



# Best Wavelet Approximation of Functions Belonging to Generalized Lipschitz Class using Haar Scaling function

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**Abstract :** In this paper, four new theorem on best wavelet approximation of the functions  $f \in Lip\alpha$ ,  $0 < \alpha < 1$ ,  $f \in Lip(\xi, p)$ ,  $1 \leq p < \infty$ , have been estimated.

**Keywords :**  $Lip\alpha$  class of functions;  $Lip(\xi, p)$  class; multiresolution analysis; scaling function  $\phi$  and best wavelet approximation.

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## 1 Introduction

In wavelet analysis, a target function is decomposed into a wavelet series of building blocks. Thus a target function is approximated by partial sums of this series. Let the number of selected terms of wavelet series depending on the function concerned then non linear approximation is introduced. This is also known as n-terms approximation. In 1907, Schmidh, at first, introduced the idea of approximation. Working in this direction, in 1979 Oskolkov discussed the n-terms

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approximation. The wavelet approximation has been studied by several researchers like Natanson [1] Meyer [2] and Morlet et al [3].

The purpose of this paper is to determine the degree of approximation of  $f \in Lip\alpha$ ,  $0 < \alpha \leq 1$  under supremum norm and to generalize this result for the function  $f \in Lip(\xi, p)$ ,  $1 \leq p < \infty$ .

## 2 Definitions and Preliminaries

### 2.1 Function of $Lip\alpha$ Class

A function  $f \in Lip\alpha$  if

$$|f(x) - f(y)| = O(|x - y|^\alpha), \text{ for } 0 < \alpha \leq 1, \text{ (Titchmarsh [4], p.406).}$$

$$f(x) = |x|^{\frac{1}{2}} \quad \forall x \in [0, 1], \quad 0 < \alpha < 1.$$

If capital 'O' is replace by little 'o' in the above definition then

$$f \in lip\alpha$$

i.e.

$$f \in lip\alpha, 0 < \alpha < 1,$$

if

$$|f(x) - f(y)| = o(|x - y|^\alpha), \text{ for } 0 < \alpha < 1.$$

Consider a function  $g(x) = x^2 \quad \forall x \in [0, 1]$ .

For this,

$$\begin{aligned} g(x+t) - g(x) &= (x+t)^2 - x^2 \\ &= t(2x+t), \end{aligned}$$

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{g(x+t) - g(x)}{t^\alpha} &= \lim_{t \rightarrow 0} \frac{t(2x+t)}{t^\alpha}, \quad 0 < \alpha < 1 \\ &= 0. \end{aligned}$$

Thus  $f \in lip\alpha, 0 < \alpha < 1$ . Next,

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{h(x+t) - h(x)}{t} = 0 &\Leftrightarrow h'(x) = 0 \quad \forall x \in [0, 1] \\ &\Leftrightarrow h \text{ is constant function.} \end{aligned}$$

In other words, if

$$|h(x+t) - h(x)| = o(t)$$

$$\text{i.e. } h \in lip(1)$$

then  $h$  is constant function.

## 2.2 Function of $Lip(\xi, p)$ Class

Let  $\xi$  be a monotonic increasing function of  $t$   
 $f \in Lip(\xi, p)$  if

$$\left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(x+t) - f(x)|^p dx \right\}^{\frac{1}{p}} = O(\xi(t)), 1 \leq p < \infty, \quad (\text{Siddiqi [5]})$$

and  $f \in lip(\xi, p)$  if

$$\left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(x+t) - f(x)|^p dx \right\}^{\frac{1}{p}} = o(\xi(t)), 1 \leq p < \infty,$$

### Remark

- It is important to note that if  $\xi(t) = t^\alpha$   $0 < \alpha \leq 1$ , in  $Lip(\xi, p)$  then  $Lip(\xi, p)$  reduces to the class  $Lip(\alpha, p)$ .
- If  $p \rightarrow \infty$  in  $Lip(\alpha, p)$  then this reduces to  $Lip\alpha$ .
- $lip(\xi, p)$  class of functions is the generalization of  $lip(\alpha, p)$  and  $lip\alpha$  classes.

## 2.3 Multiresolution Analysis and Haar Scaling Function

A multiresolution analysis of  $L^2(\mathbb{R})$  is defined as a sequence of closed subspaces  $V_j$  of  $L^2(\mathbb{R})$ ,  $j \in \mathbb{Z}$ , with the following properties:

1.  $V_j \subset V_{j+1}$ ,
2.  $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}$ ,
3.  $f(x) \in V_0 \Leftrightarrow f(x+1) \in V_0$ ,
4.  $\bigcup_{j=-\infty}^{\infty} V_j$  is dense in  $L^2(\mathbb{R})$  and  $\bigcap_{j=-\infty}^{\infty} V_j = \{0\}$ ,
5. Suppose a function  $\phi \in V_0$ , exists such that the collection  $\{\phi(x-k); k \in \mathbb{Z}\}$  is a Riesz basis of  $V_0$ .

