Thai Journal of Mathematics
Special Issue (Annual Meeting in Mathematics, 2006) : 61–73

Hyperidentities in $(xx)(yy) \approx x(yx)$ Graph Algebras of Type (2,0)

D. Boonchari and T. Poomsa-ard

Abstract : Graph algebras establish a connection between directed graphs without multiple edges and special universal algebras of type (2,0). We say that a graph G satisfies an identity $s \approx t$ if the corresponding graph algebra $\underline{A(G)}$ satisfies $s \approx t$. A graph $G = (V, E)$ is called an $(xx)(yy) \approx x(yx)$ graph if the graph algebra $\underline{A(G)}$ satisfies the equation $(xx)(yy) \approx x(yx)$. An identity $s \approx t$ of terms s and t of any type τ is called a hyperidentity of an algebra $\underline{A}$ if whenever the operation symbols occurring in s and t are replaced by any term operations of $\underline{A}$ of the appropriate arity, the resulting identities hold in $\underline{A}$.

In this paper we characterize $(xx)(yy) \approx x(yx)$ graph algebras, identities and hyperidentities in $(xx)(yy) \approx x(yx)$ graph algebras.

Keywords : identity, hyperidentity, term, normal form term, binary algebra, graph algebra, $(xx)(yy) \approx x(yx)$ graph algebra
2000 Mathematics Subject Classification :

1 Introduction

An identity $s \approx t$ of terms s, t of any type τ is called a *hyperidentity* of an algebra $\underline{A}$ if whenever the operation symbols occurring in s and t are replaced by any term operations of $\underline{A}$ of the appropriate arity, the resulting identity holds in $\underline{A}$. Hyperidentities can be defined more precisely using the concept of a hypersubstitution.

We fix a type $\tau = (n_i)_{i \in I}, n_i > 0$ for all $i \in I$, and operation symbols $(f_i)_{i \in I}$, where f_i is n_i-*ary*. Let $W_\tau(X)$ be the set of all terms of type τ over some fixed alphabet X, and let $Alg(\tau)$ be the class of all algebras of type τ. Then a mapping

$$\sigma : \{f_i | i \in I\} \longrightarrow W_\tau(X)$$

which assigns to every n_i-*ary* operation symbol f_i an n_i-*ary* term will be called a *hypersubstitution of type* τ (for short, a hypersubstitution). By $\hat{\sigma}$ we denote the extension of the hypersubstitution σ to a mapping

$$\hat{\sigma} : W_\tau(X) \longrightarrow W_\tau(X).$$

The term $\hat{\sigma}[t]$ is defined inductively by

(i) $\hat{\sigma}[x] = x$ for any variable x in the alphabet X
and
(ii) $\hat{\sigma}[f_i(t_1, ..., t_{n_i})] = \sigma(f_i)^{W_\tau(X)}(\hat{\sigma}[t_1], ..., \hat{\sigma}[t_{n_i}])$.
Here $\sigma(f_i)^{W_\tau(X)}$ on the right hand side of (ii) is the operation induced by $\sigma(f_i)$ on the term algebra $W_\tau(X)$.

Graph algebras have been invented in [11] to obtain examples of nonfinitely based finite algebras. To recall this concept, let $G = (V, E)$ be a (directed) graph with the vertex set V and the set of edges $E \subseteq V \times V$. Define the *graph algebra* $\underline{A(G)}$ corresponding to G to have the underlying set $V \cup \{\infty\}$, where ∞ is a symbol outside V, and two basic operations, a nullary operation pointing to ∞ and a binary one denoted by juxtaposition, given for $u, v \in V \cup \{\infty\}$ by

$$uv = \begin{cases} u, & \text{if} \quad (u, v) \in E, \\ \infty, & \text{otherwise.} \end{cases}$$

Graph identities were characterized in [3] by using the rooted graph of a term t where the vertices correspond to the variables occurring in t. Since on a graph algebra we have one nullary and one binary operation, $\sigma(f)$ in this case is a *binary term* in $W_\tau(X)$, i.e. a term built up from variables of a two-element alphabet and a binary operation symbol f corresponding to the binary operation of the graph algebra.

In [9] R. Pöschel has shown that any term over the class of all graph algebras can be uniquely represented by a normal form term and that there is an algorithm to construct the normal form term to every given term t.

In [1] K. Denecke and T. Poomsa-ard characterized graph hyperidentities by using normal form graph hypersubstitutions.

In [6] T. Poomsa-ard characterized associative graph hyperidentities by using normal form graph hypersubstitutions.

In [7] T. Poomsa-ard, J. Wetweerapong and C. Samartkoon characterized idempotent graph hyperidentities by using normal form graph hypersubstitutions.

In [8] T. Poomsa-ard, J. Wetweerapong and C. Samartkoon characterized transitive graph hyperidentities by using normal form graph hypersubstitutions.

A graph $G = (V, E)$ is called $(xx)(yy) \approx x(yx)$ if the graph $\underline{A(G)}$ satisfied the equation $(xx)(yy) \approx x(yx)$. In this paper we characterize $(xx)(yy) \approx x(yx)$ graph algebras, identities and hyperidentities in $(xx)(yy) \approx x(yx)$ graph algebras.

2 $(xx)(yy) \approx x(yx)$ graph algebras

We begin with a more precise definition of terms of the type of graph algebras.

