfrac{G(x_{n-1}, Tx_n, Tx_n) + G(x_n, Tx_{n-1}, Tx_{n-1})}{2}\} \\ &= \max\{G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), \\ &\qquad \tfrac{G(x_{n-1}, x_{n+1}, x_{n+1}) + G(x_n, x_n, x_n)}{2}\} \\ &= \max\{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), \tfrac{G(x_{n-1}, x_{n+1}, x_{n+1})}{2}\} \\ &\leq \max\{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), \\ &\qquad \tfrac{G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1})}{2}\} \\ &= \max\{G(x_n, x_{n-1}, x_{n-1}), G(x_n, x_{n+1}, x_{n+1})\} \text{ and} \\ N(x_{n-1}, x_n, x_n) &= \min\{G(x_{n-1}, Tx_{n-1}, Tx_{n-1}), G(x_n, Tx_n, Tx_n), \\ &\qquad G(x_n, Tx_{n-1}, Tx_{n-1}), G(x_{n-1}, Tx_n, Tx_n)\} \\ &= \min\{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), \\ &\qquad G(x_n, x_n, x_n), G(x_{n-1}, x_{n+1}, x_{n+1})\} = 0. \end{aligned}$$

If $\max\{G(x_n, x_{n-1}, x_{n-1}), G(x_n, x_{n+1}, x_{n+1})\} = G(x_n, x_{n+1}, x_{n+1})$ then from (3.1.3) we have
$G(x_{n+1}, x_n, x_n) \leq \psi(G(x_n, x_{n+1}, x_{n+1})) = \psi(G(x_{n+1}, x_n, x_n)) < G(x_{n+1}, x_n, x_n)$,
a contradiction.
Hence $\max\{G(x_n, x_{n-1}, x_{n-1}), G(x_n, x_{n+1}, x_{n+1})\} = G(x_n, x_{n-1}, x_{n-1})$ so that from (3.1.3) we have
$G(x_{n+1}, x_n, x_n) \leq \psi(G(x_n, x_{n-1}, x_{n-1}))$ for all $n \geq 1$.
Hence by induction, it follows that
$G(x_{n+1}, x_n, x_n) \leq \psi^n(G(x_0, x_1, x_1))$ (3.1.4)
From (3.1.4) and using triangular inequality, for all $k \geq 1$, we have

$$G(x_n, x_{n+k}, x_{n+k}) \leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + \ldots$$

$$+ G(x_{n+k-1}, x_{n+k}, x_{n+k})$$
$$= \Sigma_{p=n}^{n+k-1} G(x_p, x_{p+1}, x_{p+1})$$
$$\leq \Sigma_{p=n}^{+\infty} \psi^p(G(x_0, x_1, x_1)) \to 0 \text{ as } p \to \infty.$$

This implies that $\{x_n\}$ is a Cauchy sequence in X. Since (X, G) is complete, there exists $u \in X$ such that

$$\lim_{n\to\infty} x_n =^G u. \tag{3.1.5}$$

i.e., $\lim_{n\to\infty} G(x_n, u, u) = 0$. Since T is G-continuous, it follows that $\lim_{n\to\infty} G(Tx_n, Tu, Tu) = 0$. *i.e.*, $\lim_{n\to\infty} G(x_{n+1}, Tu, Tu) = 0$.
i.e., $\lim_{n\to\infty} x_{n+1} =^G Tu$.
Hence $Tu = u$ in G-metric space. □

Theorem 3.2. *Let $(X,\ G)$ be a complete G-metric space and $T : X \to X$ be an almost generalized (α, ψ)-contractive type mapping. Suppose that*

- (i) *T is α admissible;*
- (ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, Tx_0) \geq 1$;*
- (iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x, x) \geq 1$ for all n.*

Then there exists $u \in X$ such that $Tu = u$.

