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Coupled Coincidence Point for Generalized Monotone Operators in Partially Ordered Metric Spaces

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Abstract : In this note, we prove the existence and the uniqueness of coupled coincidence points theorem for $(\varphi - \psi)$ contractive condition for a generalized mapping having the mixed $g-$monotone property in complete ordered spaces. These results generalize and extend the existing fixed point results in the literature.

Keywords : coupled coincidence point; partially ordered metric space; mixed $g-$monotone property.
2010 Mathematics Subject Classification : 47H10; 54H25.

1 Introduction

The existence of fixed points in partially ordered metric spaces has been studied recently by several authors: Ran et all [1], Bhaskar and Laksmikantham [2], Agarwal et all [3], Lakshmikantham and Ciric [4], Luong Thuan [5] and Alotaibi and Alsulami [6]. The first result appeared in this direction was given by Ran and Reurings [1], who presented its applications to matrix equation. Nieto and Rodriguez-Lopez [7, 8] extended the results of Ran and Reurings [1] for nondecreasing mappings and applied them to obtain a unique solution for a first-order ordinary differential equation with periodic boundary conditions.

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Recently, Alotaibi and Alsulami [6], established the existence and uni-queness of coupled coincidence point, involving a (φ, ψ)-contraction condition for a mapping having the mixed g-monotone property. The aim of this paper is to extend the results obtained in [6] by using a more general mapping having the mixed g-monotone property and generalize the existing fixed point in the literature, [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19].

Here are some definitions, and early results we presented in the following.

Definition 1.1. ([3]). Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$. The mapping F is said to have the *mixed monotone property*, if $F(x, y)$ is monotone, nondecreasing in x and it is monotone non-increasing in y, that is, for any $x, y \in X$:

$$x_1, x_2 \in X, \quad x_1 \preceq x_2 \implies F(x_1, y) \preceq F(x_2, y),$$

and

$$y_1, y_2 \in X, \quad y_1 \preceq y_2 \implies F(x, y_1) \succeq F(x, y_2).$$

Definition 1.2. ([3]). An element $(x, y) \in X \times X$ is called a *coupled fixed point* of the mapping $F : X \times X \to X$ if

$$x = F(x, y) \quad \text{and} \quad y = F(y, x).$$

Definition 1.3. ([4]). (Mixed g-monotone Property).
Let F and g be two mappings such that

$$F : X \times X \to X \text{ and } g : X \to X.$$

The mapping F is said to have the *mixed $g-$monotone property* if $F(x, y)$ is monotone, nondecreasing in x and is monotone non-increasing in y, that is, for any $x, y \in X$:

$$x_1, x_2 \in X, \; gx_1 \preceq gx_2 \implies F(x_1, y) \preceq F(x_2, y), \tag{1.1}$$

and

$$y_1, y_2 \in X, \; gy_1 \preceq gy_2 \implies F(x, y_1) \succeq F(x, y_2). \tag{1.2}$$

Definition 1.4. ([4]). An element $(x, y) \in X \times X$ is called a *coupled coincidence point* of the mapping $F : X \times X \to X$ and $g : X \to X$ if

$$gx = F(x, y) \quad \text{and} \quad gy = F(y, x).$$

Definition 1.5. ([4]). The mappings F and g, where $F : X \times X \to X$ and $g : X \to X$, are said to be *compatible* if

$$\lim_{n\to\infty} d\left(g\left(F(x_n, y_n)\right), F(gx_n, gy_n)\right) = 0,$$

and

$$\lim_{n\to\infty} d\left(g\left(F(y_n, x_n)\right), F(gy_n, gx_n)\right) = 0,$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X, such that:

$$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} gx_n = x,$$

and

$$\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} gy_n = y,$$

are satisfied for all $x, y \in X$.

The main theoritical results presented in [2] are the following coupled point theorems.