Definition 2.1. The set $W_\tau(X)$ of all terms over the alphabet

$$X = \{x_1, x_2, x_3, ...\}$$

is defined inductively as follows:

(i) every variable $x_i, i = 1, 2, 3, ...,$ and ∞ are terms;

(ii) if t_1 and t_2 are terms, then $f(t_1, t_2)$ is a term; instead of $f(t_1, t_2)$ we will write t_1t_2, for short;

(iii) $W_\tau(X)$ is the set of all terms which can be obtained from (i) and (ii) in finitely many steps.

Terms built up from the two-element set $X_2 = \{x_1, x_2\}$ of variables are thus binary terms. We denote the set of all binary terms by $W_\tau(X_2)$. The leftmost variable of a term t is denoted by $L(t)$ and rightmost variable of a term t is denoted by $R(t)$. A term, in which the symbol ∞ occurs is called a *trivial term.*

Definition 2.2. To each non-trivial term t of type $\tau = (2, 0)$ one can define a directed graph $G(t) = (V(t), E(t))$, where the vertex set $V(t)$ is the set $var(t)$ of all variables occurring in t, and where $E(t)$ is defined inductively by

$$E(t) = \phi \text{ if } t \text{ is a variable and } E(t_1t_2) = E(t_1) \cup E(t_2) \cup \{(L(t_1), L(t_2))\},$$

when $t = t_1t_2$ is a compound term and $L(t_1), L(t_2)$ are the leftmost variables in t_1 and t_2 respectively.

$L(t)$ is called the *root* of the graph $G(t)$ and the pair $(G(t), L(t))$ is the *rooted graph* corresponding to t. Formally, to every trivial term t we assign the empty graph ϕ.

Definition 2.3. We say that a graph $G = (V, E)$ *satisfies an identity* $s \approx t$ if the corresponding graph algebra $\underline{A(G)}$ satisfies $s \approx t$ (i.e. we have $s = t$ for every assignment $V(s) \cup V(t) \to V \cup \overline{\{\infty\})}$, and in this case, we write $G \models s \approx t$.

Definition 2.4. Let $G = (V, E)$ and $G^{'} = (V^{'}, E^{'})$ be graphs. A *homomorphism* h from G into G' is a mapping $h : V \to V^{'}$ carrying edges to edges, that is, for which $(u, v) \in E$ implies $(h(u), h(v)) \in E^{'}$.

In [3] it was proved:

Proposition 2.1. *Let s and t be non-trivial terms from $W_\tau(X)$ with variables $V(s) = V(t) = \{x_0, x_1, ..., x_n\}$ and $L(s) = L(t)$. Then a graph $G = (V, E)$ satisfies $s \approx t$ if and only if the graph algebra $\underline{A(G)}$ has the following property:*

A mapping $h : V(s) \longrightarrow V$ is a $\overline{\text{homomorphism}}$ from $G(s)$ into G iff it is a homomorphism from $G(t)$ into G.

Proposition 2.1 gives a method to check whether a graph $G = (V, E)$ satisfies the equation $s \approx t$. Hence, we can check whether a graph $G = (V, E)$ has an $(xx)(yy) \approx x(yx)$ graph algebra by the following proposition.

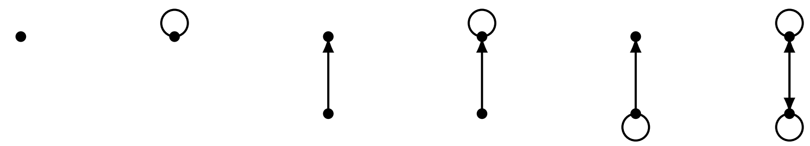
Proposition 2.2. *Let $G = (V, E)$ be a graph. Then the following condition are equivalent:*

1. *G has an $(xx)(yy) \approx x(yx)$ graph algebra.*
2. *If (a, b) inE, then $(a, a), (b, b) \in E$ iff $(b, a) \in E$.*

Proof. (1) $\Rightarrow$ (2) Suppose $G = (V, E)$ has an $(xx)(yy) \approx x(yx)$ graph algebra. Let s and t be non-trivial terms such that $s = (xx)(yy)$, $t = x(yx)$. Let $(a, a), (b, b), (a, b) \in E$ and $h : V(s) \to V$ be the restriction of an evaluation function of the variables such that $h(x) = a$ and $h(y) = b$. We see that h is a homomorphism from $G(s)$ into G. By Proposition 2.1, we have h is a homomorphism from $G(t)$ into G. Since $(y, x) \in E(t)$, we have $(b, a) = (h(y), h(x)) \in E$. Let $(a, b), (b, a) \in E$ and $h : V(t) \to V$ be the restriction of the variables such that $h(x) = a$ and $h(y) = b$. We see that h is a homomorphism from $G(t)$ into G. By Proposition 2.1, we have h is a homomorphism from $G(s)$ into G. Since $(x, x), (y, y) \in E(s)$.Then $(h(x), h(x)) = (a, a) \in E$ and $(h(y), h(y)) = (b, b) \in E$.