Proof. The Cauchy part of $\{x_n\}$ follows from the proof of Theorem 3.1.
Let $\lim_{n\to\infty} x_n = x^*,\ x^* \in X$.
We now show that $Tx^* = x^*$. By using the hypotheses (iii), we have

$$\alpha(x_n, x^*, x^*) \geq 1 \text{ for all } n \in N. \tag{3.2.1}$$

By using triangular inequality and (2.30.1), we have

$$\begin{aligned} G(Tx^*, x_n, x_n) &\leq G(Tx^*, Tx_n, Tx_n) + G(Tx_n, x_n, x_n) \\ &= G(Tx_n, Tx^*, Tx^*) + G(Tx_n, x_n, x_n) \\ &\leq \alpha(x_n, x^*, x^*) G(Tx_n, Tx^*, Tx^*) + G(Tx_n, x_n, x_n) \\ &\leq \psi(M(x_n, x^*, x^*)) + LN(x_n, x^*, x^*) + G(x_{n+1}, x_n, x_n), \end{aligned} \tag{3.2.2}$$

where

$$\begin{aligned} M(x_n, x^*, x^*) &= \max\{G(x_n, x^*, x^*), G(x_n, Tx_n, Tx_n), G(x^*, Tx^*, Tx^*), \\ &\qquad \tfrac{1}{2}(G(x_n, Tx^*, Tx^*) + G(x^*, Tx_n, Tx_n))\} \\ &= \max\{G(x_n, x^*, x^*), G(x_n, x_{n+1}, x_{n+1}), G(x^*, Tx^*, Tx^*) \\ &\qquad \tfrac{1}{2}(G(x_n, Tx^*, Tx^*) + G(x^*, x_{n+1}, x_{n+1}))\} \text{ and} \end{aligned}$$

$$\begin{aligned} N(x^*, x_n, x_n) = \min\{&G(x_n, x_{n+1}, x_{n+1}), G(x^*, Tx^*, Tx^*), \\ &G(x_n, Tx^*, Tx^*), G(x^*, x_{n+1}, x_{n+1})\}. \end{aligned}$$

On letting $n \to \infty$, we have
$\lim_{n\to\infty} M(x_n, x^*, x^*) = G(x^*, Tx^*, Tx^*)$ and $\lim_{n\to\infty} N(x_n, x^*, x^*) = 0$.
On letting $n \to \infty$ in (3.2.2), we have
$G(Tx^*, x^*, x^*) \leq \psi(G(x^*, Tx^*, Tx^*)) + L.0 + 0$.
Which implies that $G(x^*, Tx^*, Tx^*) = 0$. Therefore $Tx^* = x^*$. □

Now we prove the uniqueness of fixed point of T under 'Condition (H)' and it is the following:

Condition (H): For all $x, y \in X$ there exists $z \in X$ such that $\alpha(x, z, z) \geq 1$ and $\alpha(y, z, z) \geq 1$.

Theorem 3.3. *In addition to the hypotheses of Theorem 3.1 if Condition (H) holds, then T has a unique fixed point.*

Proof. Suppose that x^* and y^* are two fixed points of T. By Condition (H), there exists $z \in X$ such that
$\alpha(x^*, z, z) \geq 1$ and $\alpha(y^*, z, z) \geq 1$. (3.3.1)
Since T is α-admissible, from (3.3.1) we have
$\alpha(x^*, T^n z, T^n z) \geq 1$ and $\alpha(y^*, T^n z, T^n z) \geq 1$ for all $n \in N$. (3.3.2)
We define the sequence $\{z_n\}$ in X by $z_{n+1} = Tz_n$ for all $n \geq 0$.
From (2.30.1) and (3.3.2), we have

$$\begin{aligned} G(x^*, z_{n+1}, z_{n+1}) &= G(Tx^*, Tz_n, Tz_n) \\ &\leq \alpha(x^*, z_n, z_n) G(Tx^*, Tz_n, Tz_n) \\ &\leq \psi(M(x^*, z_n, z_n)) + LN(x^*, z_n, z_n), \end{aligned} \tag{3.3.3}$$