Theorem 1.6. ([2]). *Let $(X, \preceq)$ be a partially ordered set and suppose there exists a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a continuous mapping, having the mixed monotone property on X. Assume that there exists a $k \in [0, 1)$ with*

$$d\left(F(x,y), F(u,v)\right) \leq \tfrac{k}{2}[d(x,u) + d(y,v)]$$
$$\textit{for all } x \succeq u \quad \textit{and} \quad y \preceq v.$$

If there exist two elements x_0,$y_0 \in X$, with

$$x_0 \preceq F(x_0, y_0) \textit{ and } \ y_0 \succeq F(y_0, x_0),$$

then there exist $x, y \in X$ such that

$$x = F(x, y) \ \textit{ and } \ y = F(y, x).$$

Theorem 1.7. [2]. *Let $(X, \preceq)$ be a partially ordered set and suppose there exists a metric d on X such that (X, d) is a complete metric space. Assume that X has the following property:*

1. *If a nondecreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$, for all $n \geq 1$.*
2. *If a non-increasing sequence $\{y_n\} \to y$, then $y_n \succeq y$, for all $n \geq 1$.*

Let $F : X \times X \to X$ be a mapping, having the mixed monotone property on X. Assume that there exists a $k \in [0, 1)$ with

$$d\left(F(x,y), F(u,v)\right) \leq \frac{k}{2}[d(x,u) + d(y,v)]$$
$$\text{for all } x \succeq u \quad \textit{and} \quad y \preceq v.$$

If there exist two elements $x_0, y_0 \in X$, with

$$x_0 \preceq F(x_0, y_0) \ \textit{ and } \ y_0 \succeq F(y_0, x_0),$$

then there exist $x, y \in X$ such that

$$x = F(x, y) \ \textit{ and } \ y = F(y, x).$$

In [5], the authors presented some coupled fixed point theorems for a mixed monotone mapping in a partially ordered metric space, which are nondecreasing, of the results of Bhaskar and Lakshmikantham [2].

Let us denote by Φ the set of all functions $\varphi : [0,\infty[\to [0,\infty[$ which satisfy

1. φ is continuous and nondecreasing,
2. $\varphi(t) = 0$ if, and only if $t = 0$,
3. $\varphi(t+s) \leq \varphi(t) + \varphi(s)$, for all $t, s \in [0,\infty[$

and Ψ denotes the set of all functions $\psi : [0,\infty[\to [0,\infty[$, which satisfy

$$\lim_{t\to r}\psi(t) > 0 \text{ for all } r > 0 \text{ and } \lim_{t\to 0_+}\psi(t) = 0.$$

(For more details about Φ and Ψ, see [5]).

Theorem 1.8. ([2]). *Let $(X, \preceq)$ be a partially ordered set and suppose there exists a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a continuous mapping, having the mixed monotone property on X, such that there exist two elements $x_0, y_0 \in X$, with $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$. Suppose there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that*

$$\varphi\left(d\left(F(x,y), F(u,v)\right)\right) \leq \tfrac{1}{2}\varphi\left(d(x,u) + d(y,v)\right) - \psi\Big(\frac{d(x,u)+d(y,v)}{2}\Big)$$
$$\text{for all } x \succeq u \quad \text{and} \quad y \preceq v.$$

Suppose either

(a) *F is continuous or*

(b) *X has the following properties:*

   (i) *If a nondecreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$, for all $n \geq 1$,*

   (ii) *If A non-increasing sequence $\{y_n\} \to y$, then $y_n \succeq y$, for all $n \geq 1$,*

then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$. That is F has a fixed point.