(2) $\Rightarrow$ (1) Suppose $G = (V, E)$ is a graph such that if $(a, b) \in E$, then $(a, a), (b, b) \in E$ if and only if $(b, a) \in E$. Let s and t be non-trivial terms such that $s = (xx)(yy)$ and $t = x(yx)$. Suppose $h : V(s) \to V$ is a homomorphism from $G(s)$ into G. Since (x, x),(y, y),$(x, y) \in E(s)$, we have $(h(x), h(x))$,$(h(y), h(y))$,$(h(x), h(y))$ $\in E$. By assumption, we get $(h(y), h(x)) \in E$. Thus h is a homomorphism from $G(s)$ into G. Suppose that h is a homomorphism from $G(t)$ into G. Since $(x, y), (y, x) \in E(t)$, we have $(h(x), h(y)), (h(y), h(x)) \in E$. By assumption, we get $(h(x), h(x)), (h(y), h(y)) \in E$. Therefore h is a homomorphism from $G(s)$ into G. By Proposition 2.1, we get that $\underline{A(G)}$ satisfies $s \approx t$. $\square$

From Proposition 2.2, we see that graphs which have $(xx)(yy) \approx x(yx)$ graph algebras are the following graphs:



G_1 G_2 G_3 G_4 G_5 G_6

and all graphs such that each component of every induced subgraph with at most two vertices is one of these graphs.

3 Identities in $(xx)(yy) \approx x(yx)$ graph algebras

Graph identities were characterized in [3] by the following proposition:

Proposition 3.1. *A non-trivial equation $s \approx t$ is an identity in the class of all graph algebras iff either both terms s and t are trivial or none of them is trivial, $G(s) = G(t)$ and $L(s) = L(t)$.*

Further it was proved.

Proposition 3.2. *Let $G = (V, E)$ be a graph and let $h : X \cup \{\infty\} \longrightarrow V \cup \{\infty\}$ be an evaluation of the variables such that $h(\infty) = \infty$. Consider the canonical extension of h to the set of all terms. Then there holds: if t is a trivial term then $h(t) = \infty$. Otherwise, if $h : G(t) \longrightarrow G$ is a homomorphism of graphs, then $h(t) = h(L(t))$, and if h is not a homomorphism of graphs, then $h(t) = \infty$.*

In [6] the following lemma was proved:

Lemma 3.1. *Let $G = (V, E)$ be a graph, let t be a term and let*

$$h : X \longrightarrow V \cup \{\infty\}$$

be an evaluation of the variables. Then:

(i) If h is a homomorphism from $G(t)$ into G with the property that the subgraph of G induced by $h(V(t))$ is complete, then $h(t) = h(L(t))$;

(ii) If h is a homomorphism from $G(t)$ into G with the property that the subgraph of G induced by $h(V(t))$ is disconnected, then $h(t) = \infty$.

Now, we apply our results to characterize all identities in the class of all $(xx)(yy) \approx x(yx)$ graph algebras. Clearly, if s and t are trivial, then $s \approx t$ is an identity in the class of all $(xx)(yy) \approx x(yx)$ graph algebras and $(xx)(yy) \approx x(yx)$ is an identity in the class of all $(xx)(yy) \approx x(yx)$ graph algebras too. So we consider the case that s and t are non-trivial and different from variables. Then all identities in the class of all $(xx)(yy) \approx x(yx)$ graph algebras are characterized by the following theorem:

Theorem 3.1. *Let s and t are non-trivial terms and let $x_0 = L(s)$. Then $s \approx t$ is an identity in the class of all $(xx)(yy) \approx x(yx)$ graph algebras if and only if the following conditions are satisfied:*
(i) $L(s) = L(t)$;
(ii) $V(s) = V(t)$;
(iii) *for any $x \in V(s)$, x has a vertex which is both out-neighbor and in-neighbor in $G(s)$ iff x has a vertex which is both out-neighbor and in-neighbor in $G(t)$;*
(iv) *for any $x, y \in V(s)$, $x \neq y$, $(x, y) \in E(s)$ or $(y, x) \in E(s)$, each of x, y has a vertex which is both in-neighbor and out-neighbor in $E(s)$ iff $(x, y) \in E(t)$ or $(y, x) \in E(t)$, each of x, y has a vertex which is both in-neighbor and out-neighbor in $E(t)$.*

Proof. (ii). Suppose that $V(s) \neq V(t)$ there is $x \in V(s)$, $x \notin V(t)$. Consider the graphs $G = (V, E)$ such that $V = \{0\}$, E= {(0,0)}. Let $h : V(s) \cup V(t) \to V$, by $h(x) = \infty$ and $h(y) = 0$ for any $y \in V(s) \cup V(t) - \{x\}$. We see that $h(s) = \infty$ and $h(t) = 0$. Hence $\underline{A(G)}$ does not satisfy $s \approx t$.

(i). Let $G = (V, E)$ be a complete graph with $V = V(s) = V(t)$ and let $h : V(s) \to V$ be the restriction of the identity evaluation of the variables. Since G is a complete graph and by Lemma 3.1, we get $L(s) = h(L(s)) = h(s) = h(t) = h(L(t)) = L(t)$.

(iii). For any $x \in V(s)$, suppose that x has a vertex which is both out-neighbor and in-neighbor in $G(s)$ but x has no a vertex which is out-neighbor and in-neighbor in $G(t)$.

Consider the graph $G = (V, E)$, such that $V = \{0, 1, 2\}$,
$E = \{(0, 1), (1, 2), (2, 1), (2, 0), (1, 1), (2, 2)\}$. By Proposition 2.2, $\underline{A(G)}$ has an $(xx)(yy) \approx x(yx)$ graph algebras. Let $h : V(t) \to V$ be the restriction of an evaluation of variables such that $h(x) = 0$, $h(y) = 1$ for all y which are out-neighbors of x in $G(t)$ and $h(z) = 2$ for all other z in $V(t)$. We see that $h(s) = \infty$ and $h(t) = h(L(t))$.