where

$$\begin{aligned} M(x^*, z_n, z_n) &= \max\{G(x^*, z_n, z_n), G(x^*, Tx^*, Tx^*), G(z_n, Tz_n, Tz_n), \\ &\qquad \frac{G(Tz_n, Tz_n, x^*) + G(z_n, Tx^*, Tx^*)}{2}\} \\ &\leq \max\{G(x^*, z_n, z_n), 0, G(z_n, z_{n+1}, z_{n+1}), \\ &\qquad \frac{G(z_{n+1}, z_{n+1}, x^*) + G(z_n, x^*, x^*)}{2}\} \\ &= \max\{G(x^*, z_n, z_n), G(z_n, z_{n+1}, z_{n+1}), G(z_{n+1}, z_{n+1}, x^*)\} \end{aligned}$$

and

$$\begin{aligned} N(x^*, z_n, z_n) &= \min\{G(x^*, Tx^*, Tx^*), G(z_n, Tz_n, Tz_n), G(Tz_n, Tz_n, x^*), \\ &\qquad G(z_n, Tx^*, Tx^*)\} \\ &= \min\{0, G(z_n, z_{n+1}, z_{n+1}), G(z_{n+1}, z_{n+1}, x^*), G(z_n, x^*, x^*)\} = 0. \end{aligned}$$

Case (i): If

$$\begin{aligned} &\max\{G(x^*, z_n, z_n), G(z_n, z_{n+1}, z_{n+1}), G(z_{n+1}, z_{n+1}, x^*)\} \\ &\quad = G(z_{n+1}, z_{n+1}, x^*), \end{aligned}$$

then from (3.3.3), we have

$$\begin{aligned} G(x^*, z_{n+1}, z_{n+1}) &\leq \psi(G(z_{n+1}, z_{n+1}, x^*)) \\ &= \psi(G(x^*, z_{n+1}, z_{n+1})) \\ &< G(x^*, z_{n+1}, z_{n+1}), \end{aligned}$$

a contradiction.
Case (ii): If

$$\max\{G(x^*, z_n, z_n), G(z_n, z_{n+1}, z_{n+1}), G(z_{n+1}, z_{n+1}, x^*)\}$$
$$= G(z_n, z_n, x^*)$$

then from (3.3.3), we have

$$G(x^*, z_{n+1}, z_{n+1}) = G(z_{n+1}, z_{n+1}, x^*) \leq \psi(G(z_n, z_n, x^*))$$
$$\leq \psi^2(G(z_{n-1}, z_{n-1}, x^*)) \leq .... \leq \psi^n(G(z_0, z_0, x^*)).$$

Therefore $G(x^*, z_{n+1}, z_{n+1}) \to 0$ as $n \to +\infty$. Similarly, $G(y^*, z_{n+1}, z_{n+1}) \to 0$ as $n \to +\infty$. Hence $x^* = y^*$.
Case (iii): If

$$\max\{G(x^*, z_n, z_n), G(z_n, z_{n+1}, z_{n+1}), G(z_{n+1}, z_{n+1}, x^*)\}$$
$$= G(z_n, z_{n+1}, z_{n+1}),$$

then from (3.3.3), we have

$$G(x^*, z_{n+1}, z_{n+1}) \leq \psi(G(z_n, z_{n+1}, z_{n+1}))$$
$$\leq \psi^2(G(z_{n-1}, z_n, z_n)) \leq .... \leq \psi^n(G(z_0, z_0, z_1)).$$

Therefore $G(x^*, z_{n+1}, z_{n+1}) \to 0$ as $n \to +\infty$. Similarly, $G(y^* z_{n+1}, z_{n+1}) \to 0$ as $n \to +\infty$. Hence $x^* = y^*$. □

4 Corollaries and Examples

Corollary 4.1. *Let (X, G) be a complete G-metric space and $T : X \to X$ be a G-contraction map of type II. Suppose that*

- (i) *T is α admissible;*
- (ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, Tx_0) \geq 1$; either*
- (iii) *T is continuous;* (*or*)
- (iv) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x, x) \geq 1$ for all n.*

Then there exists $u \in X$ such that $Tu = u$.