Let  $\psi \in L^2(\mathbb{R})$ , and  $\psi_{j,k} := 2^{\frac{j}{2}} \psi(2^j x - k)$  and

$$W_j := \text{clos} \langle \psi_{j,k} : k \in \mathbb{Z} \rangle.$$

Then this family of subspaces of  $L^2(\mathbb{R})$  gives a direct sum decomposition of  $L^2(\mathbb{R})$  is the same that every  $f \in L^2(\mathbb{R})$  has a unique decomposition

$$f(x) = \cdots + g_{-2}(x) + g_{-1}(x) + g_0(x) + g_1(x) + g_2(x) + \cdots$$

where  $g_j \in W_j$  for all  $j \in \mathbb{Z}$  and we describe this by writing

$$L^2(\mathbb{R}) = \bigoplus_{j=-\infty}^{\infty} W_j$$

Thus

$$V_j := \bigoplus_{k=-\infty}^{j-1} W_k.$$

$\{\psi_{j,k}; k \in \mathbb{Z} \text{ where } \psi_{j,k} = 2^{\frac{j}{2}}\psi(2^j x - k)\}$ , is a Riesz basis of  $W_j$ .

Let  $\phi \in V_0$ , since  $V_0 \subset V_1$ , a sequence  $\{h_k\} \in l^2(\mathbb{Z})$  exists such that the 'phi' function satisfies,

$$\phi(x) = 2 \sum_{j=-\infty}^{\infty} h_k \phi(2x - k). \tag{2.1}$$

This functional equation is known as the refinement equation or the dilation equation or the two-scale difference equation. The collection of functions  $\{\phi_{j,k}; k \in \mathbb{Z}\}$ , with  $\phi_{j,k}(x) = 2^{\frac{j}{2}}\phi(2^j x - k)$ , is a Riesz basis of  $V_j$ . Integrating equation (2.1), and dividing by the (non-vanishing) integral of  $\phi$ , we have

$$\sum_{k=-\infty}^{\infty} h_k = 1 \tag{2.2}$$

A function  $\phi \in L^2(\mathbb{R})$  is called a scaling function, if the subspace  $V_j$ , defined by

$$V_j := \text{clos}_{L^2(\mathbb{R})} \{\phi_{j,k} : k \in \mathbb{Z}\}, j \in \mathbb{Z}$$

satisfy the properties (1) to (5) stated above in this section. It is important to note that the scaling function  $\phi$  generates a Multiresolution analysis  $\{V_j\}$  of  $L^2(\mathbb{R})$ .

Haar scaling function, denoted by  $\phi$ , is defined by

$$\phi(t) = \chi_{[0,1)} = \begin{cases} 1, & 0 \leq x < 1; \\ 0, & \text{otherwise.} \end{cases}$$

The family of functions

$$\{\phi_{j,k} = 2^{\frac{j}{2}}\phi(2^j \cdot - k) \text{ where } j, k \in \mathbb{Z}\},$$

is called the system of Haar scaling functions .

The function  $\chi_{[0,1)}$  is discontinuous at 0 and 1. For each  $j, k \in \mathbb{Z}$ ,  $\{\phi_{j,k}\}$  where

$$\phi_{j,k} = 2^{\frac{j}{2}}\phi(2^j \cdot - k) = D_{2^j} T_k \phi(t),$$

where dilation operator  $D_a f(x) = a^{\frac{1}{2}} f(ax)$  and translation operator  $T_k f(x) = f(x - k)$ . The  $\phi = \chi_{[0,1)}$  generates an MRA  $\{V_j\}$  of  $L^2(\mathbb{R})$ . If the scaling function  $\phi \in L^1(\mathbb{R})$ , then it is uniquely defined by its dilation equation and the normalization ([6])

$$\int_{-\infty}^{\infty} \phi(x) dx = 1 \tag{2.3}$$

## 2.4 Projection $P_n f$

Let  $P_n f$  be the orthogonal projection of  $L^2(\mathbb{R})$  onto  $V_n$ . Then

$$P_n f = \sum_{k=-\infty}^{\infty} a_{n,k} \phi_{n,k}, \quad n = 1, 2, 3, \dots,$$

where,  $a_{n,k} = \langle f, \phi_{n,k} \rangle$ .

