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Remark on $\mathcal{P}$-$\mathcal{D}$ Operator

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Abstract : In this short communication, we show that $\mathcal{P}$-$\mathcal{D}$ operator fall in the class of weakly compatible (respectively, occasionally weakly compatible) in the presence of a unique common fixed point (respectively, multiple common fixed points) of the given maps.

Keywords : weakly compatible mappings; $\mathcal{P}$-$\mathcal{D}$ operator; common fixed point.
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1 Introduction

In 1976, Jungck [1] extended and generalized the celebrated Banach contraction principle exploiting the idea of commuting maps. Sessa [2] coined the term weakly commuting maps. Jungck [3] extended the concept of weak commutativity by introducing compatible maps and weakly compatible maps [3, 4]. Since then a lot of research was carried out in proving the existence of unique common fixed point and multiple common fixed points of the given maps. In the literature, authors state illustrative examples to show that each generalizations of commutativity conditions is a proper extension of the previous one. Systematic comparison

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and illustration (up to 2001) can be found in Murthy [5]. Jungck and Rhoades [6] obtained several common fixed point theorems using the idea of occasionally weakly compatible maps given in [7]. Recently, D Đorić et al. [8] proved that the notion of occasionally weakly compatible maps reduce to weakly compatible maps in the presence of a unique point of coincidence (and a unique common fixed point) in case of single valued mappings. In an attempt to generalize the commuting condition, Hussain et. al [9] introduced two new classes of noncommuting self maps and obtained some common fixed point theorems on the space which is more general than metric spaces. Alghamadi, Radenović and Shahzad [10] showed that the maps considered in [9] are actually a weakly compatible. Recently, Pathak et al. [11] introduced the notion of $\mathcal{P}$-$\mathcal{D}$ operator, claiming as generalizations of weakly commuting maps and used theses conditions to obtain common fixed point results. In continuation with work of [8, 10], our main goal in this work is to take up the problem of finding the relation of recently introduced classes of maps in [11] with weakly compatible maps (respectively, occasionally weakly compatible). In this note we shall show that in the presence of a unique common fixed point (respectively, multiple common fixed points) of the given maps, $\mathcal{P}$-$\mathcal{D}$ operator reduces to weak compatibility (respectively, occasionally weakly compatible). Thus no generalizations can be obtained in this way.

2 Preliminaries

Throughout this paper (X, d) denotes a metric space.

Let X be a nonempty set. Suppose that f and g are two self-mappings defined on X. The set of fixed points of $f(resp.,\text{g})$ will be denoted by $F(\text{f})(resp., F(\text{g}))$. Moreover, a point $x \in X$ is said to be a coincidence point (CP) of the pair (f, g) if we have $fx = gx$. If there exists a point w such that $w = fx = gx$, then the point w is called a point of coincidence (POC) for (f, g). We denote by $C(f, g)$ the set of coincidence points of (f, g). Let $PC(f, g)$ represent the set of points of coincidence points of (f, g). A point $x \in X$ is called a common fixed point of f and g if $fx = gx = x$. The self maps f and g on X are called

(i) weakly compatible (WC) [4] if the pair commute at their coincidence points i.e. if $fx = gx$ for some $x \in X$ implies that $fgx = gfx$;

(ii) occasionally weakly compatible (OWC) [7] if $fgx = gfx$ for some $x \in X$ with $fx = gx$;

(iii) $\mathcal{P}$-$\mathcal{D}$ operator pair [11] if there is a point x in X such that $x \in C(f, g)$ and $d(fgx, gfx) \leq diam(PC(f, g))$.

Definition 2.1. ([11]) Let X be a non-empty set and d be a function $d : X \times X \to [0, \infty)$ such that

$$d(x, y) = 0 \quad \text{iff} \quad x = y, \ \forall \ x, y \in X.$$

For a space (X, d) satisfying above and $A \subseteq X$, the diameter of A is defined by

$$diam(A) = sup\{\max\{d(x, y), d(y, x)\} : x, y \in A\}.$$

Definition 2.2. ([11]) A symmetric on a set X is a mapping $d : X \times X \to [0, \infty)$ such that

(i) $d(x, y) = 0$ if and only if $x = y$, and

(ii) $d(x, y) = d(y, x)$.

