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On n-Absorbing Ideals and Two Generalizations of Semiprime Ideals

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Abstract : Let R be a commutative ring and n be a positive integer. A proper ideal I of R is called an *n-absorbing ideal* if whenever $x_1 \cdots x_{n+1} \in I$ for $x_1, \ldots, x_{n+1} \in R$, then there are n of the x_i's whose product is in I. We give a generalization of Prime Avoidance Theorem. Also, an n-Absorbing Avoidance Theorem is proved. Moreover, we introduce the notions of quasi-n-absorbing ideals and of semi-n-absorbing ideals. We say that a proper ideal I of R is a *quasi-n-absorbing ideal* if whenever $a^n b \in I$ for $a, b \in R$, then $a^n \in I$ or $a^{n-1}b \in I$. A proper ideal I of R is said to be a *semi-n-absorbing ideal* if whenever $a^{n+1} \in I$ for $a \in R$, then $a^n \in I$.

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1 Introduction

Throughout this paper R denotes a commutative ring with $1 \neq 0$. Recall that a proper ideal I of a ring R is said to be a *semiprime ideal* if whenever $a^2 \in I$ for some $a \in R$, then $a \in I$. Clearly, I is a semiprime ideal of R if and only if $J^2 \subseteq I$ implies

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that $J \subseteq I$ for every ideal J of R. Badawi [1] said that a proper ideal I of R is a *2-absorbing ideal of* R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. He proved that I is a 2-absorbing ideal of R if and only if whenever I_1, I_2, I_3 are ideals of R with $I_1I_2I_3 \subseteq I$, then $I_1I_2 \subseteq I$ or $I_1I_3 \subseteq I$ or $I_2I_3 \subseteq I$. Anderson and Badawi [2] generalized the notion of 2-absorbing ideals to n-absorbing ideals. A proper ideal I of R is called an *n-absorbing* (resp. *a strongly n-absorbing*) *ideal* if whenever $x_1 \cdots x_{n+1} \in I$ for $x_1, \dots, x_{n+1} \in R$ (resp. $I_1 \cdots I_{n+1} \subseteq I$ for ideals $I_1, \dots, I_{n+1}$ of R), then there are n of the x_i's (resp. n of the I_i's) whose product is in I. Clearly, every strongly n-absorbing ideal is an n-absorbing ideal, but for $n > 2$ the converse may not be holds. For more studies concerning 2-absorbing (submodules) ideals we refer to [3–12]. These concepts motivate us to introduce some generalizations of semiprime ideals. We say that a proper ideal I of R is called a *quasi-n-absorbing (resp. strongly quasi-n-absorbing) ideal* if whenever $a^nb \in I$ for $a, b \in R$ (resp. $I_1^nI_2 \subseteq I$ for ideals I_1, I_2 of R), then $a^n \in I$ or $a^{n-1}b \in I$ (resp. $I_1^n \subseteq I$ or $I_1^{n-1}I_2 \subseteq I$). It is obvious that every strongly quasi-n-absorbing ideal is a quasi-n-absorbing ideal. Also, a quasi-1-absorbing ideal is just a prime ideal. A proper ideal I of R is called a *semi-n-absorbing (resp. strongly semi-n-absorbing) ideal* if whenever $a^{n+1} \in I$ for $a \in R$ (resp. $J^{n+1} \subseteq I$ for ideal J of R), then $a^n \in I$ (resp. $J^n \subseteq I$). With these definitions a semiprime ideal is just a semi-1-absorbing (strongly semi-1-absorbing) ideal. Let M be a nonzero R-module. We say that M is *secondary* precisely when, for each $r \in R$, either $rM = M$ or there exists $n \in \mathbb{N}$ such that $r^nM = 0$. When this is the case, $P := \sqrt{(0 :_R M)}$ is a prime ideal of R: in these circumstances, we say that M is a P-secondary R-module. A *secondary ideal of* R is just a secondary submodule of the R-module R (see [13]). A domain V with the quotient field K is said to be a *valuation domain* if for every $x \in K$, $x \in V$ or $x^{-1} \in V$. A *von-Neumann regular ring* is a ring R such that for every a in R there exists an x in R such that $a = axa$.

We recall from [14] the Prime Avoidance Theorem: Let $P_1, P_2, \dots, P_n$, $n \geq 2$, be ideals of R such that at most two of $P_1, P_2, \dots, P_n$ are not prime. Let I be an ideal of R such that $I \subseteq P_1 \cup P_2 \cup \cdots \cup P_n$. Then $I \subseteq P_i$ for some i with $1 \leq i \leq n$. In [8], Payrovi and Babaei stated the 2-Absorbing Avoidance Theorem.

In section 2, we generalize the Prime Avoidance Theorem. Let I be an ideal of a ring R. We prove that if

$$I \subseteq J \cup K \cup (\cup_{i=1}^{n_1} P_{i,1}^{(1)}) \cup (\cup_{i=1}^{n_2} (P_{i,2}^{(1)} \cap P_{i,2}^{(2)})) \cup \cdots \cup (\cup_{i=1}^{n_m} (P_{i,m}^{(1)} \cap \cdots \cap P_{i,m}^{(m)}))$$

where J, K are ideals of R and $P_{i,j}^{(k)}$'s are prime ideals of R, then either $I \subseteq J$ or $I \subseteq K$ or $I \subseteq P_{i,j}^{(1)} \cap \cdots \cap P_{i,j}^{(j)}$ for some $1 \leq j \leq m$ and some $1 \leq i \leq n_j$.

We prove an n-Absorbing Avoidance Theorem as follows: Let $I_1, I_2, \dots, I_m$ $(m \geq 2)$ be ideals of R such that I_i be an n_i-absorbing ideal of R for every $3 \leq i \leq m$. Suppose that $I_i \nsubseteq (I_j :_R x^{n_j-1}) \subset R$ for every $x \in \sqrt{I_j} \backslash I_j$ with $i \neq j$. If I is an ideal of R such that $I \subseteq I_1 \cup I_2 \cup \cdots \cup I_m$, then $I \subseteq I_i$ for some $1 \leq i \leq m$. Moreover, let $I_1, I_2, \dots, I_m$ $(m \geq 2)$ be ideals of R and I_i be an n_i-absorbing ideal of R for every $2 \leq i \leq m$. Suppose that $I_i \nsubseteq (I_j :_R x^{n_j-1}) \subset R$ for every $x \in \sqrt{I_j} \backslash I_j$ with $i \neq j$. If I is an ideal of R and e is an idempotent element of R such that $I + e \subseteq I_1 \cup I_2 \cup \cdots \cup I_m$, then we show that $(I, e) \subseteq I_i$ for some $1 \leq i \leq m$.

In section 3, we give many properties of quasi-n-absorbing ideals, for example we show that if I_i is a quasi-n_i-absorbing ideal of a ring R for every $1 \leq i \leq k$, then $I_1 \cap I_2 \cap \cdots \cap I_k$ is a quasi-n-absorbing ideal for $n = n_1 + \cdots + n_k$. It is proved that for ideals $I_1, I_2, \ldots, I_t$ of a ring R:

1. If I_1 is quasi-n-absorbing and I_2 is quasi-m-absorbing for $m < n$, then $I_1 \cap I_2$ is quasi-$(n+1)$-absorbing.

2. If $I_1, I_2, \ldots, I_t$ are quasi-n-absorbing, then $I_1 \cap I_2 \cap \cdots \cap I_t$ is quasi-$(n+t)$-absorbing.

3. If I_i is quasi-n_i-absorbing for every $1 \leq i \leq t$ with $n_1 < n_2 < \cdots < n_t$ and $t > 2$, then $I_1 \cap I_2 \cap \cdots \cap I_t$ is quasi-$(n_t + 2)$-absorbing.

Also, it is shown that if I is a secondary ideal of a ring R and J is a quasi-n-absorbing ideal of R, then $I \cap J$ is secondary. For an ideal I of a Prüfer domain R we show that the following assertions hold:

1. If I is a strongly quasi-n-absorbing ideal of R, then $I[X]$ is a quasi-n-absorbing ideal of $R[X]$.

2. If $I[X]$ is a quasi-n-absorbing ideal of $R[X]$, then I is a quasi-n-absorbing ideal of R.

In section 4, it is shown that if I is a semiprime ideal of a ring R, then I is a semi-i-absorbing (resp. quasi-j-absorbing) ideal of R for every $i \geq 1$ (resp. $j > 1$). Let $R = R_1 \times R_2$ be a decomposable ring and L be a proper ideal of R. Then we prove that the following statements are equivalent:

1. L is a quasi-2-absorbing ideal of R;

2. Either $L = I_1 \times R_2$ where I_1 is a quasi-2-absorbing ideal of R_1 or $L = R_1 \times I_2$ where I_2 is a quasi-2-absorbing ideal of R_2 or $L = I_1 \times I_2$ where I_1 is a semiprime ideal of R_1 and I_2 is a semiprime ideal of R_2.

2 Properties of n-Absorbing Ideals

First, we give a generalization of the Prime Avoidance Theorem.

Theorem 2.1 (Generalized prime avoidance theorem). *Let I be an ideal of a ring R. If*

$$I \subseteq J \cup K \cup (\cup_{i=1}^{n_1} P_{i,1}^{(1)}) \cup (\cup_{i=1}^{n_2} (P_{i,2}^{(1)} \cap P_{i,2}^{(2)})) \cup \cdots \cup (\cup_{i=1}^{n_m} (P_{i,m}^{(1)} \cap \cdots \cap P_{i,m}^{(m)}))$$

where J, K are ideals of R and $P_{i,j}^{(k)}$'s are prime ideals of R, then either $I \subseteq J$ or $I \subseteq K$ or $I \subseteq P_{i,j}^{(1)} \cap \cdots \cap P_{i,j}^{(j)}$ for some $1 \leq j \leq m$ and some $1 \leq i \leq n_j$.