A. Alotaibi and S. Alsulami [6] established the existence and uniqueness of coupled coincidence point involving a (φ, ψ)-contractive condition for mappings having the mixed g-monotone property. The main theoritical result in A. Alotaibi and S. Alsulami [6], is given by the following theorem:

Theorem 1.9. [6]. *Let $(X, \preceq)$ be a partially ordered set and suppose there exists a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a continuous mapping, having the mixed monotone property on X such that there* *exist two elements* $x_0, y_0 \in X$ *with* $gx_0 \preceq F(x_0, y_0)$ *and* $gy_0 \succeq F(y_0, x_0)$. *Suppose there exist* $\varphi \in \Phi$ *and* $\psi \in \Psi$ *such that*

$$\begin{aligned}\varphi\left(d\left(F(x,y),F(u,v)\right)\right) &\leq \tfrac{1}{2}\varphi\left(d(gx,gu)+d(gy,gv)\right)\\ &\quad -\psi\Big(\frac{d(gx,gu)+d(gy,gv)}{2}\Big)\end{aligned}$$

$$\text{for all} \quad gx \succeq gu \quad \text{and} \quad gy \preceq gv.$$

Suppose $F(X \times X) \subset g(X)$, g *continuous and compatible with* F, *also suppose either*

(a) F *is continuous or*

(b) X *has the following property:*

- (i) f *a nondecreasing sequence* $\{x_n\} \to x$, *then* $x_n \preceq x$, *for all* $n \geq 1$,
- (ii) f *a non-increasing sequence* $\{y_n\} \to y$, *then* $y_n \succeq y$, *for all* $n \geq 1$

then there exist $x, y \in X$ *such that* $gx = F(x, y)$ *and* $gy = F(y, x)$. *That is* F *and* g *have a coupled coincidence point in* X.

2 Existence of Coupled Coincidence Points.

The set Ψ_1 of all lower semi-continuous functions $\psi : [0, \infty[\to [0, \infty[$, is considered, instead of Ψ, the function, ψ satisfies: $\psi(t) = 0$ if, and only if $t = 0$. We prove our main result:

Theorem 2.1 (Main Theorem)**.** *Let* $(X, \preceq)$ *be a partially ordered set and suppose there exists a metric* d *on* X *such that* (X, d) *is a complete metric space. Let* $F : X \times X \to X$ *be a continuous mapping, having the mixed monotone property on* X, *such that there exist two elements* $x_0, y_0 \in X$ *with*

$$gx_0 \preceq F(x_0, y_0) \text{ and } gy_0 \succeq F(y_0, x_0). \tag{2.1}$$

Suppose that there exist $\varphi \in \Phi$ *and* $\psi \in \Psi_1$ *such that*

$$\left.\begin{aligned}\varphi\left(d\left(F(x,y),F(u,v)\right)\right) &\leq \tfrac{1}{2}\varphi\left(\Theta(gx,gy,gu,gv)\right)\\ &\quad -\psi\left(\Theta(gx,gy,gu,gv)\right)\\ \text{for all} \quad gx \succeq gu &\quad \text{and} \quad gy \preceq gv\end{aligned}\right\} \tag{2.2}$$

$$\left.\begin{aligned}\Theta(gx,gy,gu,gv) &= \frac{\alpha}{2}\left[d(gx,gu)+d(gy,gv)\right]\\ &\quad +\frac{\beta}{2}\left[d(gx,F(x,y))+d(gy,F(y,x))\right.\\ &\qquad \left.+d(gu,F(u,v))+d(gv,F(v,u))\right]\\ &\quad +\frac{\gamma}{2}\left[d(gx,F(u,v))+d(gy,F(v,u))\right.\\ &\qquad \left.+d(gu,F(x,y)+d(gv,F(y,x))\right]\end{aligned}\right\} \tag{2.3}$$

where $\alpha > 0$, *and* $\beta, \gamma \geq 0$ *such that* $\alpha + 2\beta + 2\gamma < 2$. *Suppose* $F(X \times X) \subset g(X)$, g *continuous and compatible with* F, *also suppose either*

(a) F *is continuous or*

(b) X *has the following properties:*

(i) *If a nondecreasing sequence* $\{x_n\} \to x$, *then* $x_n \preceq x$, *for all* $n \geq 1$,

(ii) *If a non-increasing sequence* $\{y_n\} \to y$, *then* $y_n \succeq y$, *for all* $n \geq 1$,

then there exist $x, y \in X$ *such that* $gx = F(x, y)$ *and* $gy = F(y, x)$. *That is* F *and* g *have a coupled coincidence point in* X.