A set X, together with a symmetric d is called a symmetric space.

3 Main Results

We begin with the following results.

Lemma 3.1. (Abbas and Jungck [12]) If a WC pair (f, g) of self-maps on X has a unique POC, then it has a unique common fixed point.

Lemma 3.2. (Jungck and Rhoades [6]) If an OWC pair (f, g) of self-maps on X has a unique POC, then it has a unique common fixed point.

Proposition 3.3. (Đorić *et al.* [8]) Let a pair of mappings (f, g) have a unique POC. Then it is WC if and only if it is OWC.

Proposition 3.4. *If two OWC pairs (f, S) and (g, T) of self maps on X have a unique POC, then they have a unique common fixed point.*

Proof. Since (f, S) and (g, T) are OWC, there exists $x_1, x_2 \in X$ such that

$$fx_1 = Sx_1 = w_1(say), \quad gx_2 = Tx_2 = w_2(say)$$

and

$$Sw_1 = Sfx_1 = fSx_1 = fw_1, \quad Tw_2 = Tgx_2 = gTx_2 = gw_2.$$

Hence $Sw_1 = fw_1$ and $Tw_2 = gw_2$ are also POC for (f, S) and (g, T). By the hypothesis that the pairs (f, S) and (g, T) have unique POC, will give rise

$$Sw_1 = fw_1 = Tw_2 = gw_2 = w_2 = w_1$$

and hence w_1 is a unique common fixed point for the pairs (f, S) and (g, T). □

Proposition 3.5. *Let the pair of mappings (f, S) and (g, T) have unique POC. Then they are WC if and only they are OWC.*

Proof. In this case, we have only to prove that the OWC implies WC. let $w_1 = fx = Sx = gy = Ty$ be given POC, and since (f, S) and (g, T) are OWC, we have

$$fSx = Sfx \quad and \quad gTy = Tgy.$$

i.e.

$$w_2 = fw_1 = Sw_1 \quad and \quad gw_1 = Tw_1 = w_3(say). \tag{3.1}$$

Let $z \in C(f,S) \cap C(g,T)$ and $x \neq z \neq y$. we have to prove that $fSz = Sfz$ and $gTz = Tgz$.

Now, $gz = Tz = fz = Sz = w_4(say)$ is also a POC for the pairs (f,S) and (g,T). By the assumption that (f,S) and (g,T) have unique POC, we get that

$$w_4 = w_3 = w_2 = w_1$$

$$i.e. \quad fx = Sx = gy = Ty = gz = Tz = fz = Sz = w_1.$$

By (3.1), w_1 is a unique common fixed point of the pairs (f,S) and (g,T), it follows that

$$w_1 = fw_1(= fSz) = Sw_1(= Sfz) = gw_1(= gTz) = Tw_1 = Tgz.$$

Hence $fSz = Sfz$ and $gTz = Tgz$. Therefore, the pairs (f,S) and (g,T) are WC. □

Proposition 3.6. *Let $d : X \times X \to [0,\infty)$ be a mapping such that $d(x,y) = 0$ iff $x = y$. Let the pairs (f,S) and (g,T) of such maps on X have a unique POC. If (f,S) and (g,T) are each $\mathcal{P}$-$\mathcal{D}$ operator, then they are WC.*

Proof. Let (f,S) and (g,T) are each $\mathcal{P}$-$\mathcal{D}$ operator, then there exists points w_1 and w_2 in $PC(f,S)$ and $PC(g,T)$ such that

$$fu_1 = Su_1 = w_1 \quad and \quad d(fSu_1, Sfu_1) \leq diam(PC(f,S))$$

Also,

$$gu_2 = Tu_2 = w_2 \quad and \quad d(gTu_2, Tgu_2) \leq diam(PC(g,T)).$$

By the assumption $w_1 = w_2$. Also, $PC(f,S) \cap PC(g,T)$ is singleton.

If not, then $w_3 = fz = Sz = gz = Tz$ is a POC for the pairs (f,S) and (g,T). By the assumption $w_3 = w_1 = w_2$.