Proof. We use induction on m. The case when $m = 1$ is just the Prime Avoidance Theorem. Let $m > 1$ and assume that the claim holds for all positive integers less than m. Let

$$I \subseteq J \cup K \cup (\cup_{i=1}^{n_1} P_{i,1}^{(1)}) \cup (\cup_{i=1}^{n_2} (P_{i,2}^{(1)} \cap P_{i,2}^{(2)})) \cup \cdots \cup (\cup_{i=1}^{n_m} (P_{i,m}^{(1)} \cap \cdots \cap P_{i,m}^{(m)}))$$

where $J,\ K$ are ideals of R and $P_{i,j}^{(k)}$'s are prime ideals of R. If

$$I \subseteq J \cup K \cup (\cup_{i=1}^{n_1} P_{i,1}^{(1)}) \cup (\cup_{i=1}^{n_2} (P_{i,2}^{(1)} \cap P_{i,2}^{(2)})) \cup \cdots \cup (\cup_{i=1}^{n_{m-1}} (P_{i,m-1}^{(1)} \cap \cdots \cap P_{i,m-1}^{(m-1)})),$$

then by the induction hypothesis we are done. Suppose that

$$I \nsubseteq J \cup K \cup (\cup_{i=1}^{n_1} P_{i,1}^{(1)}) \cup (\cup_{i=1}^{n_2} (P_{i,2}^{(1)} \cap P_{i,2}^{(2)})) \cup \cdots \cup (\cup_{i=1}^{n_{m-1}} (P_{i,m-1}^{(1)} \cap \cdots \cap P_{i,m-1}^{(m-1)})),$$

Since

$$I \subseteq J \cup K \cup (\cup_{i=1}^{n_1} P_{i,1}^{(1)}) \cup (\cup_{i=1}^{n_2} (P_{i,2}^{(1)} \cap P_{i,2}^{(2)})) \cup \cdots$$
$$\cup (\cup_{i=1}^{n_{m-1}} (P_{i,m-1}^{(1)} \cap \cdots \cap P_{i,m-1}^{(m-1)})) \cup (\cup_{i=1}^{n_m} (P_{i,m}^{(1)} \cap \cdots \cap P_{i,m}^{(m-1)})),$$

the induction hypothesis implies that $I \subseteq P_{i,m}^{(1)} \cap \cdots \cap P_{i,m}^{(m-1)}$ for some $1 \leq i \leq n_m$. There are two cases:
Case 1. Let $I \subseteq P_{i,m}^{(1)} \cap \cdots \cap P_{i,m}^{(m-1)}$ for every $1 \leq i \leq n_m$. Notice that

$$I \subseteq J \cup K \cup (\cup_{i=1}^{n_1} P_{i,1}^{(1)}) \cup (\cup_{i=1}^{n_2} (P_{i,2}^{(1)} \cap P_{i,2}^{(2)})) \cup \cdots$$
$$\cup (\cup_{i=1}^{n_{m-1}} (P_{i,m-1}^{(1)} \cap \cdots \cap P_{i,m-1}^{(m-1)})) \cup (\cup_{i=1}^{n_m} P_{i,m}^{(m)}),$$

so we have that $I \subseteq P_{j,m}^{(m)}$ for some $1 \leq j \leq n_m$. Thus $I \subseteq P_{j,m}^{(1)} \cap \cdots \cap P_{j,m}^{(m)}$.
Case 2. Assume that there exists $1 \leq t < n_m$ such that $I \subseteq P_{i,m}^{(1)} \cap \cdots \cap P_{i,m}^{(m-1)}$ for every $1 \leq i \leq t$ and $I \nsubseteq \cup_{i=t+1}^{n_m} (P_{i,m}^{(1)} \cap \cdots \cap P_{i,m}^{(m-1)})$. Because

$$I \subseteq J \cup K \cup (\cup_{i=1}^{n_1} P_{i,1}^{(1)}) \cup (\cup_{i=1}^{n_2} (P_{i,2}^{(1)} \cap P_{i,2}^{(2)})) \cup \cdots \cup (\cup_{i=1}^{n_{m-1}} (P_{i,m-1}^{(1)} \cap \cdots \cap P_{i,m-1}^{(m-1)}))$$
$$\cup (\cup_{i=1}^{t} P_{i,m}^{(m)}) \cup (\cup_{i=t+1}^{n_m} (P_{i,m}^{(1)} \cap \cdots \cap P_{i,m}^{(m-1)})),$$

by the induction hypothesis we deduce that $I \subseteq P_{k,m}^{(m)}$ for some $1 \leq k \leq t$, whence $I \subseteq P_{k,m}^{(1)} \cap \cdots \cap P_{k,m}^{(m)}$. □

Theorem 2.2 ([2, Theorem 2.5]). *Let I be an n-absorbing ideal of a ring R. Then there are at most n prime ideals of R minimal over I.*

Theorem 2.3 ([2, Theorem 2.14]). *Let I be an n-absorbing ideal of a ring R such that I has exactly n minimal prime ideals, say $P_1, \ldots, P_n$. Then $P_1 \cdots P_n \subseteq I$.*

Corollary 2.4. *Let I be an n-absorbing ideal of a ring R such that I has exactly n minimal prime ideals. Then $(\sqrt{I})^n \subseteq I$.*

Proposition 2.5 ([2, Corollary 3.6]). *Let $n \geq 2$ and $I \subset \sqrt{I}$ be an n-absorbing ideal of a ring R. Suppose that $x \in \sqrt{I} \backslash I$ and $x^n \in I$, but $x^{n-1} \notin I$. Then $(I :_R x^{n-1})$ is a prime ideal of R.*

Theorem 2.6. *Let I be an n-absorbing ideal of a ring R with distinct minimal prime ideals $P_1, \ldots, P_n$. Suppose that $x_1, \ldots, x_{n-1} \in R$ be such that $x_i \in P_i \backslash P_n$ for $i = 1, \ldots, n-1$. Then $(I :_R x_1 \cdots x_{n-1}) = P_n$. In particular, if x is an element of R such that $x \in (P_1 \cap \cdots \cap P_{n-1}) \backslash P_n$, then $(I :_R x^{n-1}) = P_n$.*

Proof. Assume that $x_1, \ldots, x_{n-1} \in R$ be such that $x_i \in P_i \backslash P_n$ for $i = 1, \ldots, n-1$. Since $x_1 \cdots x_{n-1} \notin P_n$, then $(I :_R x_1 \cdots x_{n-1}) \subseteq P_n$. Let $y \in P_n$. By Theorem 2.3 we have that $x_1 \cdots x_{n-1} y \in P_1 \cdots P_{n-1} P_n \subseteq I$. Hence $y \in (I :_R x_1 \cdots x_{n-1})$ and so the equality holds. □

Theorem 2.7. *Let I be a strongly n-absorbing ideal of a ring R with distinct minimal prime ideals $P_1, \ldots, P_m$ $(m \leq n)$. Suppose that $x_1, \ldots, x_{m-1} \in R$ be such that $x_i \in P_i \backslash P_m$ for $i = 1, \ldots, m-1$. Then $(I :_R x_1^{n_1} \cdots x_{m-1}^{n_{m-1}}) = P_m$ for positive integers $n_1, \ldots, n_{m-1}$ with $n - 1 = n_1 + \cdots + n_{m-1}$.*

Proof. Regarding [2, Theorem 6.2] the proof is similar to that of Theorem 2.6. □

Definition 2.8. Suppose that m, n are positive integers with $m > n$. A proper ideal I of a ring R is called (m, n)-*absorbing* if whenever $a_1 a_2 \cdots a_m \in I$ for $a_1, a_2, \ldots, a_m \in R$, then the product of n of the a_i's is in I.

Theorem 2.9. *Let I be a proper ideal of a ring R and $m > n$. Then I is (m, n)-absorbing if and only if I is n-absorbing.*

Proof. The "if" part has a routin verification. For the converse, let I be (m, n)-absorbing and let $a_1, a_2, \ldots, a_{n+1} \in R$ be such that $a_1 a_2 \ldots a_{n+1} \in I$. Then $a_1 a_2 \ldots a_{n+1} \overbrace{1.1 \ldots 1}^{m-n-1 \text{ times}} \in I$. Since I is proper, then the product of n of $a_1, a_2, \ldots, a_{n+1}$ is in I. Consequently I is n-absorbing. □

Proposition 2.10. *Let V be a valuation domain with the quotient field K and let I be a proper ideal of V. Then I is an n-absorbing ideal of V if and only if whenever $x_1 x_2 \cdots x_{n+1} \in I$ with $x_1, x_2, \ldots, x_{n+1} \in K$, then there are n of $x_1, x_2, \ldots, x_{n+1}$ whose product is in I.*

Proof. Assume that I is an n-absorbing ideal of V. Let $x_1 x_2 \cdots x_{n+1} \in I$ for some $x_1, x_2, \ldots, x_{n+1} \in K$ such that $x_1 x_2 \cdots x_n \notin I$. If $x_{n+1} \notin V$, then $x_{n+1}^{-1} \in V$, since V is valuation. So $x_1 \cdots x_n x_{n+1} x_{n+1}^{-1} = x_1 \cdots x_n \in I$, a contradiction. Hence $x_{n+1} \in V$. If $x_i \in V$ for every $1 \leq i \leq n$, then there is nothing to prove. If $x_i \notin V$ for some $1 \leq i \leq n$, then $x_1 \cdots x_{i-1} x_{i+1} \cdots x_{n+1} \in I$. □

Proposition 2.11. *Let R be a von-Neumann regular ring. Then I is an n-absorbing ideal of R if and only if $e_1 e_2 \cdots e_{n+1} \in I$ for some idempotent elements $e_1, e_2, \ldots, e_{n+1} \in R$ implies that the product of n of $e_1, e_2, \ldots, e_{n+1}$ is in I.*