Proof. Let $x_0, y_0 \in X$ be such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$. Using the fact that $F(X \times X) \subset g(X)$, we construct sequences $\{x_n\}$ and $\{y_n\}$ in X as follows:

$$gx_{n+1} = F(gx_n, gy_n) \text{ and } gy_{n+1} = F(gy_n, gx_n) \text{ for all } n \geq 0. \tag{2.4}$$

We shall show that the sequence $\{gx_n\}$ is increasing and $\{gy_n\}$ is decreasing, that is:

$$gx_n \preceq gx_{n+1} \text{ for all } n \geq 0 \tag{2.5}$$

and

$$gy_n \succeq gy_{n+1} \text{ for all } n \geq 0. \tag{2.6}$$

To prove (2.5) and (2.6), we use the mathematical induction.

Let $n = 0$. Since $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$ and as $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$, we have $gx_0 \preceq gx_1$ and $gy_0 \succeq gy_1$. Thus (2.5) and (2.6) hold for $n = 0$.

Suppose now that (2.5) and (2.6) hold for some $n \geq 0$. Then, since $gx_n \preceq gx_{n+1}$ and $gy_n \succeq gy_{n+1}$, and by the mixed $g-$monotone property of F, we have

$$gx_{n+2} = F(x_{n+1}, y_{n+1}) \succeq F(x_n, y_{n+1}) \succeq F(x_n, y_n) = gx_{n+1} \tag{2.7}$$

and

$$gy_{n+2} = F(y_{n+1}, x_{n+1}) \preceq F(y_n, y_{n+1}) \preceq F(y_n, x_n) = gy_{n+1}. \tag{2.8}$$

Now, from (2.7) and (2.8), we obtain

$$gx_{n+2} \preceq gx_{n+1} \quad \text{and} \quad gy_{n+1} \succeq gy_{n+2}, \tag{2.9}$$

thus, by mathematical induction, we conclude that (2.5) and (2.6) hold for all $n \geq 0$. Therefore:

$$gx_0 \preceq gx_1 \preceq gx_2 \preceq \cdots \preceq gx_n \preceq gx_{n+1} \preceq \cdots \tag{2.10}$$

$$gy_0 \succeq gy_1 \succeq gy_2 \succeq \cdots \succeq gy_n \succeq gy_{n+1} \succeq \cdots \tag{2.11}$$

since $gx_n \succeq gx_{n-1}$ and $gy_n \preceq gy_{n-1}$. Then, from (2.3) and (2.4), we have:

$$\left.\begin{aligned}\Theta(gx_n, gy_n, gx_{n-1}, gy_{n-1}) = &\frac{\alpha}{2}\left[d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1})\right]\\ &+\frac{\beta}{2}\left[d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})\right.\\ &\left.+d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)\right]\\ &\frac{\gamma}{2}\left[d(gx_{n-1}, gx_{n+1})) + d(gy_{n-1}, gy_{n+1})\right]\end{aligned}\right\} \quad (2.12)$$

Similarly, since $gx_n \succeq gx_{n-1}$ and $gy_n \preceq gy_{n-1}$, then, from (2.3) and (2.4), we have also:

$$\left.\begin{aligned}\varphi\left(d\left(gy_{n+1}, gy_n\right)\right) &= \varphi\left(d\left(F\left(y_n, x_n\right)\right), F\left(y_{n-1}, x_{n-1}\right)\right)\\ &\leq \tfrac{1}{2}\varphi\left(\Theta(gy_n, gx_n, gy_{n-1}, gx_{n-1})\ \right)\\ &\quad -\psi\left(\Theta(gy_n, gx_n, gy_{n-1}, gx_{n-1})\right)\end{aligned}\right\} \quad (2.13)$$