As a result, we have $diam(PC(f,S) \cap PC(g,T)) = 0$, which implies that $d(fSu_1,\ Sfu_1) = 0$ and $d(gTu_2, Tgu_2) = 0$

$$i.e. \quad fSu_1 = Sfu_1 \quad and \quad gTu_2 = Tgu_2$$

and thus (f,S) and (g,T) are each OWC. By Proposition 3.5, (f,S) and (g,T) are each WC. □

Let the control function $\Phi : R^+ \to R^+$ be a continuous nondecreasing function such that $\Phi(2t) \leq 2\Phi(t)$ and $\Phi(0) = 0$ iff $t = 0$.
Let $\psi : R^+ \to R^+$ be another function such that $\psi(t) < t$, for each $t > 0$.

Proposition 3.7. *Let X be a nonempty set and $d : X \times X \to [0,\infty)$ be a function such that $d(x,y) = 0$ iff $x = y$. Suppose (f,S) and (g,T) are $\mathcal{P}$-$\mathcal{D}$ operator pairs and satisfying conditions:*

$$\Phi(d(fx, gy)) \leq \psi(M_\Phi(x,y)) \tag{3.2}$$

where

$$M_\Phi(x,y) = \max\{\Phi(d(Sx,Ty)), \Phi(d(Sx,fx)), \Phi(d(gy,Ty)), \\ 1/2[\Phi(d(fx,Ty)) + \Phi(d(Sx,gy))]\}$$

for each $x, y \in X$. Then the pairs (f,S) and (g,T) are both WC.

Proof. By the hypothesis, there exists $u_1, u_2 \in X$ such that $w_1 = fu_1 = Su_1$ and $w_2 = gu_2 = Tu_2$. We claim that $fu_1 = gu_2$, otherwise by (3.2), we have

$$\begin{aligned} 0 < \Phi(d(fu_1, gu_2)) &\le \psi(M_\Phi(u_1,u_2)) \\ &= \psi\left(\max\{\Phi(d(Su_1,Tu_2)), \Phi(d(Su_1,fu_1), \Phi(d(gu_2,Tu_2)), \right. \\ &\qquad \left. 1/2[\Phi(d(fu_1,Tu_2)) + \Phi(d(Su_1,gu_2))]\}\right) \\ &= \psi(\Phi(d(fu_1,gu_2)) \\ &< \Phi(d(fu_1,gu_2)) \end{aligned}$$

a contradiction. Thus

$$w_1 = fu_1 = Su_1 = gu_2 = Tu_2 = w_2.$$

It remains to show that (f,S) and (g,T) have a unique POC. Suppose, there exists another point w_3 such that

$$w_3 = fu_3 = Su_3 = gu_3 = Tu_3 \quad \textit{and} \quad w_3 \ne w_1.$$

Then, we have

$$\begin{aligned} 0 < \Phi(d(w_1,w_3)) &= \Phi(d(fu_1,gu_3)) \\ &\le \psi(M_\Phi(u_1,u_3)) \\ &= \psi\left(\max\{\Phi(d(Su_1,Tu_3)), \Phi(d(Su_1,fu_1)), \Phi(d(gu_3,Tu_3)), \right. \\ &\qquad \left. 1/2[\Phi(d(fu_1,Tu_3)) + \Phi(d(Su_1,gu_3))]\}\right) \\ &= \psi(\Phi(fu_1,gu_3)) \\ &< \Phi(d(fu_1,gu_3)) \end{aligned}$$

a contradiction. Thus, $w_1 = fu_1 = Su_1 = gu_3 = Tu_3 = w_3$, i.e. (f,S) and (g,T) have a unique POC. By Proposition 3.5, (f,S) and (g,T) are both WC. $\square$

Remark 3.8. The Proposition 3.7 remain true if we replace (3.2) by the following:

$$d(fx,gy) \le \phi(\max\{d(Sx,Ty), d(Sx,fx), d(Ty,gy), d(Sx,gy), d(Ty,fx)\}),$$

where $\phi : R^+ \to R^+$ be nondecreasing function satisfying the condition $\phi(t) < t$, for each $t > 0$.