Proof. Notice the fact that any finitely generated ideal of a von-Neumann regular ring R is generated by an idempotent element. □

Theorem 2.12. *Let R be a ring. Suppose that J, K are ideals of R and I_i is an n_i-absorbing ideal of R for every $1 \leq i \leq m$. If I is an ideal of R such that $I \subseteq J \cup K \cup I_1 \cup I_2 \cup \cdots \cup I_m$, then either $I \subseteq J$ or $I \subseteq K$ or $I \subseteq \sqrt{I_i}$ for some $1 \leq i \leq m$.*

Proof. Assume that I is an ideal of R such that $I \subseteq J \cup K \cup I_1 \cup I_2 \cup \cdots \cup I_m$. Then $I \subseteq J \cup K \cup \sqrt{I_1} \cup \sqrt{I_2} \cup \cdots \cup \sqrt{I_m}$. Now apply Theorem 2.2 and Theorem 2.1. □

Corollary 2.13. *Let I_i be an n_i-absorbing ideal of a ring R for every $1 \leq i \leq m$ ($m \geq 2$). Suppose that for every $1 \leq i \leq m$, I_i has exactly n_i minimal prime ideals. If I is an ideal of R such that $I \subseteq I_1 \cup I_2 \cup \cdots \cup I_m$, then $I^{n_i} \subseteq I_i$ for some $1 \leq i \leq m$.*

Proof. By Theorem 2.12 and Corollary 2.4. □

Corollary 2.14. *Let I_i be a strongly n_i-absorbing ideal of a ring R for every $1 \leq i \leq m$ ($m \geq 2$). If I is an ideal of R such that $I \subseteq I_1 \cup I_2 \cup \cdots \cup I_m$, then $I^{n_i} \subseteq I_i$ for some $1 \leq i \leq m$.*

Proof. By Theorem 2.12 and [2, Theorem 6.1]. □

Theorem 2.15 (n-Absorbing avoidance theorem)**.** *Let $I_1, I_2, \ldots, I_m$ ($m \geq 2$) be ideals of R such that I_i be an n_i-absorbing ideal of R for every $3 \leq i \leq m$. Suppose that $I_i \nsubseteq (I_j :_R x^{n_j - 1}) \subset R$ for every $x \in \sqrt{I_j} \backslash I_j$ with $i \neq j$. If I is an ideal of R such that $I \subseteq I_1 \cup I_2 \cup \cdots \cup I_m$, then $I \subseteq I_i$ for some $1 \leq i \leq m$.*

Proof. Suppose that I is an ideal of R such that $I \subseteq I_1 \cup I_2 \cup \cdots \cup I_m$. By Theorem 2.12, either $I \subseteq I_1$ or $I \subseteq I_2$ or $I \subseteq \sqrt{I_i}$ for some $3 \leq i \leq m$. If $I \subseteq I_1$ or $I \subseteq I_2$, then we are done. So, we assume that $I \subseteq \sqrt{I_j}$ for some $3 \leq j \leq m$. Let $I \nsubseteq I_j$. Hence there exists $x \in I \backslash I_j$, and so $x \in \sqrt{I_j} \backslash I_j$. Then we may assume that $I \subseteq I_1 \cup I_2 \cup \cdots \cup I_m$ is an efficient covering of ideals of R. Therefore $I = \cup_{i=1}^m (I_i \cap I)$ is an efficient union. So, [15, Lemma 2.1] implies that $(\cap_{i \neq k} I_i) \cap I \subseteq I_k \cap I$. Thus by hypothesis, $I_i \nsubseteq (I_j :_R x^{n_j - 1}) \subset R$ for every $i \neq j$. Then, for every $i \neq j$ there exists $r_i \in I_i \backslash (I_j :_R x^{n_j - 1})$. Set $r = \prod_{i \neq j} r_i$. Thus $rx \in (\cap_{i \neq j} I_i) \cap I \subseteq I_j \cap I$. Therefore $r \in (I_j :_R x^{n_j - 1})$ which is a contradiction, because by Proposition 2.5, $(I_j :_R x^{n_j - 1})$ is a prime ideal of R. Consequently $I \subseteq I_i$ for some $1 \leq i \leq m$. □

Theorem 2.16. *Let I be an ideal of a ring R and let $a \in R$. If*

$$I + a \subseteq J \cup (\cup_{i=1}^{n_1} P_{i,1}^{(1)}) \cup (\cup_{i=1}^{n_2} (P_{i,2}^{(1)} \cap P_{i,2}^{(2)})) \cup \cdots \cup (\cup_{i=1}^{n_m} (P_{i,m}^{(1)} \cap \cdots \cap P_{i,m}^{(m)}))$$

where J is an ideal of R and $P_{i,j}^{(k)}$'s are prime ideals of R, then either $(I, a) \subseteq J$ or $(I, a) \subseteq P_{i,j}^{(1)} \cap \cdots \cap P_{i,j}^{(j)}$ for some $1 \leq j \leq m$ and some $1 \leq i \leq n_j$.

Proof. By using [16, Theorem 12] and by a similar manner to that of Theorem 2.1. □

Corollary 2.17. *Let R be a ring, J be an ideal of R and I_i be an n_i-absorbing ideal of R for every $1 \leq i \leq m$ $(m \geq 1)$. If I is an ideal of R and a is an element of R such that $I + a \subseteq J \cup I_1 \cup I_2 \cup \cdots \cup I_m$, then either $(I, a) \subseteq J$ or $(I, a) \subseteq \sqrt{I_i}$ for some $1 \leq i \leq m$.*

Corollary 2.18. *Let I_i be an n_i-absorbing ideal of a ring R for every $1 \leq i \leq m$ $(m \geq 2)$. Suppose that for every $1 \leq i \leq m$, I_i has exactly n_i minimal prime ideals. If I is an ideal of R and $a \in R$ such that $I + a \subseteq I_1 \cup I_2 \cup \cdots \cup I_m$, then $(I, a)^{n_i} \subseteq I_i$ for some $1 \leq i \leq m$.*

Corollary 2.19. *Let I_i be a strongly n_i-absorbing ideal of a ring R for every $1 \leq i \leq m$ $(m \geq 2)$. If I is an ideal of R and $a \in R$ such that $I + a \subseteq I_1 \cup I_2 \cup \cdots \cup I_m$, then $(I, a)^{n_i} \subseteq I_i$ for some $1 \leq i \leq m$.*

Theorem 2.20. *Let $I_1, I_2, \ldots, I_m$ $(m \geq 2)$ be ideals of R and I_i be an n_i-absorbing ideal of R for every $2 \leq i \leq m$. Suppose that $I_i \nsubseteq (I_j :_R x^{n_j - 1}) \subset R$ for every $x \in \sqrt{I_j} \backslash I_j$ with $i \neq j$. If I is an ideal of R and e is an idempotent element of R such that $I + e \subseteq I_1 \cup I_2 \cup \cdots \cup I_m$, then $(I, e) \subseteq I_i$ for some $1 \leq i \leq m$.*

Proof. By Corollary 2.17, we deduce that either $(I, e) \subseteq I_1$ or $(I, e) \subseteq \sqrt{I_j}$ for some $2 \leq j \leq m$. The first case leads us to the claim. In the second case we have $I \subseteq \sqrt{I_j}$ and $e \in I_j$. If $I \subseteq I_j$, then there is nothing to prove. So, we assume that $I \nsubseteq I_j$. Hence, there exists $x \in I \backslash I_j$ and then $x \in \sqrt{I_j} \backslash I_j$. If $I + e \subseteq I_1 \cup I_2 \cup \cdots \cup I_m$ is an efficient covering of I, then $(\cap_{i \neq j} I_i) \cap I \subseteq I_j \cap I$. On the other hand, by our hypothesis we have that for every $i \neq j$ there exists $r_i \in I_i \backslash (I_j :_R x^{n_j - 1})$. Set $r = \prod_{i \neq j} r_i$. Thus $rx \in (\cap_{i \neq j} I_i) \cap I \subseteq I_j$. Therefore $r \in (I_j :_R x^{n_j - 1})$ which is a contradiction, since $(I_j :_R x^{n_j - 1})$ is a prime ideal of R. Consequently $I + e \subseteq I_i$ for some $1 \leq i \leq m$, and so $(I, e) \subseteq I_i$. □

3 Quasi-n-Absorbing Ideals

We begin this section with the following proposition.

Proposition 3.1. *Let I be an ideal of a ring R. Then the following statements are equivalent:*

1. *I is quasi-n-absorbing;*
2. *For each $a \in R$ with $a^n \notin I$, $(I :_R a^n) = (I :_R a^{n-1})$;*
3. *For every $a \in R$ and every ideal J of R with $a^n J \subseteq I$, either $a^n \in I$ or $a^{n-1} J \subseteq I$.*

Proof. The proof is easy. □

Corollary 3.2. *Let R be a ring. Then 0 is a quasi-n-absorbing ideal of R if and only if for each $a \in R$, either $a^n = 0$ or $ann_R(a^n) = ann_R(a^{n-1})$.*

Proof. By Proposition 3.1. □

Proposition 3.3. *Let $f : R \to R'$ be a homomorphism of rings.*

1. *If I' is a quasi-n-absorbing (resp. semi-n-absorbing) ideal of R', then $f^{-1}(I')$ is a quasi-n-absorbing (resp. semi-n-absorbing) ideal of R.*

2. *If f is an epimorphism and I is a quasi-n-absorbing (resp. semi-n-absorbing) ideal of R containing $Ker(f)$, then $f(I)$ is a quasi-n-absorbing (resp. semi-n-absorbing) ideal of R'.*

Proof. 1. Let I' is a quasi-n-absorbing ideal of R' and $a^n b \in f^{-1}(I')$ for some $a, b \in R$. Then $f(a)^n f(b) \in I'$. Hence either $f(a)^n \in I'$ or $f(a)^{n-1}f(b) \in I'$, and thus either $a^n \in f^{-1}(I')$ or $a^{n-1}b \in f^{-1}(I')$. So $f^{-1}(I')$ is a quasi-n-absorbing ideal of R.
2. Assume that f is an epimorphism and I is a quasi-n-absorbing ideal of R such that $Ker(f) \subseteq I$. Let $a', b' \in R'$ and $(a')^n b' \in f(I)$. So there exist $a, b \in R$ such that $f(a) = a'$ and $f(b) = b'$, and $f(a^n b) = (a')^n b' \in f(I)$. Since $\mathrm{Ker}(f) \subseteq I$, then $a^n b \in I$. It implies that either $a^n \in I$ or $a^{n-1}b \in I$. Therefore either $(a')^n \in f(I)$ or $(a')^{n-1}b' \in f(I)$. Consequently $f(I)$ is a quasi-n-absorbing ideal of R'. □

As an immediate consequence of Proposition 3.3 we have the following result.