$$\begin{aligned}\varphi\left(d\left(gx_{n+1}, gx_n\right)\right) &= \varphi\left(d\left(F\left(x_n, y_n\right)\right), F\left(x_{n-1}, y_{n-1}\right)\right)\\ &\leq \frac{1}{2}\varphi\left(\Theta(gx_n, gy_n, gx_{n-1}, gy_{n-1})\ \right)\\ &\quad -\psi\left(\Theta(gx_n, gy_n, gx_{n-1}, gy_{n-1})\ \right)\end{aligned}$$

where

$$\left.\begin{aligned}\Theta(gy_n, gx_n, gy_{n-1}, gx_{n-1}) = &\frac{\alpha}{2}\left[d(gy_n, gy_{n-1}) + d(gx_n, gx_{n-1})\right]\\ &+\frac{\beta}{2}\left[d(gy_n, gy_{n+1}) + d(gx_n, gx_{n+1})\right.\\ &\left.+d(gy_{n-1}, gy_n) + d(gx_{n-1}, gx_n)\right]\\ &+\frac{\gamma}{2}\left[d(gy_{n-1}, gy_{n+1})) + d(gx_{n-1}, gx_{n+1})\right]\end{aligned}\right\} \quad (2.14)$$

$$\left.\begin{aligned}\Theta(gx_n, gy_n, gx_{n-1}, gy_{n-1}) = &\frac{\alpha}{2}\left[d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1})\right]\\ &+\frac{\beta}{2}\left[d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})\right.\\ &\left.+d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)\right]\\ &+\frac{\gamma}{2}\left[d(gx_{n-1}, gx_{n+1})) + d(gy_{n-1}, gy_{n+1})\right]\end{aligned}\right\} \quad (2.15)$$

From (2.15) and (2.14), we remark that, for all $n \geq 1$:

$$\Theta(gx_n, gy_n, gx_{n-1}, gy_{n-1}) = \Theta(gy_n, gx_n, gy_{n-1}, gx_{n-1}). \quad (2.16)$$

Now, let us set

$$\delta_{n-1} = d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1}). \quad (2.17)$$

From (2.16), (2.17) and triangular inequality, we get

$$\left.\begin{aligned}\frac{\alpha}{2}\delta_{n-1} \leq \Theta(gx_n, gy_n, gx_{n-1}, gy_{n-1}) \leq \frac{\alpha}{2}\delta_{n-1} + \frac{\beta}{2}\left[\delta_n + \delta_{n-1}\right]\\ +\frac{\gamma}{2}\left[\delta_{n-1} + \delta_n\right]\end{aligned}\right\} \quad (2.18)$$

Now, from (2.18), we have

$$\Theta(gx_n, gy_n, gx_{n-1}, gy_{n-1}) \leq \frac{\alpha+\beta+\gamma}{2}\delta_{n-1} + \frac{\beta+\gamma}{2}\delta_n\,. \tag{2.19}$$

If there exists $n_0 \in \mathbb{N}$ such that $\Theta(gx_{n_0}, gy_{n_0}, gx_{n_0-1}, gy_{n_0-1}) = 0$, then $\delta_{n_0-1} = 0$. Therefore:

$$gx_{n_0} = gx_{n_0-1} = F\left(x_{n_0-1}, y_{n_0-1}\right)$$

and

$$gy_{n_0} = gy_{n_0-1} = F\left(y_{n_0-1}, x_{n_0-1}\right),$$

so, the proof is finished.