Proof. The conclusion of this corollary follows by taking $\psi(t) = kt,\ t \geq 0$ where $k \in [0, 1)$ and $L = 0$ in Theorem 3.1. □

Remark 4.2. *Theorem 2.29 follows as a corollary to Theorem 3.1 by choosing $L = 0$.*

The following is an example in support of Theorem 3.1.

Example 4.3. Let $X = [0, 1]$. We define $G : X^3 \to \mathbb{R}_+$ by $G(x, y, z) = d(x, y) + d(y, z) + d(z, x)$. Then (X, G) is a complete G-metric space.

We define mapping $T : X \to X$ by $T(x) = \begin{cases} \frac{x^2}{2} & if\ x \in [0, \frac{1}{4}] \\ \frac{1}{32} & if\ x \in [\frac{1}{4}, \frac{1}{2}) \\ \frac{1}{16}(31x - 15) & if\ x \in [\frac{1}{2}, 1]. \end{cases}$

We define $\alpha : X^3 \to [0, +\infty)$ and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ by

$\alpha(x, y, z) = \begin{cases} \frac{3}{2} & if\ x, y = z \in [0, \frac{1}{4}] \\ 0 & otherwise \end{cases}$ and $\psi(t) = \frac{t}{2}$ for all $t > 0$.

We choose $x_0 = \frac{1}{4} \in X$, then $\alpha(x_0, Tx_0, Tx_0) = \frac{3}{2} > 1$. Then, it is easy to verify that T is continuous, α-admissible and satisfies the inequality (2.30.1). Here $x_n = \frac{1}{2^{2n+1}} \to 0$ as $n \to \infty$. Thus T satisfies all the hypotheses of Theorem 3.1 and fixed points of T are 0 and 1. Here we observe that for $x = 0$ and $y = 1$ we have $\alpha(x, z, z) = \begin{cases} \frac{3}{2} & if\ z \in [0, \frac{1}{4}] \\ 0 & otherwise \end{cases}$ and $\alpha(y, z, z) = 0$ for all $z \in X$ so that Condition (H) fails to hold.

The following is an example in support of Theorem 3.2 when T is not continuous.

Example 4.4. Let $X = [0, 2]$. We define $G : X^3 \to \mathbb{R}_+$ by $G(x, y, z) = d(x, y) + d(y, z) + d(z, x)$. Then (X, G) is a complete G-metric space.

We define mapping $T : X \to X$ by $T(x) = \begin{cases} x^2 & if\ x \in [0, \frac{1}{4}] \\ x & if\ x \in (\frac{1}{4}, 2]. \end{cases}$

We define $\alpha : X^3 \to [0, +\infty)$ and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ by

$\alpha(x, y, z) = \begin{cases} 1 & if\ x, y = z \in [0, \frac{1}{4}] \\ 0 & otherwise \end{cases}$ and $\psi(t) = \frac{t}{t+1}$ for all $t > 0$.

We choose $x_0 = \frac{1}{4} \in X$, then $\alpha(x_0, Tx_0, Tx_0) = 1$. Then, it is easy to verify that T is α-admissible and satisfies the inequality (2.30.1). Here $x_n = \frac{1}{2^{2n}} \to 0$ as $n \to \infty$. Thus T satisfies all the hypotheses of Theorem 3.2 and fixed points of T are 0 and every point of the interval $[\frac{1}{4}, 2]$. Here we observe that T is not continuous. In this example also, it is easy to see that α fails to satisfy Condition (H).

Example 4.5. Let X, T, Ψ and α be as in Example 2.31. Then T is continuous and α-admissible. We choose $x_0 = 1$, then $\alpha(x_0, Tx_0, Tx_0) = \alpha(1, 3, 3) \geq 1$. Also T is an almost generalized (α, ψ)-contractive type map (*Example* 2.31) and T satisfies condition (H). We note that T satisfies all the hypotheses of the Theorem 3.3 and 3 is the unique fixed point of T.

Remark 4.6. *Remark* 2.32 *and Example* 4.5 *suggest that Theorem* 3.3 *is a generalization of Theorem* 2.29.

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