Thus,

$$P_n f = \sum_{k=-\infty}^{\infty} \langle f, \phi_{n,k} \rangle \phi_{n,k} \quad (\text{Sweldens and Piessens [7]}).$$

## 2.5 Wavelet Approximation

The wavelet approximation under supremum norm is defined by

$$\begin{aligned} E_n(f) &= \|f - P_n f\|_{\infty} \\ &= \sup_x |(f(x) - P_n f(x))| \quad (\text{Zygmund [8], p.114}). \end{aligned}$$

We define,

$$\|f\|_p = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^p dx \right\}^{\frac{1}{p}}, \quad 1 \leq p < \infty.$$

The degree of wavelet approximation  $E_n(f)$  of  $f$  by  $P_n f$  under norm  $\|\cdot\|_p$  is given by

$$E_n(f) = \min_{P_n f} \|f - P_n f\|_p.$$

If  $E_n(f) \rightarrow 0$  as  $n \rightarrow \infty$  then  $E_n(f)$  is called the best approximation of  $f$  of order  $n$  (Zygmund [8], p.115).

## 2.6 Generalized Minkowski's Inequality:

Generalized Minkowski Inequality:

$$\left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| \int_a^b g(t, x) dt \right|^p dx \right\}^{\frac{1}{p}} \leq \int_a^b \left\{ \frac{1}{2\pi} \int_0^{2\pi} |g(t, x)|^p dx \right\}^{\frac{1}{p}} dt, \quad 1 \leq p < \infty$$

(Chui [9] p.37).

## 3 Main Results

In this paper, four new theorems have been established in the following forms:

**Theorem 1.** *If a function  $f \in Lip_\alpha[0, 1]$ ,  $0 < \alpha \leq 1$ , then the best wavelet approximation  $E_n(f)$  of  $f$  is given by*

$$E_n(f) = \|f - P_n f\|_\infty = O\left(\frac{1}{2^{n\alpha}}\right), 0 < \alpha \leq 1 \quad n = 1, 2, 3, \dots$$

**Theorem 2.** *Let  $\xi$  be a monotonic increasing function of  $t$  such that*

$$\left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(x+t) - f(x)|^p \right\}^{\frac{1}{p}} = O(\xi(t)), \quad 1 \leq p < \infty,$$

and  $\xi(t) \rightarrow 0$  as  $t \rightarrow 0^+$ . Then the best wavelet approximation  $E_n(f)$  of a function  $f \in Lip_{(\xi,p)}[0, 1]$  satisfies:

$$E_n(f) = \min_{P_n f} \|f - P_n f\|_p = O\left(\xi\left(\frac{1}{2^n}\right)\right).$$

**Theorem 3.**  $f \in lip_\alpha[0, 1]$ ,  $0 < \alpha < 1$ , i.e.

$$|f(x+t) - f(x)| = o(t^\alpha) \iff E_n(f) = o\left(\left(\frac{1}{2^n}\right)^\alpha\right).$$

**Theorem 4.**  $f \in lip_{(\xi,p)}[0, 1]$ ,  $1 \leq p < \infty$

$$\iff E_n(f) = o\left(\xi\left(\frac{1}{2^n}\right)\right), \quad n = 1, 2, 3, \dots$$

### 3.1 Proof of Theorem 1

Projection operator  $P_n f : L^2(\mathbb{R}) \rightarrow V_n$  is defined as

$$P_n f = \sum_{k \in \mathbb{Z}} a_{n,k} \phi_{n,k}, \quad n = 1, 2, 3, \dots$$

where

$$\begin{aligned} a_{n,k} &= \langle f, \phi_{n,k} \rangle \\ &= \int_{-\infty}^{\infty} f(y) \overline{\phi_{n,k}(y)} dy \end{aligned}$$

Thus

$$\begin{aligned}
(P_n f)(x) &= \sum_{k \in \mathbb{Z}} \left\{ \int_{-\infty}^{\infty} f(y) \overline{\phi_{n,k}(y)} dy \right\} \phi_{n,k}(x) \\
&= \int_{-\infty}^{\infty} f(y) \left( \sum_{k=-\infty}^{\infty} \phi_{n,k}(x) \overline{\phi_{n,k}(y)} \right) dy \\
&= \int_{-\infty}^{\infty} f(y) \left( \sum_{k=-\infty}^{\infty} 2^{\frac{n}{2}} \phi(2^n x - k) 2^{\frac{n}{2}} \overline{\phi(2^n y - k)} \right) dy \\
&= 2^n \int_{-\infty}^{\infty} f(y) \left( \sum_{k=-\infty}^{\infty} \phi(2^n x - k) \overline{\phi(2^n y - k)} \right) dy \\
&= 2^n \int_{-\infty}^{\infty} f(y) K(2^n x, 2^n y) dy.
\end{aligned}$$