From now on, we suppose that

$$\Theta(gx_n, gy_n, gx_{n-1}, gy_{n-1}) > 0 \quad \text{for} \quad \text{all } n \geq 1. \tag{2.20}$$

From (2.12), we obtain

$$\left.\begin{array}{l}\varphi\left(d\left(gx_{n+1}, gx_n\right)\right) + \varphi\left(d\left(gy_{n+1}, gy_n\right)\right) \leq \\ \quad \varphi\left(\Theta(gx_n, gy_n, gx_{n-1}, gy_{n-1})\right) - 2\psi\left(\Theta(gx_n, gy_n, gx_{n-1}, gy_{n-1})\right)\end{array}\right\} \tag{2.21}$$

Using the property of φ, we have

$$\left.\begin{array}{rcl}\varphi\left(d\left(gx_{n+1}, gx_n\right) + d(gy_{n+1}, gy_n)\right) & \leq & \varphi\left(d\left(gx_{n+1}, gx_n\right)\right) \\ & & +\varphi\left(d\left(gy_{n+1}, gy_n\right)\right)\end{array}\right\} \tag{2.22}$$

From (2.21) and (2.22), we have

$$\left.\begin{array}{rcl}\varphi\left(\delta_n\right) & \leq & \varphi\left(\Theta(gx_n, gy_n, gx_{n-1}, gy_{n-1})\right) \\ & & -2\psi\left(\Theta(gx_n, gy_n, gx_{n-1}, gy_{n-1})\right)\end{array}\right\} \tag{2.23}$$

Therefore, from (2.19) and (2.23) and by the property of φ, we obtain

$$\left.\begin{array}{rcl}\varphi\left(\delta_n\right) & \leq & \varphi\Big(\dfrac{\alpha+\beta+\gamma}{2}\delta_{n-1} + \dfrac{\beta+\gamma}{2}\delta_n\Big) \\ & & -2\psi\left(\Theta(gx_n, gy_n, gx_{n-1}, gy_{n-1})\right)\end{array}\right\} \tag{2.24}$$

Now, we claim that

$$\delta_n \leq \delta_{n-1} \qquad \text{for all } n \geq 1. \tag{2.25}$$

Suppose that (2.25) is not true, in this case, there exists $n_0 \geq 1$ such that

$$\delta_{n_0} > \delta_{n_0-1}, \tag{2.26}$$

then from (2.24) and (2.26) we have

$$\left.\begin{array}{rcl}\varphi\left(\delta_{n_0}\right) & \leq & \varphi\Big(\dfrac{\alpha+2\beta+2\gamma}{2}\delta_{n_0}\Big) \\ & & -2\psi\left(\Theta(gx_{n_0}, gy_{n_0}, gx_{n_0-1}, gy_{n_0-1})\right)\end{array}\right\} \tag{2.27}$$

since $\frac{\alpha+2\beta+2\gamma}{2}<1$ and since φ is nondecreasing, we get

$$\varphi\left(\delta_{n_0}\right)\leq\varphi\left(\delta_{n_0}\right)-2\psi\left(\Theta(gx_{n_0},gy_{n_0},gx_{n_0-1},gy_{n_0-1})\right), \tag{2.28}$$

which implies:

$$\psi\left(\Theta(gx_{n_0},gy_{n_0},gx_{n_0-1},gy_{n_0-1})\right)=0,$$

and by the property of ψ, we have

$$\Theta(gx_{n_0},gy_{n_0},gx_{n_0-1},gy_{n_0-1})=0,$$

which is a contradiction to (2.20). Therefore, (2.25) is true.

Now we shall prove that

$$\lim_{n\to\infty}\delta_{n-1}=0. \tag{2.29}$$

From (2.25), the sequence $\{\delta_n\}$ is non-increasing, with 0 as lower bound, thus, there exists $\delta\geq 0$ such that $\delta_n\to\delta$. We shall show that $\delta=0$. We suppose that $\delta>0$, then from (2.25), (2.21) and (2.27), we have:

$$\left.\begin{array}{rcl}\lim\limits_{n\to\infty}\frac{\alpha}{2}\delta_n & \leq & \limsup\limits_{n\to\infty}\Theta(gx_n,gy_n,gx_{n-1},gy_{n-1})\\ & \leq & \limsup\limits_{n\to\infty}\Big[\frac{\alpha+2\beta+2\gamma}{2}\delta_{n-1}\Big].\end{array}\right\} \tag{2.30}$$