(iv). Suppose that for any $x, y \in V(s)$, $x \neq y$, $(x, y) \in E(s)$ or $(y, x) \in E(s)$, each of x, y has a vertex which is both in-neighbor and out-neighbor in $G(s)$ but $(x, y) \notin E(t)$ and $(y, x) \notin E(t)$ or at least one of x and y has no a vertex which is both in-neighbor and out-neighbor in $G(t)$. In the case $(x, y) \notin E(t)$ and at least one of x and y has no a vertex which is both in-neighbor and out-neighbor in $G(t)$. Consider the graph $G = (V, E)$ such that $V = \{0, 1, 2\}$,$E = \{(0, 1), (1, 2), (2, 0), (0, 2), (2, 2), (0, 0)\}$. By Proposition 2.2, $\underline{A(G)}$ has an $(xx)(yy) \approx x(yx)$ graph algebras, let $h : V(t) \to V$ be the restriction of an evaluation of variable such that $h(x) = 1$, $h(y) = 0$, $h(w) = 0$ for all w are in-neighbor of x and $h(z) = 2$ for all other $z \in V(t)$. We get that $h(s) = \infty$ and $h(t) = h(L(t))$.

In the case $(x, y) \notin E(t)$ and $(y, x) \notin E(t)$. Consider the graph $G = (V, E)$ such that $V = \{0, 1, 2\}$ and $E = \{(0, 0), (1, 1), (2, 2), (0, 2), (2, 0), (1, 2), (2, 1)\}$. By Proposition 2.2, $\underline{A(G)}$ has an $(xx)(yy) \approx x(yx)$ graph algebras and let $h : V(t) \to V$ be the restriction of an evaluation of variable such that $h(x) = 0$, $h(y) = 1$ and $h(z) = 2$ for all other $z \in V(t)$. We see that $h(s) = \infty$ and $h(t) = h(L(t))$. Hence $\underline{A(G)}$ does not satisfy $s \approx t$.

Conversely, suppose that s and t are non-trivial terms satisfy $(i), (ii), (iii)$ and (iv). Let $G = (V, E)$ be an $(xx)(yy) \approx x(yx)$ graph and let $h : V(s) \to V$ be the restriction of evaluation of the variable. Suppose h is an homomorphism from $G(s)$ into G and let $(x, y) \in E(t)$. If $x = y$, then by (iii), there is $z \in V(s)$ such that $(x, z), (z, x) \in E(s)$. Since h is an homomorphism from $G(s)$ into G, we get $(h(x), h(z)), (h(z), h(x)) \in E$. Therefor by Proposition 2.2, $(h(x), h(x)) \in E$. In the case $x \neq y$, by (iv), $(x, y) \in E(s)$ or $(y, x) \in E(s)$ and each of x, y has a vertex which is both in-neighbor and out-neighbor. If $(x, y) \in E(s)$, then $(h(x), h(y)) \in E$. If $(x, y) \notin E(s)$, then $(y, x) \in E(s)$ and each of x, y has a vertex which is both in-neighbor and out-neighbor. So there are $z_1, z_2 \in V(s)$ such that $(z_1, x), (x, z_1), (z_2, y), (y, z_2) \in E(s)$. Since h is an homomorphism from $G(s)$

into G, we get $(h(y), h(x)), (h(z_1), h(x)), (h(x), h(z_1)), (h(z_2), h(y)), (h(y), h(z_2)) \in E(s)$. By Proposition 2.2, we have $(h(x), h(y)) \in E$. Hence h is a homomorphism from $G(t)$ into G. By the same way, we can prove that if h is a homomorphism from $G(s)$ into G, then h is a homomorphism from $G(s)$ into G. Hence, by Proposition 2.1, we get $\underline{A(G)}$ satisfy $s \approx t$. □

4 Hyperidentities in $(xx)(yy) \approx x(yx)$ graph algebras

Let $\mathcal{H}$ be the classes of all $(xx)(yy) \approx x(yx)$ graph algebras and let $Id\mathcal{H}$ be the set of all identities satisfied in $\mathcal{H}$. Now we want to make precise the concept of a hypersubstitution for graph algebras.

Definition 4.1. A mapping $\sigma : \{f, \infty\} \to W_\tau(X_2)$, where f is the operation symbol corresponding to the binary operation of a graph algebra is called *graph hypersubstitution* if $\sigma(\infty) = \infty$ and $\sigma(f) = s \in W_\tau(X_2)$. The graph hypersubstitution with $\sigma(f) = s$ is denoted by σ_s.

Definition 4.2. An identity $s \approx t$ is a $(xx)(yy) \approx x(yx)$ *graph hyperidentity* iff for all graph hypersubstitutions σ, the equations $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ are identities in $\mathcal{H}$.

If we want to check that $s \approx t$ is a hyperidentity in $\mathcal{H}$, we can restrict ourselves to a (small) subset of $Hyp\mathcal{G}$ - the set of all graph hypersubstitutions.

In [4] the following relation between hypersubstitutions was defined:

Definition 4.3. Two graph hypersubstitutions σ_1, σ_2 are called $\mathcal{H}$-*equivalent* iff $\sigma_1(f) \approx \sigma_2(f)$ is an identity in $\mathcal{H}$. In this case we write $\sigma_1 \sim_{\mathcal{H}} \sigma_2$.