$$\text{Since } K(x, y) = \sum_{k=-\infty}^{\infty} \phi(x - k) \overline{\phi(y - k)},$$

$$\text{and } \int_{-\infty}^{\infty} K(x, y) dy = 1, \quad x \in \mathbb{R},$$

therefore, replacing  $y \rightarrow 2^n y$  and  $x \rightarrow 2^n x$ , we have

$$2^n \int_{-\infty}^{\infty} K(2^n x, 2^n y) dy = 1.$$

Next,

$$\begin{aligned}
f(x) &= f(x) 2^n \int_{-\infty}^{\infty} K(2^n x, 2^n y) dy \quad (\because 2^n \int_{-\infty}^{\infty} K(2^n x, 2^n y) dy = 1) \\
&= 2^n \int_{-\infty}^{\infty} K(2^n x, 2^n y) f(x) dy.
\end{aligned}$$

Therefore

$$\begin{aligned}
(P_n f)(x) - f(x) &= 2^n \int_{-\infty}^{\infty} f(y) K(2^n x, 2^n y) dy - 2^n \int_{-\infty}^{\infty} K(2^n x, 2^n y) f(x) dy, \\
&= 2^n \int_{-\infty}^{\infty} K(2^n x, 2^n y) [f(y) - f(x)] dy, \\
&= \int_{-\infty}^{\infty} K(2^n x, y) [f(2^{-n} y) - f(x)] dy, \text{ replacing } y \text{ by } 2^{-n} y, \\
&= \int_{-\infty}^{\infty} K(2^n x, 2^n x - u) [f(x - 2^{-n} y) - f(x)] du, \quad 2^{-n} y = x - 2^{-n} u, \\
&= \int_{-\infty}^{\infty} K(2^n x, 2^n x - y) [f(x - 2^{-n} y) - f(x)] dy, \text{ replacing } u \text{ by } y.
\end{aligned}$$

Haar Scaling function, denoted by  $\phi_H$ , is defined by

$$\chi_{[0,1]} = \phi_H = \begin{cases} 1, & 0 \leq x < 1; \\ 0, & \text{otherwise.} \end{cases}$$

Then,

$$\begin{aligned} |(P_n f)(x) - f(x)| &\leq \int_{-\infty}^{\infty} |K(2^n x, 2^n x - y)| |f(x - 2^{-n}y) - f(x)| dy, \\ &\leq \int_{-\infty}^{\infty} |K(2^n x, 2^n x - y)| dy \sup_y |f(x - 2^{-n}y) - f(x)|, \\ &\quad \text{by Hölder's inequality,} \\ &= 2^n \int_{-\infty}^{\infty} |K(2^n x, 2^n v)| dy \sup_y |f(x - 2^{-n}y) - f(x)|, \\ &\quad \text{taking } 2^n x - y = 2^n v, \text{ in first factor only,} \\ &= 2^n \int_{-\infty}^{\infty} |K(2^n x, 2^n y)| dy \sup_y |f(x - 2^{-n}y) - f(x)|, \\ &\quad \text{replacing } v \text{ by } y \text{ in first factor only,} \\ &= \sup_y |f(x - 2^{-n}y) - f(x)|, \\ &\quad \text{since } 2^n \int_{-\infty}^{\infty} |K(2^n x, 2^n y)| dy = O(1), \\ &= \sup_{y \in [0,1]} (O|2^{-n}y|^\alpha), \\ &\quad |f(x - 2^{-n}y) - f(x)| = O(|2^{-n}y|), \quad f \in \text{Lip}_\alpha[0, 1], \\ &= O\left(\int_0^1 (2^{-n}y)^\alpha dy\right) \\ &= O\left(\left(\frac{1}{2^n}\right)^\alpha \int_0^1 y^\alpha dy\right) \\ &= O\left(\frac{1}{2^{n\alpha}}\right) \left(\frac{1}{1+\alpha}\right) \\ &= O\left(\frac{1}{2^{n\alpha}}\right). \end{aligned}$$

Thus

$$\sup_x \|(P_n f)(x) - f(x)\|_\infty = \|P_n f - f\|_\infty = O\left(\frac{1}{2^{n\alpha}}\right),$$

Hence

$$E_n(f) = O\left(\frac{1}{2^{n\alpha}}\right).$$

**Remark**



Converse of Theorem (1) is also true.

$$E_n(f) = O\left(\frac{1}{2^{n\alpha}}\right), \quad n = 1, 2, 3, \dots$$

Let  $T := \{0 = t_0 < t_1 < t_2 < \dots < t_{2^n-1} < t_{2^n} = 1\}$  be a division/partition of  $[0, 1]$  and  $I_k = [t_{2^k-1}, t_{2^k}]$ ,  $1 \leq k \leq n$ .  $x, y \in I_k$  for some  $1 \leq k \leq n$ .