Corollary 3.4. *Let R be a ring and I be an ideal of R.*

1. *If R' is a subring of R and I is a quasi-n-absorbing (resp. semi-n-absorbing) ideal of R, then $I \cap R'$ is a quasi-n-absorbing (resp. semi-n-absorbing) ideal of R'.*

2. *Let J be an ideal of R with $J \subseteq I$. Then I is a quasi-n-absorbing (resp. semi-n-absorbing) ideal of R if and only if I/J is a quasi-n-absorbing (resp. semi-n-absorbing) ideal of R/J.*

Corollary 3.5. *Let I be an ideal of a ring R. Then $\langle I, X\rangle$ is a quasi-n-absorbing (resp. semi-n-absorbing) ideal of $R[X]$ if and only if I is a quasi-n-absorbing (resp. semi-n-absorbing) ideal of R.*

Proof. By Corollary 3.4(2) and regarding the isomorphism $\langle I, X\rangle/\langle X\rangle \simeq I$ in $R[X]/\langle X\rangle \simeq R$ we have the result. □

Let M be an R-module. The set of all zero divisors on M is:
$\mathrm{Z}_R(M) = \{r \in R \mid \text{there exists an element } 0 \neq x \in M \text{ such that } rx = 0\}$.

Proposition 3.6. *Let R be a ring, S be a multiplicatively closed subset of R, and I a proper ideal of R. Then*

1. *If I is a quasi-n-absorbing (resp. semi-n-absorbing) ideal of R with $I \cap S = \emptyset$, then $S^{-1}I$ is a quasi-n-absorbing (resp. semi-n-absorbing) ideal of $S^{-1}R$.*

2. *If $S^{-1}I$ is a quasi-n-absorbing (resp. semi-n-absorbing) ideal of $S^{-1}R$, $S \cap Z_R(R/I) = \emptyset$, then I is a quasi-n-absorbing (resp. semi-n-absorbing) ideal of R.*

Proof. 1. Let $a, b \in R$, $s, t \in S$ such that $(\frac{a}{s})^n \frac{b}{t} \in S^{-1}I$. Then there exists $u \in S$ such that $ua^n b \in I$. Since I is quasi-n-absorbing and $ua^n b \in I$, then $a^n \in I$ or $ua^{n-1}b \in I$. So $(\frac{a}{s})^n \in S^{-1}I$ or $(\frac{a}{s})^{n-1}\frac{b}{t} = \frac{ua^{n-1}b}{us^{n-1}t} \in S^{-1}I$. Therefore $S^{-1}I$ is a quasi-n-absorbing ideal of $S^{-1}R$.
2. Let $a, b \in R$ such that $a^n b \in I$. Then $\frac{a^n b}{1} = (\frac{a}{1})^n \frac{b}{1} \in S^{-1}I$. Therefore $(\frac{a}{1})^n \in S^{-1}I$ or $(\frac{a}{1})^{n-1}\frac{b}{1} \in S^{-1}I$. If $(\frac{a}{1})^n \in S^{-1}I$, then for some $\nu \in S$, $\nu a^n \in I$. Since $\nu \in S$ and $S \cap Z_R(R/I) = \emptyset$, then $a^n \in I$. Similarly, if $(\frac{a}{1})^{n-1}\frac{b}{1} \in S^{-1}I$, then $a^{n-1}b \in I$. Consequently I is a quasi-n-absorbing ideal of R. □

Definition 3.7. Let $m > n$ be positive integers. A proper ideal I of a ring R is called *quasi-(m, n)-absorbing* if whenever $a^{m-1}b \in I$ for $a, b \in R$, then $a^n \in I$ or $a^{n-1}b \in I$.

Proposition 3.8. *Let I be a proper ideal of R and $m > n$ be positive integers. Then I is quasi-(m, n)-absorbing if and only if I is quasi-n-absorbing.*

Proof. Assume that I is quasi-(m, n)-absorbing. Let $a^n b \in I$ for some $a, b \in R$. Since $n \leq m - 1$, then $a^{m-1}b \in I$. Therefore $a^n \in I$ or $a^{n-1}b \in I$. Consequently I is quasi-n-absorbing. Now, suppose that I is quasi-n-absorbing. Let $a^{m-1}b \in I$ for some $a, b \in R$. Therefore $a^n a^{(m-1-n)}b \in I$. Hence $a^n \in I$ or $a^{n-1}a^{(m-1-n)}b = a^{(m-2)}b \in I$. Repeating this method implies that $a^n \in I$ or $a^{n-1}b \in I$. Thus I is quasi-(m, n)-absorbing. □

Proposition 3.9. *Let I be a proper ideal of a ring R.*

1. *I is prime if and only if I is quasi-1-absorbing if and only if I is 1-absorbing.*

2. *If I is quasi-n-absorbing, then it is quasi-i-absorbing for all $i \geq n$.*

3. *If I is prime, then it is quasi-n-absorbing for all $n \geq 1$.*

4. *If I is n-absorbing, then it is quasi-n-absorbing.*

5. *If I is quasi-n-absorbing for some $n \geq 1$, then there exists the least $n_0 \geq 1$ such that I is quasi-n_0-absorbing. In this case, I is quasi-n-absorbing for all $n \geq n_0$ and it is not quasi-i-absorbing for $n_0 > i > 0$.*

Proof. Every statement has a routin verification. □

Proposition 3.10. *Let I be a proper ideal of a ring R. If I is a quasi-n-absorbing ideal of R, then $\sqrt{I} = \{x \in R \mid x^n \in I\}$, the converse holds if I is primary.*

Proof. First, assume that I is quasi-n-absorbing. Clearly $\{x \in R \mid x^n \in I\} \subseteq \sqrt{I}$. Now, let $x \in \sqrt{I}$. Then there exists $m \geq 1$ such that $x^m \in I$. If $m \leq n$, then $x^n \in I$. If $m > n$, then by Proposition 3.8 we have that I is quasi-(m,n)-absorbing. So, $x^{m-1}x \in I$ implies that $x^n \in I$. Therefore $\sqrt{I} = \{x \in R \mid x^n \in I\}$. Conversely, assume that $\sqrt{I} = \{x \in R \mid x^n \in I\}$ and I is primary. Let $a^n b \in I$ for some $a, b \in R$. If $a^n \in I$, then we are done. Therefore, suppose that $a^n \notin I$. Hence $a \notin \sqrt{I}$, and so $a^{n-1}b \in I$. Consequently I is quasi-n-absorbing. □

The following remark shows that the two concepts of quasi-$(n+1)$-absorbing ideals ($(n+1)$-absorbing ideals) and of quasi-n-absorbing ideals are different in general.

Remark 3.11. *Let p, q be distinct prime numbers. By* [2, p. 1650]*, $p^n\mathbb{Z}$ is an n-absorbing ideal of $\mathbb{Z}$. So $p^n\mathbb{Z} \cap q\mathbb{Z}$ is an $(n+1)$-absorbing ideal,* [2, Theorem 2.1](c)*. Then $p^n\mathbb{Z} \cap q\mathbb{Z}$ is quasi-$(n+1)$-absorbing. If $p^n\mathbb{Z} \cap q\mathbb{Z}$ is a quasi-n-absorbing ideal, then $p^n q \in p^n\mathbb{Z} \cap q\mathbb{Z}$ implies that either $p^n \in q\mathbb{Z}$ or $p^{n-1}q \in p^n\mathbb{Z}$, which is a contradiction.*

Proposition 3.12. *Let $\{P_\lambda\}_{\lambda \in \Lambda}$ be a family of prime ideals of a ring R. Then $\bigcap_{\lambda \in \Lambda} P_\lambda$ is a quasi-i-absorbing ideal for every $i \geq 2$.*

Proof. Let $I = \bigcap_{\lambda \in \Lambda} P_\lambda$. By Proposition 3.9(2), it is sufficient to we show that I is a quasi-2-absorbing ideal. Suppose that $a^2 b \in I$ for some $a, b \in R$. Since every P_λ is prime and $a^2 b \in P_\lambda$, then $ab \in P_\lambda$. Therefore $ab \in I$, and so we conclude that I is a quasi-2-absorbing ideal. □

Remark 3.13. *Let $p_1, p_2, \ldots, p_{n+1}$ be distinct prime numbers. Then by Proposition 3.12, $\mathbb{Z}(p_1p_2\ldots p_{n+1}) = \mathbb{Z}p_1 \cap \mathbb{Z}p_2 \cap \cdots \cap \mathbb{Z}p_{n+1}$ is a quasi-i-absorbing ideal of $\mathbb{Z}$ for every $i \geq 2$. But, clearly $\mathbb{Z}(p_1p_2\ldots p_{n+1})$ is not an n-absorbing ideal. This remark shows that the two concepts of quasi-n-absorbing ideals and of n-absorbing ideals are different in general.*

A commutative ring R is called *semiprimitive* if $Jac(R) = 0$, [17]. A commutative ring is semiprimitive if and only if it is a subdirect product of fields, [18, p. 179].