In [2] (see also [4]) the following lemma was proved:

Lemma 4.1. *If $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in Id\mathcal{H}$ and $\sigma_1 \sim_{\mathcal{H}} \sigma_2$ then $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in Id\mathcal{H}$.*

Therefore it is enough to consider the quotient set $Hyp\mathcal{G}/\sim_{\mathcal{H}}$.

In [7] it was shown that any non-trivial term t over the class of graph algebras has a uniquely determined normal form term $NF(t)$ and there is an algorithm to construct the normal form term to a given term t. Now we want to describe how to construct the normal form term . Let t be a non-trivial term. The normal form term of t is the term $NF(t)$ constructed by the following algorithm:

(i) Construct $G(t) = (V(t), E(t))$.

(ii) Construct for every $x \in V(t)$ the list $l_x = (x_{i_1}, ..., x_{i_{k(x)}})$ of all out-neighbors (i.e. $(x, x_{i_j}) \in E(t), 1 \leq j \leq k(x)$) ordered by increasing indices $i_1 \leq ... \leq i_{k(x)}$ and let s_x be the term $(...((xx_{i_1})x_{i_2})...x_{i_{k(x)}})$.

(iii) Starting with $x := L(t), Z := V(t), s := L(t)$, choose the variable $x_i \in Z \cap V(s)$ with the least index i, substitute the first occurrence of x_i by the term s_{x_i}, denote the resulting term again by s and put $Z := Z \setminus \{x_i\}$. While $Z \neq \phi$ continue this procedure. The resulting term is the normal form $NF(t)$.

The algorithm stops after a finite number of steps, since $G(t)$ is a rooted graph . Without difficulties one shows $G(NF(t)) = G(t), L(NF(t)) = L(t)$.

In [1] the following definition was given:

Definition 4.4. The graph hypersubstitution $\sigma_{NF(t)}$, is called *normal form graph hypersubstitution*. Here $NF(t)$ is the normal form of the binary term t.

Since for any binary term t the rooted graphs of t and $NF(t)$ are the same, we have $t \approx NF(t) \in Id\mathcal{H}$. Then for any graph hypersubstitution σ_t with $\sigma_t(f) = t \in W_\tau(X_2)$, one obtains $\sigma_t \sim_{\mathcal{H}} \sigma_{NF(t)}$.

In [1] all rooted graphs with at most two vertices were considered. Then we formed the corresponding binary terms and used the algorithm to construct normal form terms. The result is given in the following table:

normal form term	graph hypers.	normal form term	graph hypers.
x_1x_2	σ_0	x_1	σ_1
x_2	σ_2	x_1x_1	σ_3
x_2x_2	σ_4	x_2x_1	σ_5
$(x_1x_1)x_2$	σ_6	$(x_2x_1)x_2$	σ_7
$x_1(x_2x_2)$	σ_8	$x_2(x_1x_1)$	σ_9
$(x_1x_1)(x_2x_2)$	σ_{10}	$(x_2(x_1x_1))x_2$	σ_{11}
$x_1(x_2x_1)$	σ_{12}	$x_2(x_1x_2)$	σ_{13}
$(x_1x_1)(x_2x_1)$	σ_{14}	$x_2((x_1x_1)x_2)$	σ_{15}
$x_1((x_2x_1)x_2)$	σ_{16}	$(x_2(x_1x_2))x_2$	σ_{17}
$(x_1x_1)((x_2x_1)x_2)$	σ_{18}	$(x_2((x_1x_1)x_2))x_2$	σ_{19}

By Theorem 3.1, we have the following relations:

(i) $\sigma_{10} \sim_{\mathcal{H}} \sigma_{12} \sim_{\mathcal{H}} \sigma_{14} \sim_{\mathcal{H}} \sigma_{16} \sim_{\mathcal{H}} \sigma_{18}$,

(ii) $\sigma_{11} \sim_{\mathcal{H}} \sigma_{13} \sim_{\mathcal{H}} \sigma_{15} \sim_{\mathcal{H}} \sigma_{17} \sim_{\mathcal{H}} \sigma_{19}$.

Let $M_{\mathcal{H}}$ be the set of all normal form graph hypersubstitutions in $\mathcal{H}$. Then we get

$$M_{\mathcal{H}} = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}, \sigma_{11}\}$$

We define the product of two normal form graph hypersubstitutions in $M_{\mathcal{H}}$ as follows:

Definition 4.5. *The product $\sigma_{1N} \circ_N \sigma_{2N}$ of two normal form graph hypersubstitutions is defined by $(\sigma_{1N} \circ_N \sigma_{2N})(f) := NF(\sigma_{1N}[\sigma_{2N}(f)])$.*

The following table gives the multiplication of elements in $M_{\mathcal{H}}$.