Then

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - (P_n f)(x) + (P_n f)(x) - (P_n f)(y) + (P_n f)(y) - f(y)| \\ &\leq |f(x) - (P_n f)(x)| + |(P_n f)(x) - (P_n f)(y)| + |(P_n f)(y) - f(y)| \\ &\leq \|f - (P_n f)\|_\infty + \sup_{x, y \in T} |(P_n f)(x) - (P_n f)(y)| + \|(P_n f) - f\|_\infty \\ &= O\left(\frac{1}{2^{n\alpha}}\right) + 0 + O\left(\frac{1}{2^{n\alpha}}\right) \\ &= O\left(\frac{1}{2^{n\alpha}}\right) \\ &= O(|x - y|^\alpha) \end{aligned}$$

so  $f \in Lip_\alpha[0, 1]$ .

Thus the Theorem 1 is completely established.

### 3.2 Proof of Theorem 2

Following the proof of theorem (1),

$$\begin{aligned} |(P_n f)(x) - f(x)| &= \int_{-\infty}^{\infty} |K(2^n x, 2^n x - y)| |f(x - 2^{-n}y) - f(x)| dy \\ &\leq O(1) \int_0^1 |f(x - 2^{-n}y) - f(x)| dy \end{aligned}$$

Applying generalized Minkoski's inequality in above expression, we have

$$\begin{aligned} \|P_n f - f\|_p &= O(1) \int_0^1 \|f(x - 2^{-n}y) - f(x)\|_p dy \\ &= O(1) \int_0^1 \xi(2^{-n}y) dy \\ &= O\left(\xi\left(\frac{1}{2^n}\right)\right) \int_0^1 dy, \quad f \in Lip(\xi, p) \\ &= O\left(\xi\left(\frac{1}{2^n}\right)\right) \end{aligned}$$

$$E_n(f) = \min_{P_n f} \|f - P_n f\|_p = O\left(\xi\left(\frac{1}{2^n}\right)\right), \quad n = 1, 2, 3, \dots$$

**Remark**

Converse of Theorem 2 is also, true.

$$\begin{aligned} |f(x+t) - f(x)| &= |f(x+t) - (P_n f)(x+t) + (P_n f)(x+t) - (P_n f)(x) \\ &\quad + (P_n f)(x) - f(x)| \\ &\leq |f(x+t) - (P_n f)(x+t)| + |(P_n f)(x+t) - (P_n f)(x)| \\ &\quad + |(P_n f)(x) - f(x)| \end{aligned}$$

so

$$\begin{aligned} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(x+t) - f(x)|^p dx \right\}^{\frac{1}{p}} &\leq \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(x+t) - (P_n f)(x+t)|^p dx \right\}^{\frac{1}{p}} \\ &\quad + \left\{ \frac{1}{2\pi} \int_0^{2\pi} |(P_n f)(x+t) - (P_n f)(x)|^p dx \right\}^{\frac{1}{p}} \\ &\quad + \left\{ \frac{1}{2\pi} \int_0^{2\pi} |(P_n f)(x) - f(x)|^p dx \right\}^{\frac{1}{p}} \\ &= O\left(\xi\left(\frac{1}{2^n}\right)\right) + 0 + O\left(\xi\left(\frac{1}{2^n}\right)\right) \\ &= O\left(\xi\left(\frac{1}{2^n}\right)\right) \end{aligned}$$

so  $f \in Lip_{(\xi,p)}[0, 1]$ .

Thus the Theorem 2 is completely established.

**3.3 Proof of Theorems 3 and 4**

Proofs of Theorem 3 & 4 can be developed on the lines of proofs of Theorems 1 & 2 considering  $f \in lip_\alpha[0, 1]$  and  $f \in lip_{(\xi,p)}[0, 1]$  respectively.

**4 Notes**

$E_n(f) \rightarrow 0$  as  $n \rightarrow \infty$  in Theorems 1, 2, 3 & 4 the wavelet approximations determined in these theorems are best possible.

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