Since $\frac{\alpha+2\beta+2\gamma}{2}<1$, (2.30) implies that

$$\frac{\alpha}{2}\delta\leq\limsup_{n\to\infty}\Theta(gx_n,gy_n,gx_{n-1},gy_{n-1})<\delta. \tag{2.31}$$

So, there exists $\delta_1>0$ and a subsequence $\{\delta_{n_k}\}$ of $\{\delta_n\}$ such that

$$\lim_{k\to\infty}\Theta(gx_{n_k},gy_{n_k},gx_{n_k-1},gy_{n_k-1})=\delta_1\leq\delta. \tag{2.32}$$

By the lower semicontinuity of ψ, we have

$$\psi\left(\delta_1\right)\leq\liminf_{k\to\infty}\psi\left(\Theta(gx_{n_k},gy_{n_k},gx_{n_k-1},gy_{n_k-1})\right). \tag{2.33}$$

From (2.23), we have

$$\left.\begin{array}{rcl}\varphi\left(\delta_{n_k}\right) & \leq & \varphi\left(\Theta(gx_{n_k},gy_{n_k},gx_{n_k-1},gy_{n_k-1})\right)\\ & & -2\psi\left(\Theta(gx_{n_k},gy_{n_k},gx_{n_k-1},gy_{n_k-1})\right)\end{array}\right\} \tag{2.34}$$

thus

$$\left.\begin{array}{rcl}\limsup\limits_{k\to\infty}\varphi\left(\delta_{n_k}\right) & \leq & \limsup\limits_{k\to\infty}\varphi\left(\Theta(gx_{n_k},gy_{n_k},gx_{n_k-1},gy_{n_k-1})\right)\\ & & -2\liminf\limits_{k\to\infty}\psi\left(\Theta(gx_{n_k},gy_{n_k},gx_{n_k-1},gy_{n_k-1})\right),\end{array}\right\} \tag{2.35}$$

which implies

$$\varphi(\delta) \leq \varphi(\delta) - 2\psi(\delta_1). \tag{2.36}$$

From (2.36) we obtain $\psi(\delta_1) = 0$, thus, by the property of ψ, we get $\delta_1 = 0$, which is a contradiction with the fact that $\delta > 0$, therefore $\delta = 0$. Now we will prove that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences. Suppose to the contrary, that at least one of $\{gx_n\}$ or $\{gy_n\}$ is not Cauchy sequence. Then there exists an $\varepsilon > 0$ for which we can find subsequences $\{gx_{n(k)}\}$, $\{gx_{m(k)}\}$ of $\{gx_n\}$ and $\{gy_{n(k)}\}$, $\{gy_{m(k)}\}$ of $\{gy_n\}$, with $n(k) > m(k) \geq k$, such that

$$d\left(gx_{n(k)}, gx_{m(k)}\right) + d\left(gy_{n(k)}, gy_{m(k)}\right) \geq \varepsilon. \tag{2.37}$$

Furthermore, corresponding to $m(k)$, we can choose $n(k)$ as the smallest integer with $n(k) > m(k) \geq k$ and satisfying (2.37). Then

$$d\left(gx_{n(k)-1}, gx_{m(k)}\right) + d\left(gy_{n(k)-1}, gy_{m(k)}\right) \leq \varepsilon. \tag{2.38}$$

Using (2.37), (2.38) and the triangular inequality, we have

$$\begin{aligned}
\varepsilon &\leq r_k := d\left(gx_{n(k)}, gx_{m(k)}\right) + d\left(gy_{n(k)}, gy_{m(k)}\right) \\
&\leq d\left(gx_{n(k)}, gx_{n(k)-1}\right) + d\left(gx_{n(k)-1}, gx_{m(k)}\right) \\
&\quad + d\left(gy_{n(k)}, gy_{n(k)-1}\right) + d\left(gy_{n(k)-1}, gy_{m(k)}\right) \\
&\leq d\left(gx_{n(k)}, gx_{n(k)-1}\right) + d\left(gy_{n(k)}, gy_{n(k)-1}\right) + \varepsilon.
\end{aligned}$$