$\circ_N$	σ_0	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}
σ_0	σ_0	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}
σ_1	σ_1	σ_1	σ_2	σ_1	σ_2	σ_2	σ_1	σ_2	σ_1	σ_2	σ_1	σ_2
σ_2	σ_2	σ_1	σ_2	σ_1	σ_2	σ_1	σ_2	σ_2	σ_2	σ_1	σ_2	σ_2
σ_3	σ_3	σ_1	σ_2	σ_3	σ_4	σ_4	σ_3	σ_4	σ_3	σ_4	σ_3	σ_4
σ_4	σ_4	σ_1	σ_2	σ_3	σ_4	σ_3	σ_4	σ_4	σ_4	σ_3	σ_4	σ_4
σ_5	σ_5	σ_1	σ_2	σ_3	σ_4	σ_0	σ_9	σ_{11}	σ_7	σ_6	σ_{11}	σ_{11}
σ_6	σ_6	σ_1	σ_2	σ_3	σ_4	σ_7	σ_6	σ_7	σ_{10}	σ_{11}	σ_{10}	σ_{11}
σ_7	σ_7	σ_1	σ_2	σ_3	σ_4	σ_6	σ_{11}	σ_{11}	σ_7	σ_6	σ_{11}	σ_{11}
σ_8	σ_8	σ_1	σ_2	σ_3	σ_4	σ_9	σ_{10}	σ_{11}	σ_8	σ_9	σ_{10}	σ_{11}
σ_9	σ_9	σ_1	σ_2	σ_3	σ_4	σ_8	σ_9	σ_{11}	σ_{11}	σ_{10}	σ_{11}	σ_{11}
σ_{10}	σ_{10}	σ_1	σ_2	σ_3	σ_4	σ_{11}	σ_{10}	σ_{11}	σ_{10}	σ_{11}	σ_{10}	σ_{11}
σ_{11}	σ_{11}	σ_1	σ_2	σ_3	σ_4	σ_{10}	σ_{11}	σ_{11}	σ_{11}	σ_{10}	σ_{11}	σ_{11}

In [2] the concept of leftmost normal form graph hypersubsttutions was defined.

Definition 4.6. *A graph hyperstitution σ is called leftmost hypersubstitution if $L(\sigma(f)) = x_1$.*

The set $M_{L(\mathcal{H})}$ of all leftmost normal form graph hypersubstitutions in $M_{\mathcal{H}}$ contains exactly the following elements.

$$M_{L(\mathcal{H})} = \{\sigma_0, \sigma_1, \sigma_3, \sigma_6, \sigma_8, \sigma_{10}\}.$$

In [5] the concept of a proper hypersubstitution of a class of algebras was introduced.

Definition 4.7. *A hypersubstitution σ is called proper with respect to a class $\mathcal{K}$ of algebras if $\widehat{\sigma}[s] \approx \widehat{\sigma}[t] \in Id\mathcal{K}$ for all $s \approx t \in Id\mathcal{K}$.*

A graph hypersubstitution with the property that $\sigma(f)$ contains both variables x_1 and x_2 is called regular. It is easy to check that the set of all regular graph hypersubstitutions forms a groupoid M_{reg}.

We want to prove that $\{\sigma_0, \sigma_6, \sigma_8, \sigma_{10}\}$ is the set of all proper normal form graph hypersubstitutions with respect to $\mathcal{H}$.

In [2] the following lemma was proved.

Lemma 4.2. *For each non-trivial term s ($s \neq x \in X_2$) and for all $u, v \in X_2$, we have :*

(i) $E(\hat{\sigma}_6[s]) = E(s) \cup \{(u,u) \mid (u,v) \in E(s)\}$,

and

(ii) $E(\hat{\sigma}_8[s]) = E(s) \cup \{(v,v) \mid (u,v) \in E(s)\}$.

Then we obtain

Theorem 4.1. $\{\sigma_0, \sigma_6, \sigma_8, \sigma_{10}\}$ *is the set of all proper graph hypersubstitution with respect to the class $\mathcal{H}$ of all $(xx)(yy) \approx x(yx)$ graph algebras.*

Proof. If $s \approx t \in Id\mathcal{H}$ and s, t are trivial terms, then for every graph hypersubstitution $\sigma \in \{\sigma_0, \sigma_6, \sigma_8, \sigma_{10}\}$ the term $\hat{\sigma}[s]$ and $\hat{\sigma}[t]$ are also trivial and thus $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{H}$. In the same manner, we see that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{H}$ for every $\sigma \in \{\sigma_0, \sigma_6, \sigma_8, \sigma_{10}\}$, if $s = t = x$.

Now, assume that s and t are non-trivial terms, different from variables, and $s \approx t \in Id\mathcal{H}$. Then (i)-(iv) of Theorem 3.1 hold.

For σ_6, σ_8 we obtain

$$L(\widehat{\sigma}_6[s]) = L(s) = L(t) = L(\widehat{\sigma}_6[t]),$$

and

$$L(\widehat{\sigma}_8[s]) = L(s) = L(t) = L(\widehat{\sigma}_8[t]).$$

Since σ_6, σ_8 are regular, we have

$$V(\widehat{\sigma}_6[s]) = V(s) = V(t) = V(\widehat{\sigma}_6[t]),$$

and

$$V(\widehat{\sigma}_8[s]) = V(s) = V(t) = V(\widehat{\sigma}_8[t]).$$

By Lemma 4.2, we have

$$\begin{aligned} E(\widehat{\sigma}_6[s]) &= E(s) \cup \{(u,u) \mid (u,v) \in E(s)\}, \\ E(\widehat{\sigma}_6[t]) &= E(t) \cup \{(u,u) \mid (u,v) \in E(t)\}, \\ E(\widehat{\sigma}_8[s]) &= E(s) \cup \{(v,v) \mid (u,v) \in E(s)\}, \\ E(\widehat{\sigma}_8[t]) &= E(t) \cup \{(v,v) \mid (u,v) \in E(t)\}. \end{aligned}$$

