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Some Geometric Properties of Generalized Difference Cesàro Sequence Spaces

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Abstract : In this paper, we define the generalized Cesàro difference sequence space $C_{(p)}(\Delta^m)$ and consider it equipped with the Luxemburg norm under which it is a Banach space and we show that in the space $C_{(p)}(\Delta^m)$ every weakly convergent sequence on the unit sphere converges is the norm, where $p = (p_n)$ is a bounded sequence of positive real numbers with $p_n > 1$ for all $n \in \mathbb{N}$.

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1 Introduction

Let X be a real Banach space and let $B(X)$ and $S(X)$ be the closed unit ball and the unit sphere of X, respectively. A point $x \in S(X)$ is called an *extreme point* if for any $y, z \in B(X)$ the equality $2x = y + z$ implies $y = z$.

A Banach space X is said to *have property* (H) if every weakly convergent sequence on the unit sphere is convergent in norm.

For a real vector space X, a function $\varrho : X \longrightarrow [0, \infty]$ is called a *modular* if it satisfies the following conditions:

(i) $\varrho(x) = 0$ if and only if $x = 0$,

(ii) $\varrho(\alpha x) = \varrho(x)$ for all scalar α with $|\alpha| = 1$,

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(iii) $\varrho(\alpha x+\beta y)\leq \varrho(x)+\varrho(y)$ for all $x,y\in X$ and all $\alpha,\beta\geq 0$ with $\alpha+\beta=1$.

The modular ϱ is called *convex* if

(iv) $\varrho(\alpha x+\beta y)\leq \alpha\varrho(x)+\beta\varrho(y)$ for all $x,y\in X$ and all $\alpha,\beta\geq 0$ with $\alpha+\beta=1$.

For any modular ϱ on X, the space

$$X_\varrho=\{x\in X:\varrho(\lambda x)<\infty \text{ for some } \lambda>0\}$$

is called the *modular space*. If ϱ is a convex modular, the functions

$$\|x\|=\inf\left\{\lambda>0:\varrho\left(\frac{x}{\lambda}\right)\leq 1\right\},$$

$$\|x\|_0=\inf_{k>0}\frac{1}{k}\left(1+\varrho(kx)\right)$$

are two norms on X_ϱ, which are called the *Luxemburg norm* and the *Amemiya norm*, respectively. These norms are equivalent (see [1]).

Let us denote by ℓ^0 the space of all real sequences. The Cesàro sequence spaces

$$ces_p=\left\{x\in\ell^0:\sum_{n=1}^{\infty}\left(n^{-1}\sum_{i=1}^{n}|x(i)|\right)^p<\infty\right\},\ 1\leq p<\infty$$

and

$$ces_\infty=\left\{x\in\ell^0:\sup_n n^{-1}\sum_{i=1}^{n}|x(i)|<\infty\right\}$$

have been introduced by Shiue [2]. Jagers [3] has determined the Köthe duals of the sequence space ces_p $(1<p<\infty)$. It can be shown that the inclusion $\ell_p\subset ces_p$ is strict for $1<p<\infty$ although it does not hold for $p=1$. Some geometric properties of the Cesàro sequence space have been studied by Cui and Hudzik [4,5], Cui *et al.* [6], Karakaya [7], Lee [8], Leibowitz [9], Maligranda [10], Maligranda *et al.* [11], Mursaleen et al. [12], Musielak [1], Petrot and Suantai [13,14], Sanhan and Suantai [15], Şimşek *et al.* [16], Suantai [17,18] and many others.

The difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$, consisting of all real valued sequences $x=(x(k))$ such that $\Delta x=(x(k)-x(k+1))$ in the sequence spaces ℓ_∞, c and c_0, were defined by Kızmaz [19]. The idea of difference sequences was generalized by Et and Çolak [20]. Later on difference sequence spaces have been studied by Altın [21], Altay and Basar [22], Bhardwaj and Bala [23], Et *et al.* [24,25], Işık [26], Srivastava and Kumar [27], Tripathy *et al.* [28–36] and many others. Recently, Et [37] defined the Cesàro difference sequence space $C_p(\Delta^m)$ as follows:

$$C_p(\Delta^m)=\left\{x\in\ell^0:\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|\Delta^m x(k)|\right)^p<\infty,\quad 1\leq p<\infty\right\},$$

where $m\in\mathbb{N}$ (the set of positive integers), $\Delta^0x=(x(k))$, $\Delta x=(x(k)-x(k+1))$, $\Delta^m x=(\Delta^m x(k))=(\Delta^{m-1}x(k)-\Delta^{m-1}x(k+1))$

and so that $\Delta^m x(k) = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x(k+v)$. The space $C_p(\Delta^m)$ is a Banach space for $1 \le p < \infty$ normed by

$$\|x\|_p = \sum_{i=1}^{m} |x(i)| + \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |\Delta^m x(k)| \right)^p \right)^{\frac{1}{p}}.$$

Let $p = (p_n)$ be a sequence of positive real numbers with $p_n \ge 1$ for all $n \in \mathbb{N}$. Now we define the generalized Cesàro difference sequence space $C_{(p)}(\Delta^m)$ by

$$C_{(p)}(\Delta^m) = \left\{ x \in \ell^0 : \rho_{\Delta^m}(\lambda x) < \infty \text{ for some } \lambda > 0 \right\},$$

where

$$\rho_{\Delta^m}(x) = \sum_{i=1}^{m} |x(i)| + \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |\Delta^m x(k)| \right)^{p_n}.$$

We consider the space $C_{(p)}(\Delta^m)$ equipped with Luxemburg norm

$$\|x\| = \inf \left\{ \lambda > 0 : \rho_{\Delta^m}\left(\frac{x}{\lambda} \right) \le 1 \right\}. \tag{1.1}$$

If $p = (p_n)$ is bounded, then we have

$$C_{(p)}(\Delta^m) = \left\{ x = x(k) : \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |\Delta^m x(k)| \right)^{p_n} < \infty \right\}.$$

Throughout this paper we assume that $p = (p_n)$ is bounded with $p_n > 1$ for all $n \in \mathbb{N}$ and $M = \sup_n p_n$.

2 Main Results

We begin establishing some basic properties of modular on the space $C_{(p)}(\Delta^m)$.

Theorem 2.1. *The functional ρ_{Δ^m} on $C_{(p)}(\Delta^m)$ is a convex modular.*

Proof. We have

$$\begin{aligned} \rho_{\Delta^m}(x) = 0 &\iff \rho_{\Delta^m}(x) = \sum_{i=1}^{m} |x(i)| + \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |\Delta^m x(k)| \right)^{p_n} = 0 \\ &\iff \sum_{i=1}^{m} |x(i)| = 0 \text{ and } \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |\Delta^m x(k)| \right)^{p_n} = 0 \\ &\iff x = 0. \end{aligned}$$

It is obvious that $\rho_{\Delta^m}(\alpha x) = \rho_{\Delta^m}(x)$ for all scalar α with $|\alpha| = 1$. If $x, y \in C_{(p)}(\Delta^m)$ and $\alpha \geq 0, \beta \geq 0$ with $\alpha + \beta = 1$, by the convexity of the function $t \to |t|^{p_n}$ for every $n \in \mathbb{N}$ and the linearity of the operator Δ^m, we have

$$\begin{aligned}
\rho_{\Delta^m}(\alpha x + \beta y) &= \sum_{i=1}^{m} |\alpha x(i) + \beta y(i)| + \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |\Delta^m (\alpha x(k) + \beta y(k))| \right)^{p_n} \\
&\leq \sum_{i=1}^{m} (\alpha |x(i)| + \beta |y(i)|) + \sum_{n=1}^{\infty} \left(\alpha \left(\frac{1}{n} \sum_{k=1}^{n} |\Delta^m x(k)| \right) \right. \\
&\quad \left. + \beta \left(\frac{1}{n} \sum_{k=1}^{n} |\Delta^m y(k)| \right) \right)^{p_n} \\
&\leq \alpha \sum_{i=1}^{m} |x(i)| + \beta \sum_{i=1}^{m} |y(i)| + \alpha \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |\Delta^m x(k)| \right)^{p_n} \\
&\quad + \beta \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |\Delta^m y(k)| \right)^{p_n} \\
&= \alpha \rho_{\Delta^m}(x) + \beta \rho_{\Delta^m}(y).
\end{aligned}$$

□

The proofs of the following two theorems can be established using known and standard techniques. Therefore we state the theorems without proof.