As a direct consequence of Proposition 3.12 we have the following result.

Corollary 3.14. *Let R be a ring.*

1. *For every proper ideal I of R, $\sqrt{I}$ is a quasi-i-absorbing ideal of R for every $i \geq 2$.*
2. *$Nil(R)$ and $Jac(R)$ are quasi-i-absorbing ideals of R for every $i \geq 2$.*
3. *If R is a semiprimitive ring, then 0 is a quasi-i-absorbing ideal in R for every $i \geq 2$.*

Proposition 3.15. *Let R be a ring. The following statements are equivalent:*

1. *For every elements* $a, b \in R$, $a^n = ra^n b$ *for some* $r \in R$ *or* $a^{n-1}b = sa^n b$ *for some* $s \in R$*;*
2. *Every proper ideal of* R *is quasi-n-absorbing.*

Proof. Straightforward. □

Proposition 3.16. *Let* R *be a ring. The following statements are equivalent:*

1. *For every ideals* I, J *of* R, $I^n = I^n J$ *or* $I^{n-1}J = I^n J$*;*
2. *For every ideals* $I_1, I_2, \ldots, I_{n+1}$ *of* R, $(I_1 \cap I_2 \cap \cdots \cap I_n)^n \subseteq I_1 I_2 \cdots I_{n+1}$ *or* $(I_1 \cap I_2 \cap \cdots \cap I_n)^{n-1} I_{n+1} \subseteq I_1 I_2 \cdots I_{n+1}$*;*
3. *Every proper ideal of* R *is strongly quasi-n-absorbing.*

Proof. (1)⇒(2) Suppose that $I_1, I_2, \ldots, I_{n+1}$ are ideals of R. By part (1),

$$(I_1 \cap I_2 \cap \cdots \cap I_n)^n = (I_1 \cap I_2 \cap \cdots \cap I_n)^n I_{n+1} \subseteq I_1 I_2 \cdots I_{n+1},$$

or

$$(I_1 \cap I_2 \cap \cdots \cap I_n)^{n-1} I_{n+1} = (I_1 \cap I_2 \cap \cdots \cap I_n)^n I_{n+1} \subseteq I_1 I_2 \cdots I_{n+1}.$$

(2)⇒(1) For ideals I, J of R, we have from (2), $I^n = (\overbrace{I \cap \cdots \cap I}^{n \text{ times}})^n \subseteq I^n J$ or $I^{n-1}J = (\overbrace{I \cap \cdots \cap I}^{n \text{ times}})^{n-1} J \subseteq I^n J$.
(1)⇔(3) Is trivial. □

Proposition 3.17. *Let* I *be a proper ideal of* R.

1. *If for every ideals* I_1, I_2 *of* R, *we have* $I_1^n I_2 \subseteq I \subseteq I_1 \cap I_2 \Rightarrow [I_1^n \subseteq I$ *or* $I_1^{n-1} I_2 \subseteq I]$, *then* I *is strongly quasi-n-absorbing.*
2. *If for every ideals* $I_1, I_2, \ldots, I_{n+1}$ *of* R, *we have*

$$I_1 I_2 \cdots I_{n+1} \subseteq I \quad \textit{and} \quad I \subseteq I_1 \cap I_2 \cap \cdots \cap I_{n+1} \Rightarrow$$

$$[I_1 \cdots \widehat{I_i} \cdots I_{n+1} \subseteq I, \quad \textit{for some } 1 \leq i \leq n+1]$$

then I *is a strongly* n*-absorbing ideal.*

Proof. 1. Assume that I is an ideal that satisfies the hypothesis atated in 1. Let $J_1^n J_2 \subseteq I$ for some ideals J_1, J_2 of R. Then $(J_1 + I)^n (J_2 + I) \subseteq I$, so we have $(J_1 + I)^n \subseteq I$ or $(J_1 + I)^{n-1}(J_2 + I) \subseteq I$. Thus $J_1^n \subseteq I$ or $J_1^{n-1} J_2 \subseteq I$.

2. The proof is similar to that of 1. □

Notice that in Remark 3.11 we can observe that the intersection of two quasi-n-absorbing ideals may not be quasi-n-absorbing.

Proposition 3.18. *Let $\{I_\lambda\}_{\lambda\in\Lambda}$ be a chain of quasi-n-absorbing ideals. Then $\bigcap_{\lambda\in\Lambda} I_\lambda$ is a quasi-n-absorbing ideal.*

Proof. Let $I=\bigcap_{\lambda\in\Lambda} I_\lambda$ and suppose that $a^nb\in I$ for some $a,b\in R$. If $a^n\in I_\lambda$ for each $\lambda\in\Lambda$, then $a^n\in I$. So we may assume that there is $\lambda_0\in\Lambda$ such that $a^n\notin I_{\lambda_0}$, then $a^n\notin I_\lambda$ for each $I_\lambda\subseteq I_{\lambda_0}$. Since all ideals in $\{I_\lambda\}_{\lambda\in\Lambda}$ are quasi-n-absorbing, it follows that $a^{n-1}b\in I_\lambda$ for each $I_\lambda\subseteq I_{\lambda_0}$. Thus $a^{n-1}b\in I_\lambda$ for each $\lambda\in\Lambda$, so that $a^{n-1}b\in I$. We deduce that I is a quasi-n-absorbing ideal. □

Proposition 3.19. *Let $I_1,I_2,\ldots,I_k$ be ideals of R. If I_i is a quasi-n_i-absorbing ideal of R for every $1\le i\le k$, then $I_1\cap I_2\cap\cdots\cap I_k$ is a quasi-n-absorbing ideal for $n=n_1+\cdots+n_k$.*

Proof. Let $a,b\in R$ be such that $a^nb\in I_1\cap I_2\cap\cdots\cap I_k$. Since I_i's are quasi-n_i-absorbing, then, for every $1\le i\le k$, either $a^{n_i}\in I_i$ or $a^{n_i-1}b\in I_i$. If for every $1\le i\le k$, $a^{n_i}\in I_i$, then $a^n\in I_1\cap I_2\cap\cdots\cap I_k$. If for every $1\le i\le k$, $a^{n_i-1}b\in I_i$, then $a^{n-1}b\in I_1\cap I_2\cap\cdots\cap I_k$. Otherwise, without loss of generality we may assume that there exists $1\le j<k$ such that $a^{n_i}\in I_i$ for every $1\le i\le j$ and $a^{n_i-1}b\in I_i$ for every $j+1\le i\le k$. Hence $a^{n-1}b\in I_1\cap I_2\cap\cdots\cap I_k$ which shows that $I_1\cap I_2\cap\cdots\cap I_k$ is a quasi-n-absorbing ideal. □

Theorem 3.20. *Let $I_1,I_2,\ldots,I_t$ be ideals of R.*

1. *If I_1 is quasi-n-absorbing and I_2 is quasi-m-absorbing for $m<n$, then $I_1\cap I_2$ is quasi-$(n+1)$-absorbing.*
2. *If $I_1,I_2,\ldots,I_t$ are quasi-n-absorbing, then $I_1\cap I_2\cap\cdots\cap I_t$ is quasi-$(n+t)$-absorbing.*
3. *If I_i is quasi-n_i-absorbing for every $1\le i\le t$ with $n_1<n_2<\cdots<n_t$ and $t>2$, then $I_1\cap I_2\cap\cdots\cap I_t$ is quasi-(n_t+2)-absorbing.*

Proof. 1. Let $a,b\in R$ be such that $a^{n+1}b\in I_1\cap I_2$. We show that $a^{n+1}\in I_1\cap I_2$ or $a^nb\in I_1\cap I_2$. Since I_1 is quasi-n-absorbing, then by Proposition 3.8 it is quasi-$(n+2,n)$-absorbing. Therefore either $a^n\in I_1$ or $a^{n-1}b\in I_1$. Also, I_2 is quasi-m-absorbing, again by Proposition 3.8 either $a^m\in I_2$ or $a^{m-1}b\in I_2$. There are four cases.
Case 1. Suppose that $a^n\in I_1$ and $a^m\in I_2$. Then $a^n\in I_1\cap I_2$.
Case 2. Suppose that $a^n\in I_1$ and $a^{m-1}b\in I_2$. Then $a^nb\in I_1\cap I_2$.
Case 3. Suppose that $a^{n-1}b\in I_1$ and $a^m\in I_2$. Then $a^{n-1}b\in I_1\cap I_2$.
Case 4. Suppose that $a^{n-1}b\in I_1$ and $a^{m-1}b\in I_2$. Then $a^{n-1}b\in I_1\cap I_2$. Consequently $I_1\cap I_2$ is quasi-$(n+1)$-absorbing.
2. Induction on t: For $t=1$ there is nothing to prove. Let $t>1$ and assume that for $t-1$ the claim holds. Then $I_1\cap I_2\cap\cdots\cap I_{t-1}$ is quasi-$(n+t-1)$-absorbing. Since I_t is quasi-n-absorbing, then it is quasi-$(n+t-2)$-absorbing, by Proposition 3.9(2). Therefore $I_1\cap I_2\cap\cdots\cap I_t$ is quasi-$(n+t)$-absorbing, by part 1.
3. Induction on t: For $t=3$ apply parts 1 and 2. Let $t>3$ and suppose that for $t-1$ the claim holds. Hence $I_1\cap I_2\cap\cdots\cap I_{t-1}$ is quasi-$(n_{t-1}+2)$-absorbing. We

have the following cases:
Case 1. Let $n_{t-1}+2 < n_t$. In this case $I_1 \cap I_2 \cap \cdots \cap I_t$ is quasi-(n_t+1)-absorbing, by part 1. Therefore $I_1 \cap I_2 \cap \cdots \cap I_t$ is quasi-(n_t+2)-absorbing.
Case 2. Let $n_{t-1}+2 = n_t$. Thus $I_1 \cap I_2 \cap \cdots \cap I_t$ is quasi-(n_t+2)-absorbing, by part 2.
Case 3. Let $n_{t-1}+2 > n_t$. Then $I_1 \cap I_2 \cap \cdots \cap I_t$ is quasi-$(n_{t-1}+3)$-absorbing, by part 1. Since $n_{t-1}+3 = n_t+2$, then $I_1 \cap I_2 \cap \cdots \cap I_t$ is quasi-(n_t+2)-absorbing. □