For any $x \in V(\widehat{\sigma}_6[s])$, suppose that x has a vertex which is both in-neighbor and out-neighbor in $G(\widehat{\sigma}_6[s])$. Then there exists $y \in V(\widehat{\sigma}_6[s])$ such that $(x,y), (y,x) \in E(\widehat{\sigma}_6[s])$. In the case $x = y$, if $(x,x) \in E(s)$, then x has a vertex which is both in-neighbor and out-neighbor in $G(t)$. By (iii) x has a vertex which is both in-neighbor and out-neighbor in $G(t)$. Thus x has a vertex which is both in-neighbor and out-neighbor in $G(\widehat{\sigma}_6[t])$. If $(x,x) \notin E(s)$, then there is $m \in E(s)$ such that $(x,m) \in E(s)$, $x \neq m$. By Theorem 3.1, $(x,m) \in E(t)$ or $(m,x) \in E(t)$, each of x, m has a vertex which is both in-neighbor and out-neighbor in $E(t)$. By (iii) x has a vertex which is both in-neighbor and out-neighbor in $G(t)$. Thus x has a vertex which is both in-neighbor and out-neighbor in $G(\widehat{\sigma}_6[t])$.

In the case $x \neq y$, then $(x,y),(y,x) \in E(s)$. By (iv), we get x has a vertex which is both in-neighbor and out-neighbor in $G(t)$. Hence x has a vertex which is both in-neighbor and out-neighbor in $G(\widehat{\sigma}_6[t])$. By the same way, we can prove that, if x has a vertex which is both in-neighbor and out-neighbor in $G(\widehat{\sigma}_6[t])$, then x has a vertex which is both in-neighbor and out-neighbor in $G(\widehat{\sigma}_6[s])$.

For any $x,y \in V(\widehat{\sigma}_6[s])$, $x \neq y$. Suppose that $(x,y) \in E(\widehat{\sigma}_6[s])$ or $(y,x) \in E(\widehat{\sigma}_6[s])$, each of x,y has a vertex which is both in-neighbor and out-neighbor in $G(\widehat{\sigma}_6[s])$. In the case $(x,y) \in E(\widehat{\sigma}_6[s])$, then $(x,y) \in E(s)$. By (iv), we have $(x,y) \in E(t)$ or $(y,x) \in E(t)$, each of x,y has a vertex which is both in-neighbor and out-neighbor in $G(t)$. If $(x,y) \in E(t)$, then we get the result. Suppose that $(y,x) \in E(t)$. If x has a vertex which is both in-neighbor and out-neighbor in $G(t)$, then by (iii) we get the result. Otherwise there exists $z \in V(t)$ such that $(x,z) \in E(t)$. Then by (iv)$(x,z) \in E(t)$ or $(z,x) \in E(t)$, each of x,z has a vertex which is both in-neighbor and out-neighbor in $G(t)$. For all cases we get the result.By the same way, we can prove that if $(x,y) \in E(\widehat{\sigma}_6[t])$ or $(y,x) \in E(\widehat{\sigma}_6[t])$, each of x,y has a vertex which is both in-neighbor and out-neighbor in $E(\widehat{\sigma}_6[t])$, then $(x,y) \in E(\widehat{\sigma}_6[s])$ or $(y,x) \in E(\widehat{\sigma}_6[s])$, each of x,y has a vertex which is both in-neighbor and out-neighbor in $E(\widehat{\sigma}_6[s])$. Hence $\widehat{\sigma}_6[s] \approx \widehat{\sigma}_6[s] \in Id\mathcal{H}$. By the similar way, we can prove that $\widehat{\sigma}_8[s] \approx \widehat{\sigma}_8[s] \in Id\mathcal{H}$.

For σ_{10}, since $\sigma_6 \circ_N \sigma_8 = \sigma_{10}$ and σ_6, σ_8 are proper. Then $\widehat{\sigma}_{10}$ is a proper.

For any $\sigma \notin \{\sigma_0, \sigma_6, \sigma_8, \sigma_{10}\}$, we give an identity $s \approx t$ in $\mathcal{H}$ such that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \notin Id\mathcal{G}'$. Clearly, if s and t are terms with different leftmost and different rightmost, then $\hat{\sigma_1}[s] \approx \hat{\sigma_1}[t] \notin Id\mathcal{G}'$, $\hat{\sigma_2}[s] \approx \hat{\sigma_2}[t] \notin Id\mathcal{G}'$ and $\hat{\sigma_3}[s] \approx \hat{\sigma_3}[t] \notin Id\mathcal{G}'$, $\hat{\sigma_4}[s] \approx \hat{\sigma_4}[t] \notin Id\mathcal{G}'$.

Now, let $s = (x_1x_1)(x_2x_2), t = x_1(x_2x_1)$. By Theorem 3.1, we get $s \approx t \in Id\mathcal{H}$. If $\sigma \in \{\sigma_5, \sigma_7, \sigma_9\sigma_{11}\}$, then $L(\sigma(f)) = x_2$. We see that $L(\hat{\sigma}[s]) = x_2$ and $L(\hat{\sigma}[t]) = x_1$ for $\sigma \in \{\sigma_5, \sigma_7, \sigma_9, \sigma_{11}\}$. Thus $\hat{\sigma}[s] \approx \hat{\sigma}[t] \notin Id\mathcal{H}$. □

Now we apply our results to characterize all hyperidentities in the class of all transitive graph algebras. Clearly, if s and t are trivial terms, then $s \approx t$ is a hyperidentity in $\mathcal{H}$ if and only if they have the same leftmost and the same rightmost and $x \approx x, x \in X$ is a hyperidentity in $\mathcal{H}$ too. So we consider the case that s and t are non-trivial and different from variables.