Theorem 2.2. *For $x \in C_{(p)}(\Delta^m)$, the modular ρ_{Δ^m} on $C_{(p)}(\Delta^m)$ satisfies the following properties:*

(*i*) *if $0 < a < 1$, then $a^M \rho_{\Delta^m}\left(\frac{x}{a}\right) \leq \rho_{\Delta^m}(x)$ and $\rho_{\Delta^m}(ax) \leq a\rho_{\Delta^m}(x)$,*

(*ii*) *if $a \geq 1$, then $\rho_{\Delta^m}(x) \leq a^M \rho_{\Delta^m}\left(\frac{x}{a}\right)$,*

(*iii*) *if $a \geq 1$, then $\rho_{\Delta^m}(x) \leq a\rho_{\Delta^m}(x) \leq \rho_{\Delta^m}(ax)$.*

Theorem 2.3. *For any $x \in C_{(p)}(\Delta^m)$, we have*

(*i*) *if $\|x\| < 1$, then $\rho_{\Delta^m}(x) \leq \|x\|$,*

(*ii*) *if $\|x\| > 1$, then $\rho_{\Delta^m}(x) \geq \|x\|$,*

(*iii*) *$\|x\| = 1$ if and only if $\rho_{\Delta^m}(x) = 1$,*

(*iv*) *$\|x\| < 1$ if and only if $\rho_{\Delta^m}(x) < 1$,*

(*v*) *$\|x\| > 1$ if and only if $\rho_{\Delta^m}(x) > 1$,*

(*vi*) *if $0 < a < 1$ and $\|x\| > a$, then $\rho_{\Delta^m}(x) > a^M$,*

(*vii*) *if $a \geq 1$ and $\|x\| < a$, then $\rho_{\Delta^m}(x) < a^M$.*

Theorem 2.4. *The sequence space $C_{(p)}(\Delta^m)$ is a Banach space normed by* (1.1).

Proof. It is a routine verification that $C_{(p)}(\Delta^m)$ is a normed space normed by (1.1). To show that $C_{(p)}(\Delta^m)$ is complete, let (x_s) be a Cauchy sequence in $C_{(p)}(\Delta^m)$ and $\varepsilon \in (0,1)$. For $H = \max\{1, M\}$, there exists n_0 such that

$$\|x_s - x_t\| = \inf\left\{\lambda > 0 : \rho_{\Delta^m}\left(\frac{x_s - x_t}{\lambda}\right) \le 1\right\} < \varepsilon^H$$

for all $s, t \ge n_0$. By Theorem 2.3(i) we have

$$\rho_{\Delta^m}(x_s - x_t) < \|x_s - x_t\| < \varepsilon^H \tag{2.1}$$

for all $s, t \ge n_0$, which means that

$$\sum_{i=1}^{m} |x_s(i) - x_t(i)| + \sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} |\Delta^m(x_s(k) - x_t(k))|\right)^{p_n} < \varepsilon$$

for all $s, t \ge n_0$. We have

$$\sum_{i=1}^{m} |x_s(i) - x_t(i)| < \frac{\varepsilon}{2}$$

and

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} |\Delta^m(x_s(k) - x_t(k))|\right)^{p_n} < \frac{\varepsilon}{2}.$$