Proposition 3.21. *Let $R = R_1 \times R_2$ be a decomposable ring, I_1 a proper ideal of R_1 and I_2 a proper ideal of R_2. Then I_1 (resp. I_2) is a quasi-n-absorbing ideal of R_1 (resp. R_2) if and only if $I_1 \times R_2$ (resp. $R_1 \times I_2$) is a quasi-n-absorbing ideal of R.*

Proof. $(\Rightarrow)$ Suppose that I_1 is a quasi-n-absorbing ideal of R_1. Let $(a_1, a_2)^n(b_1, b_2) \in I_1 \times R_2$ for some $(a_1, a_2), (b_1, b_2) \in R$. Hence $a_1^n b_1 \in I_1$, and so either $a_1^n \in I_1$ or $a_1^{n-1} b_1 \in I_1$. Therefore either $(a_1, a_2)^n \in I_1 \times R_2$ or $(a_1, a_2)^{n-1}(b_1, b_2) \in I_1 \times R_2$. Consequently $I_1 \times R_2$ is a quasi-n-absorbing ideal of R.
$(\Leftarrow)$ Assume that $I_1 \times R_2$ is a quasi-n-absorbing ideal of R. Let $a^n b \in I_1$ for some $a, b \in R_1$. Then $(a, 1)^n(b, 1) \in I_1 \times R_2$. Hence $(a, 1)^n \in I_1 \times R_2$ or $(a, 1)^{n-1}(b, 1) \in I_1 \times R_2$. Therefore $a^n \in I_1$ or $a^{n-1} b \in I_1$. So I_1 is a quasi-n-absorbing ideal of R_1. □

A strategy similar to Theorem 3.20 leads us to the following theorem:

Theorem 3.22. *Let $I_1, I_2, \ldots, I_t$ be ideals of rings $R_1, R_2, \ldots, R_t$, respectively.*

1. *If I_1 is a quasi-n-absorbing ideal of R_1 and I_2 is a quasi-m-absorbing ideal of R_2 for $m < n$, then $I_1 \times I_2$ is a quasi-$(n+1)$-absorbing ideal of $R_1 \times R_2$.*
2. *If $I_1, I_2, \ldots, I_t$ are quasi-n-absorbing ideals of $R_1, R_2, \ldots, R_t$, respectively, then $I_1 \times I_2 \times \cdots \times I_t$ is a quasi-$(n+t)$-absorbing ideal of $R_1 \times R_2 \times \cdots \times R_t$.*
3. *If I_i is a quasi-n_i-absorbing ideal of R_i for every $1 \leq i \leq t$ with $n_1 < n_2 < \cdots < n_t$ and $t > 2$, then $I_1 \times I_2 \times \cdots \times I_t$ is a quasi-(n_t+2)-absorbing ideal of $R_1 \times R_2 \times \cdots \times R_t$.*

Proof. 1. Let $(a_1, a_2), (b_1, b_2) \in R_1 \times R_2$ be such that $(a_1, a_2)^{n+1}(b_1, b_2) \in I_1 \times I_2$. Therefore $a_1^{n+1} b_1 \in I_1$ and $a_2^{n+1} b_2 \in I_2$. Since I_1 is a quasi-n-absorbing ideal of R_1, then $a_1^n \in I_1$ or $a_1^{n-1} b_1 \in I_1$. Also, I_2 is a quasi-m-absorbing ideal of R_2 and $a_2^{n+1} b_2 = a_2^m(a_2^{n+1-m} b_2) \in I_2$, so $a_2^m \in I_2$ or $a_2^{m-1}(a_2^{n+1-m} b_2) = a_2^n b_2 \in I_2$. Consider the following cases.
Case 1. Assume that $a_1^n \in I_1$ and $a_2^m \in I_2$. Then $(a_1, a_2)^n \in I_1 \times I_2$.
Case 2. Assume that $a_1^n \in I_1$ and $a_2^n b_2 \in I_2$. Then $(a_1, a_2)^n(b_1, b_2) \in I_1 \times I_2$.
Case 3. Assume that $a_1^{n-1} b_1 \in I_1$ and $a_2^m \in I_2$. Then $(a_1, a_2)^{n-1}(b_1, b_2) \in I_1 \times I_2$.
Case 4. Assume that $a_1^{n-1} b_1 \in I_1$ and $a_2^n b_2 \in I_2$. Then $(a_1, a_2)^n(b_1, b_2) \in I_1 \times I_2$.
Consequently $I_1 \times I_2$ is a quasi-$(n+1)$-absorbing ideal of $R_1 \times R_2$.
2. We use induction on t. For $t = 1$ there is nothing to prove. Let $t > 1$ and assume that for $t-1$ the claim holds. Then $I_1 \times I_2 \times \cdots \times I_{t-1}$ is a quasi-$(n+t-1)$-absorbing

ideal of $R_1 \times R_2 \times \cdots \times R_{t-1}$. Since I_t is a quasi-n-absorbing ideal of R_t, then it is quasi-$(n+t-2)$-absorbing, by Proposition 3.9(2). Therefore $I_1 \times I_2 \times \cdots \times I_t$ is a quasi-$(n+t)$-absorbing ideal of $R_1 \times R_2 \times \cdots \times R_t$ by 1.
3. Induction on t: For $t = 3$ apply parts 1 and 2. Let $t > 3$ and suppose that for $t-1$ the claim holds. Hence $I_1 \times I_2 \times \cdots \times I_{t-1}$ is a quasi-$(n_{t-1}+2)$-absorbing ideal of $R_1 \times R_2 \times \cdots \times R_{t-1}$. We consider the following cases:
Case 1. Let $n_{t-1} + 2 < n_t$. In this case $I_1 \times I_2 \times \cdots \times I_t$ is a quasi-$(n_t + 1)$-absorbing ideal of $R_1 \times R_2 \times \cdots \times R_t$ by part 1. Therefore $I_1 \times I_2 \times \cdots \times I_t$ is a quasi-$(n_t + 2)$-absorbing ideal of $R_1 \times R_2 \times \cdots \times R_t$.
Case 2. Let $n_{t-1} + 2 = n_t$. Thus $I_1 \times I_2 \times \cdots \times I_t$ is a quasi-$(n_t + 2)$-absorbing ideal of $R_1 \times R_2 \times \cdots \times R_t$ by part 2.
Case 3. Let $n_{t-1} + 2 > n_t$. Then $I_1 \times I_2 \times \cdots \times I_t$ is a quasi-$(n_{t-1} + 3)$-absorbing of $R_1 \times R_2 \times \cdots \times R_t$ by part 1. Since $n_{t-1} + 3 = n_t + 2$, then $I_1 \times I_2 \times \cdots \times I_t$ is quasi-$(n_t + 2)$-absorbing. □

Theorem 3.23. *Let I be a secondary ideal of a ring R. If J is a quasi-n-absorbing ideal of R, then $I \cap J$ is secondary.*

Proof. Assume that I is a P-secondary ideal of R, and let $a \in R$. If $a \in P = \sqrt{(0 :_R I)}$, then clearly $a \in \sqrt{(0 :_R I \cap J)}$. If $a \notin P$, then $a^n \notin P$, and so $a^n I = I$. We show that $a(I \cap J) = I \cap J$. Suppose that $x \in I \cap J$. There is an element $b \in I$ such that $x = a^n b \in J$. Since J is quasi-n-absorbing we get $a^n \in J$ or $a^{n-1}b \in J$. If $a^n \in J$, then $I = a^n I \subseteq J$ and so $a(I \cap J) = aI = I = I \cap J$. If $a^{n-1}b \in J$, then $x = a^n b \in a(I \cap J)$ and we are done. □

Let R be a ring with identity. We recall that if $f = a_0 + a_1X + \cdots + a_tX^t$ is a polynomial on the ring R, then *content* of f is defined as the ideal of R, generated by the coefficients of f, i.e. $c(f) = (a_0, a_1, \ldots, a_n)$. Let T be an R-algebra and c the function from T to the ideals of R defined by $c(f) = \cap\{I \mid I$ is an ideal of R and $f \in IT\}$ known as the content of f. Note that the content function c is nothing but the generalization of the content of a polynomial $f \in R[X]$. The R-algebra T is called a *content R-algebra* if the following conditions hold:

1. For all $f \in T$, $f \in c(f)T$.
2. (Faithful flatness) For any $r \in R$ and $f \in T$, the equation $c(rf) = rc(f)$ holds and $c(1_T) = R$.
3. (Dedekind-Mertens content formula) For each $f, g \in T$, there exists a natural number n such that $c(f)^n c(g) = c(f)^{n-1} c(fg)$.

For more information on content algebras and their examples we refer to [19–21]. In [7] Nasehpour gave the definition of a Gaussian R-algebra as follows: Let T be an R-algebra such that $f \in c(f)T$ for all $f \in T$. T is said to be a Gaussian R-algebra if $c(fg) = c(f)c(g)$, for all $f, g \in T$.