In [1] the concept of a *dual term* s^d of the non-trivial term s was defined in the following way:

If $s = x \in X$, then $x^d = x$; if $s = t_1t_2$, then $s^d = t_2^d t_1^d$. The dual term s^d can be obtained by application of the graph hypersubstitution σ_5, namely, $\hat{\sigma}_5[s] = s^d$. Then, we can prove the following lemma:

Theorem 4.2. *An identity $s \approx t$ in $\mathcal{H}$, where s,t are non-trivial and $s \neq x, t \neq x$, is a hyperidentity in $\mathcal{H}$ if and only if the dual equation $s^d \approx t^d$ is also an identity in $\mathcal{H}$.*

Proof. If $s \approx t$ is a hyperidentity in $\mathcal{H}$, then $\hat{\sigma}_5[s] \approx \hat{\sigma}_5[t]$ is an identity in $\mathcal{H}$, i.e. $s^d \approx t^d$ is an identity in $\mathcal{H}$. Conversely, assume that $s \approx t$ is an identity in $\mathcal{H}$ and that $s^d \approx t^d$ is an identity in $\mathcal{H}$ too. We have to prove that $s \approx t$ is closed under all graph hypersubstitutions from $M_{\mathcal{H}}$.

If $\sigma \in \{\sigma_0, \sigma_6, \sigma_8, \sigma_{10}\}$, then σ is a proper and we get that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{H}$. By assumption, $\hat{\sigma}_5[s] = s^d \approx t^d = \hat{\sigma}_5[t]$ is an identity in $\mathcal{H}$.

For $\sigma_1, \sigma_2, \sigma_3$ and σ_4, we have $\hat{\sigma}_1[s] = L(s) = L(t) = \hat{\sigma}_1[t]$, $\hat{\sigma}_2[s] = L(s^d) = L(t^d) = \hat{\sigma}_2[t]$, $\hat{\sigma}_3[s] = L(s)L(s) = L(t)L(t) = \hat{\sigma}_3[t]$ and $\hat{\sigma}_4[s] = L(s^d)L(s^d) = L(t^d)L(t^d) = \hat{\sigma}_4[t]$.

Because of $\sigma_6 \circ_N \sigma_5 = \sigma_7$, $\sigma_8 \circ_N \sigma_5 = \sigma_9$, $\sigma_{10} \circ_N \sigma_5 = \sigma_{11}$ and $\hat{\sigma}[\hat{\sigma}_5[t']] = \hat{\sigma}[t'^d]$ for all $\sigma \in M_{\mathcal{H}}$, $t' \in W_\tau(X)$, we have that $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t]$, $\hat{\sigma}_9[s] \approx \hat{\sigma}_9[t]$ and $\hat{\sigma}_{11}[s] \approx \hat{\sigma}_{11}[t]$ are identities in $\mathcal{H}$. □

References

[1] K. Denecke and T. Poomsa-ard, Hyperidentities in graph algebras, Contributions to General Algebra and Aplications in Discrete Mathematics, Potsdam 1997, 59-68.

[2] K. Denecke and M. Reichel, Monoids of Hypersubstitutions and M-solid varieties, Contributions to General Algebra, Wien 1995, 117-125.

[3] E. W. Kiss, R. Pöschel and P. Pröhle, Subvarieties of varieties generated by graph algebras, *Acta Sci. Math.* **54**(1990), 57-75.

[4] J. Płonka, Hyperidentities in some of vareties, General Algebra and discrete Mathematics ed. by K. Denecke and O. Lüders, Lemgo 1995, 195-213.

[5] J. Płonka, Proper and inner hypersubstitutions of varieties, Proceedings of the International Conference: Summer School on General Algebra and Ordered Sets 1994, Palacký University Olomouce 1994, 106-115.

[6] T. Poomsa-ard, Hyperidentities in associative graph algebras, *Discussiones Mathematicae General Algebra and Applications* **20**(2000), 169-182.

[7] T. Poomsa-ard, J. Wetweerapong and C. Samartkoon Hyperidentities in idempotent graph algebras, *Thai journal of Mathematics* **2**(2004), 171-181.

[8] T. Poomsa-ard, J. Wetweerapong and C. Samartkoon Hyperidentities in transitive graph algebras, *Discussiones Mathematicae General Algebra and Applications* **25**(2005), 23-37.

[9] R. Pöschel, The equational logic for graph algebras, *Zeitschr.f.math. Logik und Grundlagen d. Math. Bd.* **35**(1989), 273-282.

[10] R. Pöschel, Graph algebras and graph varieties, *Algebra Universalis* **27**(1990), 559-577.

[11] C. R. Shallon, *Nonfinitely based finite algebras derived from lattices*, Ph. D. Dissertation, Uni. of California, Los Angeles, 1979.

(Received 25 May 2006)

D. Boonchari and T. Poomsa-ard
Mathematics Program
Faculty of Science
Mahasarakham University
Mahasarakham 44150, THAILAND.
e-mail: `boonchari@hotmail.com`