For fixed $i \in \mathbb{N}$, we can write

$$|x_s(i) - x_t(i)| < \frac{\varepsilon}{2}.$$

Hence we obtain that the sequence $(x_t(i))$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, $x_t(i) \longrightarrow x(i)$ as $t \longrightarrow \infty$. We have

$$|x_s(i) - x(i)| < \frac{\varepsilon}{2}$$

and

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} |\Delta^m(x_s(k) - x(k))|\right)^{p_n} < \frac{\varepsilon}{2}$$

for all $s \ge n_0$. Now we show that the sequence $(x(i))$ is an element of $C_{(p)}(\Delta^m)$. From the inequality (2.1), we can write

$$\rho_{\Delta^m}(x_s - x_t) \longrightarrow \rho_{\Delta^m}(x_s - x),$$

as $t \longrightarrow \infty$ for all $s \ge n_0$. Thus we have $\rho_{\Delta^m}(x_s - x) < \|x_s - x\| < \varepsilon$ for all $s \ge n_0$. Since $C_{(p)}(\Delta^m)$ is a linear space, we have $x = x_{n_0} - (x_{n_0} - x) \in C_{(p)}(\Delta^m)$. This completes the proof. □

We state the following result without proof.

Theorem 2.5. *Let (x_s) be a sequence in $C_{(p)}(\Delta^m)$*

(*i*) *If $\|x_s\| \longrightarrow 1$ as $s \longrightarrow \infty$, then $\rho_{\Delta^m}(x_s) \longrightarrow 1$ as $s \longrightarrow \infty$,*

(*ii*) *If $\rho_{\Delta^m}(x_s) \longrightarrow 0$ as $s \longrightarrow \infty$, then $\|x_s\| \longrightarrow 0$ as $s \longrightarrow \infty$.*

Now we show that $C_{(p)}(\Delta^m)$ has the property (H). First we prove the following.

Lemma 2.6. *Let $x \in C_{(p)}(\Delta^m)$ and $(x_s) \subseteq C_{(p)}(\Delta^m)$. If $\rho_{\Delta^m}(x_s) \longrightarrow \varrho_{\Delta^m}(x)$ as $s \longrightarrow \infty$ and $\Delta^m x_s(k) \longrightarrow \Delta^m x(k)$ as $s \longrightarrow \infty$ for all $k \in \mathbb{N}$, then $x_s \longrightarrow x$ as $s \longrightarrow \infty$.*

Proof. Let $\varepsilon > 0$ be given. Since $\varrho_{\Delta^m}(x) = \sum_{i=1}^{m} |x(i)| + \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |\Delta^m x(k)| \right)^{p_n} < \infty$, there exists $k_0 \in \mathbb{N}$ such that

$$\sum_{i=1}^{m} |x(i)| + \sum_{n=k_0+1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |\Delta^m x(k)| \right)^{p_n} < \frac{\varepsilon}{3} \frac{1}{2^{M+1}}. \tag{2.2}$$

Since $\rho_{\Delta^m}(x_s) \longrightarrow \varrho_{\Delta^m}(x)$ as $s \longrightarrow \infty$ and $\Delta^m x_s(k) \longrightarrow \Delta^m x(k)$ as $s \longrightarrow \infty$ for all $k \in \mathbb{N}$, we have

$$\varrho_{\Delta^m}(x_s) - \sum_{n=1}^{k_0} \left(\frac{1}{n} \sum_{k=1}^{n} |\Delta^m x_s(k)| \right)^{p_n} \longrightarrow \varrho_{\Delta^m}(x) - \sum_{n=1}^{k_0} \left(\frac{1}{n} \sum_{k=1}^{n} |\Delta^m x(k)| \right)^{p_n}.$$

Thus there exists $n_0 \in \mathbb{N}$ such that

$$\rho_{\Delta^m}(x_s) - \sum_{n=1}^{k_0} \left(\frac{1}{n} \sum_{k=1}^{n} |\Delta^m x_s(k)| \right)^{p_n} < \rho_{\Delta^m}(x) - \sum_{n=1}^{k_0} \left(\frac{1}{n} \sum_{k=1}^{n} |\Delta^m x(k)| \right)^{p_n} + \frac{\varepsilon}{3} \frac{1}{2^M}, \text{ for all } s \geq n_0 \tag{2.3}$$

and

$$\sum_{n=1}^{k_0} \left(\frac{1}{n} \sum_{k=1}^{n} |\Delta^m x_s(k) - \Delta^m x(k)| \right)^{p_n} < \frac{\varepsilon}{3}, \text{ for all } s \geq n_0 \,. \tag{2.4}$$