Let $\mathcal{F}$ denote all functions $f : [0,\infty) \to [0,\infty)$ such that $f(t) = 0$ if and only if $t = 0$. We denote by Ψ and Φ be subsets of $\mathcal{F}$ such that

$$\begin{aligned} \Psi &= \{\psi \in \mathcal{F} : \psi \text{ is continuous and nondecreasing}\}, \\ \Phi &= \{\phi \in \mathcal{F} : \phi \text{ is lower semi-continuous}\}. \end{aligned}$$

Proposition 3.9. *Let X be a nonempty set and $d : X \times X \to [0, \infty)$ be a function such that $d(x, y) = 0 \iff x = y$. Suppose (S, T) is a $\mathcal{P}$-$\mathcal{D}$ operator pair and satisfying conditions:*

$$\psi(d(Tx, Ty)) \leq \psi(M(x, y)) - \phi(M(x, y)) + LN(x, y) \tag{3.3}$$

where $\phi \in \Phi$, $\psi \in \Psi$, $L \geq 0$ and

$$M(x, y) = \max\{d(Sx, Sy), d(Sx, Ty), d(Sy, Tx), d(Sy, Ty), d(Sx, Tx)\}$$

$$N(x, y) = \min\{d(Sx, Ty), d(Sy, Tx), d(Sy, Ty), d(Sx, Tx)\}$$

for each $x, y \in X$. Then, the pair (S, T) is WC.

Proof. Since the pair (S, T) is $\mathcal{P}$-$\mathcal{D}$ operator, then there exists u in $PC(S, T)$ such that $u = Sp = Tp, (p \in X)$. Now, we show that the pair (S, T) is with the unique point of coincidence. For this, we suppose that $v \neq u$ is an arbitrary point of coincidence of the pair (S, T). It follows that there exists $q \in X, q \neq p$ such that $Sq = Tq = v$ and $Sp = Tp = u$. Then from (3.3) for $x = p, y = q$ we obtain

$$\psi(d(u, v)) \leq \psi(d(u, v) - \phi(d(u, v)$$

a contradiction unless $v = u$. Thus the pair (S, T) has unique point of coincidence. According to Proposition 3.6 the pair (S, T) is WC. □

Remark 3.10. The conclusion of Propositions 3.6, 3.7 and 3.9, suggest that there is no need to find common fixed point of $\mathcal{P}$-$\mathcal{D}$ operators using the contractive type conditions. Thus the results from [11] (Theorems 4.1, 4.4, 4.6, 4.8, 4.10, 4.12, 5.1, 5.4, 5.5, 5.7, 5.9, 5.11, 5.13 and Corollaries 4.2, 5.3) are not generalizations (extension) of some common fixed point theorems due to Bhatt et al. [13], Jungck and Rhoades [14] and Hussain et al. [9]. Moreover, all mappings in these results are WC.

The following example shows that the assumption about the uniqueness of POC in Proposition 3.4, 3.5 and 3.6 cannot be removed.

Example 3.11. [11] Let $X = [0, 1]$ and defined $d : X \times X \to [0, \infty)$ by

$$d(x, y) = \begin{cases} e^{x-y} - 1 & \text{if } x \geq y \\ e^{y-x} & \text{if } x < y. \end{cases}$$

Define $f, g : X \to X$ by

$$f(x) = \begin{cases} 1 & \text{if } x = 0, \\ x^2 & \text{if } 0 < x \leq 1/2, \\ 1 & \text{if } 1/2 < x \leq 1,, \end{cases} \qquad g(x) = \begin{cases} 1 & \text{if } x = 0, \\ x/2 & \text{if } 0 < x \leq 1. \end{cases}$$

Here $C(f,g) = \{0, 1/2\}$ and $POC(f,g) = \{1, 1/4\}$.
Clearly, (f,g) is $\mathcal{P}$-$\mathcal{D}$ operator pair, but not commuting, not weakly compatible and not occasionally weakly compatible. Note that the esteemed pair has no common fixed point.

Remark 3.12. It is very clear from the definitions of occasionally weakly compatible and common fixed point that even in case of existence of multiple common fixed points of the involve maps $\mathcal{P}$-$\mathcal{D}$ always imply occasionally weakly compatible (indeed, not necessarily weakly compatible). Thus no generalizations can be obtained in this way.

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