Example 3.24 ([7])**.** Let T be a content R-algebra such that R is a Prüfer domain. Since every nonzero finitely generated ideal of R is a cancellation ideal of R, the Dedekind-Mertens content formula causes T to be a Gaussian R-algebra.

Theorem 3.25. *Let R be a Prüfer domain, T a content R-algebra and I an ideal of R.*

1. *If I is a strongly quasi-n-absorbing (resp. strongly semi-n-absorbing) ideal of R, then IT is a quasi-n-absorbing (resp. semi-n-absorbing) ideal of T.*
2. *If IT is a quasi-n-absorbing (resp. semi-n-absorbing) ideal of T, then I is a quasi-n-absorbing (resp. semi-n-absorbing) ideal of R.*

Proof. 1. Assume that I is a strongly quasi-n-absorbing ideal of R. Let $f^n g \in IT$ for some $f, g \in T$. Then $c(f^n g) \subseteq I$. Since R is a Prüfer domain and T is a content R-algebra, then T is a Gaussian R-algebra. Therefore $c(f^n g) = c(f)^n c(g) \subseteq I$. Since I is a strongly quasi-n-absorbing ideal of R, $c(f)^n \subseteq I$ or $c(f)^{n-1}c(g) \subseteq I$. So $f^n \in c(f^n)T \subseteq IT$ or $f^{n-1}g \in c(f^{n-1}g)T \subseteq IT$. Consequently IT is a quasi-n-absorbing ideal of T.

2. Note that since T is a content R-algebra, $IT \cap R = I$ for every ideal I of R. Now, apply Corollary 3.4(1). □

The algebra of all polynomials over an arbitrary ring with an arbitrary number of indeterminates is an example of content algebras.

Corollary 3.26. *Let R be a Prüfer domain and I be an ideal of R.*

1. *If I is a strongly quasi-n-absorbing (resp. strongly semi-n-absorbing) ideal of R, then $I[X]$ is a quasi-n-absorbing (resp. semi-n-absorbing) ideal of $R[X]$.*
2. *If $I[X]$ is a quasi-n-absorbing (resp. semi-n-absorbing) ideal of $R[X]$, then I is a quasi-n-absorbing (resp. semi-n-absorbing) ideal of R.*

4 Semi-n-Absorbing Ideals

Suppose that m, n are positive integers with $m > n$. A more general concept than semi-n-absorbing ideals is the concept of semi-(m, n)-absorbing ideals. A proper ideal I of a ring R is called a *semi-(m, n)-absorbing ideal* if whenever $a^m \in I$ for $a \in R$, then $a^n \in I$. It is easy to see that every semi-(m, n)-absorbing ideal is a semi-n-absorbing ideal.

Note that a semiprime ideal is just a semi-1-absorbing ideal.

Theorem 4.1. *Let I be a proper ideal of R and m, n be positive integers with $m > n$.*

1. *If I is quasi-n-absorbing, then it is semi-(m, n)-absorbing.*
2. *I is semi-(m, n)-absorbing if and only if I is semi-(m, k)-absorbing for each $m > k \geq n$ if and only if I is semi-(i, j)-absorbing for each $m \geq i > j \geq n$.*
3. *If I is semi-(m, n)-absorbing, then it is semi-(mk, nk)-absorbing for every positive integer k.*

4. *If I is semi-(m,n)-absorbing and semi-(r,s)-absorbing for some positive integers $r > s$, then it is semi-(mr, ns)-absorbing.*

Proof. 1. Is trivial.

2. Straightforward.

3. Suppose that I is a semi-(m,n)-absorbing ideal of R. Let $a \in R$ and k be a positive integer such that $a^{mk} \in I$. Then $\left(a^k\right)^m \in I$. Since I is semi-(m,n)-absorbing, $\left(a^k\right)^n = a^{nk} \in I$, and so I is semi-(mk, nk)-absorbing.

4. Assume that I is semi-(m,n)-absorbing and semi-(r,s)-absorbing for some positive integers $r > s$. Let $a^{mr} \in I$. Since I is semi-(m,n)-absorbing, $a^{nr} \in I$, and since I is semi-(r,s)-absorbing, $a^{ns} \in I$. Hence I is semi-(mr, ns)-absorbing. □

Corollary 4.2. *Let I be a proper ideal of R.*

1. *If I is quasi-n-absorbing, then it is semi-n-absorbing.*
2. *Let $t \leq n$ be an integer. If I is semi-$(n+1,t)$-absorbing, then it is semi-$(nk+i, tk)$-absorbing for all $k \geq i \geq 1$.*
3. *If I is semi-n-absorbing, then it is semi-$(nk+i, nk)$-absorbing for all $k \geq i \geq 1$.*
4. *If I is semi-n-absorbing, then it is semi-$(nk+j)$-absorbing for all $k > j \geq 0$.*
5. *If I is semi-n-absorbing, then it is semi-(nk)-absorbing for every positive integer k.*
6. *If I is semiprime, then it is semi-k-absorbing for every positive integer k.*
7. *If I is semiprime, then for every $k \geq 1$ and every $a \in R$, $a^k \in I$ implies that $a \in I$.*
8. *If I is semi-n-absorbing, then it is semi-$((n+1)^t, n^t)$-absorbing for all $t \geq 1$.*
9. *If I is semiprime, then it is quasi-k-absorbing for every $k > 1$.*

Proof. 1. By Theorem 4.1(1).

2. Suppose that I is semi-$(n+1,t)$-absorbing. Then by Theorem 4.1(3), I is semi-$(nk+k, tk)$-absorbing, for every positive integer k. Again by Theorem 4.1(2), I is semi-$(nk+i, tk)$-absorbing for every $k \geq i \geq 1$.

3. In part 2 get $t = n$.

4. By part 3.

5. Is a special case of 4.

6. Is a direct consequence of 5.

7. By part 6.

8. By Theorem 4.1(4).

9. Assume that I is semiprime. Let $a^k b \in I$ for some $a, b \in R$ and some $k > 1$. Then $(ab)^k \in I$. Therefore $ab \in I$, by part 7. So I is quasi-k-absorbing. □

Proposition 4.3. *Let $I_1, I_2, \ldots, I_n$ be ideals of R. If for every $1 \leq i \leq n$, I_i is a semiprime ideal, then $I_1 I_2 \cdots I_n$ is a semi-n-absorbing ideal. In particular, if I is a semiprime ideal of R, then I^n is a semi-n-absorbing ideal.*

Proof. Use Corollary 4.2(7). □

Remark 4.4. *Let I be an ideal of a ring R. If I^{n+1} is a strongly semi-n-absorbing ideal, then $I^{n+1} = I^n$. In particular, if I^2 is a semiprime ideal, then I is idempotent.*

The following remark shows that the two concepts of semi-n-absorbing ideals and of semi-$(n+1)$-absorbing ideals are different in general.

Remark 4.5. *Let $n > 1$, R be a ring and P be a prime ideal of R. By Proposition 4.3, P^{n+1} is a semi-$(n+1)$-absorbing ideal. If P^{n+1} is a semi-n-absorbing ideal, then $P^{n+1} = P^n$. Consequently, for any prime number p, $p^{n+1}\mathbb{Z}$ is a semi-$(n+1)$-absorbing ideal of $\mathbb{Z}$ which is not a semi-n-absorbing ideal.*

Proposition 4.6. *Let I be an ideal of a ring R. If I is such that for every ideal J of R, we have $J^{n+1} \subseteq I \subseteq J \Rightarrow J^n \subseteq I$, then I is strongly semi-n-absorbing.*

Proof. The proof is similar to that of Proposition 3.17(1). □

Proposition 4.7. *Let $I_1, I_2, \ldots, I_n$ be semi-2-absorbing ideals of R. Then $I_1 I_2 \cdots I_n$ is a semi-$(3^n - 1)$-absorbing ideal.*

Proof. Suppose that $a^{3^n} \in I_1 I_2 \cdots I_n$ for some $a \in R$. For every $1 \leq i \leq n$, $a^{3^n} \in I_i$ and I_i is semi-2-absorbing, then $a^{2^n} \in I_i$. Therefore $a^{n2^n} \in I_1 I_2 \cdots I_n$. On the other hand $n2^n \leq 3^n - 1$. So $a^{3^n - 1} \in I_1 I_2 \cdots I_n$ which shows that $I_1 I_2 \cdots I_n$ is semi-$(3^n - 1)$-absorbing. □

Theorem 4.8. *Let $I_1, I_2, \ldots, I_k$ be ideals of R. If I_i is a semi-n_i-absorbing ideal of R for every $1 \leq i \leq k$, then $I_1 \cap I_2 \cap \cdots \cap I_k$ is a semi-$(n-1)$-absorbing ideal for $n = \prod_{i=1}^{k} (n_i + 1)$.*

Proof. Let $a \in R$ be such that $a^n \in I_1 \cap I_2 \cap \cdots \cap I_k$. Then for every $1 \leq i \leq k$,

$$\left(a^{\prod_{j=1, j \neq i}^{k} (n_j + 1)} \right)^{(n_i + 1)} \in I_i.$$

Since I_i's are semi-n_i-absorbing, then, for each $1 \leq i \leq k$,

$$a^{\left[n_i \prod_{j=1,j\neq i}^{k}(n_j+1)\right]} \in I_i.$$

Note that for every $1 \leq i \leq k$,

$$n_i \prod_{j=1,j\neq i}^{k}(n_j+1) \leq \prod_{i=1}^{k}(n_i+1) - 1 = n-1.$$

So we have $a^{n-1} \in I_i$ for every $1 \leq i \leq k$. Hence $a^{n-1} \in I_1 \cap I_2 \cap \cdots \cap I_k$ which implies that $I_1 \cap I_2 \cap \cdots \cap I_k$ is a semi-$(n-1)$-absorbing ideal. □