It follows from (2.2), (2.3) and (2.4) that for $s \geq n_0$,

$$\begin{aligned} \rho_{\Delta^m}(x_s - x) &= \sum_{i=1}^{m} |x_s(i) - x(i)| + \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |\Delta^m x_s(k) - \Delta^m x(k)| \right)^{p_n} \\ &\leq \sum_{i=1}^{m} |x_s(i)| + \sum_{i=1}^{m} |x(i)| + \sum_{n=1}^{k_0} \left(\frac{1}{n} \sum_{k=1}^{n} |\Delta^m x_s(k) - \Delta^m x(k)| \right)^{p_n} \\ &\quad + \sum_{n=k_0+1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |\Delta^m x_s(k) - \Delta^m x(k)| \right)^{p_n} \end{aligned}$$

$$\leq \frac{\varepsilon}{3} + 2^M \left(\sum_{i=1}^{m} |x_s(i)| + \sum_{i=1}^{m} |x(i)| + \sum_{n=k_0+1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |\Delta^m x_s(k)| \right)^{p_n} \right.$$
$$\left. + \sum_{n=k_0+1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |\Delta^m x(k)| \right)^{p_n} \right)$$
$$= \frac{\varepsilon}{3} + 2^M \left(\rho_{\Delta^m}(x_s) - \sum_{n=1}^{k_0} \left(\frac{1}{n} \sum_{k=1}^{n} |\Delta^m x_s(k)| \right)^{p_n} + \sum_{i=1}^{m} |x(i)| \right.$$
$$\left. + \sum_{n=k_0+1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |\Delta^m x(k)| \right)^{p_n} \right)$$
$$< \frac{\varepsilon}{3} + 2^M \left(\rho_{\Delta^m}(x) - \sum_{n=1}^{k_0} \left(\frac{1}{n} \sum_{k=1}^{n} |\Delta^m x(k)| \right)^{p_n} + \frac{\varepsilon}{3}\frac{1}{2^M} + \sum_{i=1}^{m} |x(i)| \right.$$
$$\left. + \sum_{n=k_0+1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |\Delta^m x(k)| \right)^{p_n} \right)$$
$$= \frac{\varepsilon}{3} + 2^M \left(2\sum_{i=1}^{m} |x(i)| + 2 \sum_{n=k_0+1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |\Delta^m x(k)| \right)^{p_n} + \frac{\varepsilon}{3}\frac{1}{2^M} \right)$$
$$\leq \frac{\varepsilon}{3} + 2^M \left(2\frac{\varepsilon}{3}\frac{1}{2^{M+1}} + \frac{\varepsilon}{3}\frac{1}{2^M} \right)$$
$$= \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

This show that $\rho_{\Delta^m}(x_s - x) \longrightarrow 0$ as $s \longrightarrow \infty$. Hence, by Theorem 2.5(ii), we have $\|x_s - x\| \longrightarrow 0$ as $s \longrightarrow \infty$. □

Theorem 2.7. *The space $C_{(p)}(\Delta^m)$ has the property (H).*

Proof. Let $x \in S\left(C_{(p)}(\Delta^m)\right)$ and $(x_s) \subseteq C_{(p)}(\Delta^m)$ such that $\|x_s\| \longrightarrow 1$ and $x_s \xrightarrow{w} x$ as $s \longrightarrow \infty$. From Theorem 2.3(iii), we have $\rho_{\Delta^m}(x) = 1$, so it follows from Theorem 2.5 (i) that $\rho_{\Delta^m}(x_s) \longrightarrow \rho_{\Delta^m}(x)$ as $s \longrightarrow \infty$. Since the mapping $p_k : C_{(p)}(\Delta^m) \longrightarrow \mathbb{R}$, defined by $p_k(y) = \Delta^m y(k)$, is a continuous linear functional on $C_{(p)}(\Delta^m)$, it follows that $\Delta^m x_s(k) \longrightarrow \Delta^m x(k)$ as $s \longrightarrow \infty$ for all $k \in \mathbb{N}$. Thus, we obtain by *Lemma* 2.6 that $x_s \longrightarrow x$ as $s \longrightarrow \infty$. □

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