Proposition 4.9. *Let I_1, I_2 be ideals of R and m, n be positive integers.*

1. *If I_1 is quasi-m-absorbing and I_2 is semi-n-absorbing, then I_1I_2 is semi-$(n(m+1)+m)$-absorbing.*
2. *If I_1 is quasi-$(2m)$-absorbing and I_2 is semi-m-absorbing, then I_1I_2 is semi-$(m(m+2))$-absorbing.*

Proof. 1. Assume that $a^{(n+1)(m+1)} \in I_1I_2$ for some $a \in R$. Since I_1 is quasi-m-absorbing and $a^{(n+1)(m+1)} \in I_1$, then $a^m \in I_1$. On the other hand I_2 is semi-n-absorbing and $a^{(n+1)(m+1)} \in I_2$, then $a^{n(m+1)} \in I_2$. Consequently $a^{n(m+1)+m} \in I_1I_2$, and so I_1I_2 is semi-$(n(m+1)+m)$-absorbing.
2. Suppose that $a^{(m+1)^2} \in I_1I_2$ for some $a \in R$. Since I_1 is quasi-$(2m)$-absorbing and $a^{(m+1)^2} \in I_1$, then $a^{2m} \in I_1$. Since I_2 is semi-m-absorbing and $a^{(m+1)^2} \in I_2$, then $a^{m^2} \in I_2$. Hence $a^{m^2+2m} \in I_1I_2$ which shows that I_1I_2 is semi-$(m(m+2))$-absorbing. □

Let R be a ring and I be an ideal of R. We denote by $I^{[n]}$ the ideal of R generated by the n-th powers of all elements of I. If $n!$ is a unit in R, then $I^{[n]} = I^n$, see [22].

Proposition 4.10. *Let I be an ideal of a ring R. Then I is semi-n-absorbing if and only if $J^{[n+1]} \subseteq I$ implies that $J^{[n]} \subseteq I$ for every ideal J of R.*

Proof. The proof is easy. □

Corollary 4.11. *Let R be a ring such that $n!$ is a unit in R. Then every semi-n-absorbing ideal of R is strongly semi-n-absorbing.*

Proposition 4.12. *Let R be a ring. The following statements are equivalent:*

1. *For every ideal I of R, $I^{[n]} \subseteq I^{n+1}$;*
2. *For all ideals $I_1, I_2, \ldots, I_{n+1}$ of R we have $(I_1 \cap I_2 \cap \cdots \cap I_{n+1})^{[n]} \subseteq I_1I_2 \cdots I_{n+1}$;*

3. *For every elements $a \in R$, $a^n = ra^{n+1}$ for some $r \in R$;*
4. *Every ideal I of R is semi-n-absorbing.*

Proof. (1)⇒(2) For ideals $I_1, I_2, \ldots, I_{n+1}$ of R, we get from 1,

$$(I_1 \cap I_2 \cap \cdots \cap I_{n+1})^{[n]} \subseteq (I_1 \cap I_2 \cap \cdots \cap I_{n+1})^{n+1} \subseteq I_1 I_2 \cdots I_{n+1}.$$

(2)⇒(1) For an ideal I of R, by 2 we have that $I^{[n]} = (\overbrace{I \cap \cdots \cap I}^{n+1 \text{ times}})^{[n]} \subseteq I^{n+1}$. So we have $I^{[n]} \subseteq I^{n+1}$.
(1)⇔(3) and (3)⇔(4) are easy. □

Proposition 4.13. *Let R be a ring. The following statements are equivalent:*

1. *For every ideal I of R, $I^{n+1} = I^n$;*
2. *For every ideals $I_1, I_2, \ldots, I_{n+1}$ of R we have $(I_1 \cap I_2 \cap \cdots \cap I_{n+1})^n \subseteq I_1 I_2 \cdots I_{n+1}$;*
3. *Every ideal I of R is strongly semi-n-absorbing.*

Proof. Similar to the proof of Proposition 4.12. □

Remark 4.14. *Let $\{I_\lambda\}_{\lambda \in \Lambda}$ be a family of semi-n-absorbing ideals of R. Then $\bigcap_{\lambda \in \Lambda} I_\lambda$ is semi-n-absorbing.*

The following remark shows that the two concepts of semi-n-absorbing ideals and of quasi-n-absorbing (n-absorbing) ideals are different in general.

Remark 4.15. *Let p, q be distinct prime numbers. By Proposition 4.3, $p^n\mathbb{Z}$ is a semi-n-absorbing ideal of $\mathbb{Z}$. Therefore Remark 4.14 implies that $p^n\mathbb{Z} \cap q\mathbb{Z}$ is a semi-n-absorbing ideal of $\mathbb{Z}$, but it is not quasi-n-absorbing, by Remark 3.11.*

Proposition 4.16. *For any ring R there exists a unique least semi-n-absorbing ideal.*

Proof. Set $\mathcal{I}^{(n)} = \bigcap\{I \mid I \text{ is a semi-}n\text{-absorbing ideal of } R\}$. By Remark 4.14, $\mathcal{I}^{(n)}$ is the least semi-n-absorbing ideal. □

By notation in the the proof of the previous proposition we have the following remark:

Remark 4.17. *Let R be a ring. First of all, we know that $Nil(R)$ (the set of all nilpotent elements of R) is the intersection of all prime ideals of R, then $\mathcal{I}^{(1)} \subseteq Nil(R)$. Suppose that $x \in Nil(R)$, then there is a positive integer m such that $x^m = 0 \in \mathcal{I}^{(1)}$. Hence $\mathcal{I}^{(1)}$ semiprime implies that $x \in \mathcal{I}^{(1)}$. Thus $\mathcal{I}^{(1)} = Nil(R)$.*

Proposition 4.18. *The following statements hold:*

1. $\mathcal{I}^{(1)} = \sum_{n\geq 1} \mathcal{I}^{(n)}$.

2. $\mathcal{I}^{(nk)} \subseteq \mathcal{I}^{(n)}$ *for every positive integer* k.

3. $\mathcal{I}^{(n)} \subseteq I^n$ *for every semiprime ideal* I.

Proof. 1. By Corollary 4.2(6) every semiprime ideal is semi-n-absorbing for every $n \geq 1$. Then $\mathcal{I}^{(n)} \subseteq \mathcal{I}^{(1)}$ for every $n \geq 1$.
2. By Corollary 4.2(5).
3. By Proposition 4.3. □

Proposition 4.19. *Let* R_1, R_2 *be rings. If* I_1 *is a semi-*n*-absorbing ideal of* R_1 *and* I_2 *is a semi-*n*-absorbing ideal of* R_2*, then* $I_1 \times I_2$ *is a semi-*n*-absorbing ideal of* $R_1 \times R_2$*.*

Proof. Let $(a, b)^{n+1} \in I_1 \times I_2$ for some $a \in R_1$ and $b \in R_2$. Then $a^{n+1} \in I_1$ and $b^{n+1} \in I_2$. Since I_1 is semi-n-absorbing, then $a^n \in I_1$, and since I_2 is semi-n-absorbing, then $b^n \in I_2$. Hence $(a, b)^n \in I_1 \times I_2$ which shows that $I_1 \times I_2$ is semi-n-absorbing. □

Proposition 4.20. *Let* $R = R_1 \times R_2$ *be a decomposable ring and* L *be a quasi-*n*-absorbing ideal of* R*. Then either* $L = I_1 \times R_2$ *where* I_1 *is a quasi-*n*-absorbing ideal of* R_1 *or* $L = R_1 \times I_2$ *where* I_2 *is a quasi-*n*-absorbing ideal of* R_2 *or* $L = I_1 \times I_2$ *where* I_1 *is a semi-*$(n-1)$*-absorbing ideal of* R_1 *and* I_2 *is a semi-*$(n-1)$*-absorbing ideal of* R_2*.*

Proof. Regarding Proposition 3.21 we only investigate the case when $L = I_1 \times I_2$ in which I_1 is a proper ideal of R_1 and I_2 is a proper ideal of R_2. Let $a^n \in I_1$ for some $a \in R_1$. Therefore $(a, 1)^n(1, 0) \in I_1 \times I_2$. Since I_2 is proper, then $(a, 1)^n \notin I_1 \times I_2$. Hence $(a, 1)^{n-1}(1, 0) \in I_1 \times I_2$, because $I_1 \times I_2$ is a quasi-n-absorbing ideal of R. Thus $a^{n-1} \in I_1$ which shows that I_1 is a semi-$(n-1)$-absorbing ideal of R_1. Similarly we can show that I_2 is a semi-$(n-1)$-absorbing ideal of R_2. □

Proposition 4.21. *Let* $R = R_1 \times R_2$ *be a decomposable ring and* L *be a proper ideal of* R*. Then the following statements are equivalent:*

1. L *is a quasi-2-absorbing ideal of* R*;*

2. *Either* $L = I_1 \times R_2$ *where* I_1 *is a quasi-2-absorbing ideal of* R_1 *or* $L = R_1 \times I_2$ *where* I_2 *is a quasi-2-absorbing ideal of* R_2 *or* $L = I_1 \times I_2$ *where* I_1 *is a semiprime ideal of* R_1 *and* I_2 *is a semiprime ideal of* R_2*.*

Proof. (1)⇒(2) By Proposition 4.20.
(2)⇒(1) Assume that $L = I_1 \times I_2$ for some semiprime ideal I_1 of R_1 and some semiprime ideal I_2 of R_2. Then, by Proposition 4.19, $L = I_1 \times I_2$ is a semiprime ideal of $R = R_1 \times R_2$. Thus $L = I_1 \times I_2$ is a quasi-2-absorbing ideal of $R = R_1 \times R_2$, by Corollary 4.2(9). □

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