Letting $k \to \infty$ and using (2.17), we get

$$\lim_{k\to\infty} r_k = \lim_{k\to\infty} \left[d\left(gx_{n(k)}, gx_{m(k)}\right) + d\left(gx_{n(k)}, gx_{m(k)}\right)\right] = \varepsilon. \tag{2.39}$$

From (2.3) we have

$$\begin{aligned}
&\frac{\alpha}{2}\left[d\left(gx_{n(k)}, gx_{m(k)}\right) + d\left(gy_{n(k)}, gy_{m(k)}\right)\right] \\
&\quad \leq \Theta(gx_{n_k}, gy_{n_k}, gx_{n_k-1}, gy_{n_k-1}) \\
&\quad = \tfrac{\alpha}{2}[d\left(gx_{n(k)}, gx_{m(k)}\right) + d\left(gy_{n(k)}, gy_{m(k)}\right)] \\
&\qquad + \tfrac{\beta}{2}[d\left(gx_{n(k)}, F(x_{n(k)}, gy_{n(k)})\right) + d\left(gy_{n(k)}, F(y_{n(k)}, gx_{n(k)})\right) \\
&\qquad + d\left(gx_{m(k)}, F(x_{m(k)}, y_{m(k)})\right) + d\left(gy_{m(k)}, F(y_{m(k)}, x_{m(k)})\right)] \\
&\qquad + \tfrac{\gamma}{2}[d\left(gx_{m(k)}, F(x_{m(k)}, y_{m(k)})\right) + d\left(gy_{n(k)}, F(y_{m(k)}, x_{m(k)})\right)] \\
&\qquad + d\left(gx_{n(k)}, F(x_{n(k)}, y_{n(k)})\right) + d\left(gy_{n(k)}, F(y_{n(k)}, x_{n(k)})\right)].
\end{aligned}$$

Then

$$\left.\begin{array}{l}\frac{\alpha}{2}\left[d\left(gx_{n(k)},gx_{m(k)}\right)+d\left(gy_{n(k)},gy_{m(k)}\right)\right]\\ \qquad\leq\ \Theta(gx_{n_k},gy_{n_k},gx_{n_k-1},gy_{n_k-1})\\ \qquad=\ \frac{\alpha}{2}[d\left(gx_{n(k)},gx_{m(k)}\right)+d\left(gy_{n(k)},gy_{m(k)}\right)]\\ \qquad\quad+\frac{\beta}{2}[d(gx_{n(k)},gx_{n(k)+1})+d(gy_{n(k)},gy_{n(k)+1})\\ \qquad\quad d(gx_{m(k)},gx_{m(k)+1})+d(gy_{m(k)},gy_{m(k)+1})]\\ \qquad\quad+\frac{\gamma}{2}[d(gx_{m(k)},gx_{m(k)+1})+d(gy_{m(k)},gy_{m(k)+1})\\ \qquad\quad+d(gx_{m(k)},gx_{n(k)+1})+d(gy_{m(k)},gy_{n(k)+1})]\end{array}\right\} \tag{2.40}$$

From triangular inequality, we obtain

$$\left.\begin{array}{l}d(gx_{m(k)},gx_{n(k)+1})\leq d(gx_{m(k)},gx_{m(k)+1})+d(gx_{m(k)+1},gx_{n(k)+1}\\ d(gy_{m(k)},gx_{n(k)+1})\leq d(gy_{m(k)},gy_{m(k)+1})+d(gy_{m(k)+1},gy_{n(k)+1})\end{array}\right\} \tag{2.41}$$

From (2.40) and (2.41) we get

$$\left.\begin{array}{l}\frac{\alpha}{2}r_{n_k}\leq\Theta(gx_{n_k},gy_{n_k},gx_{n_k-1},gy_{n_k-1})\leq\frac{\alpha}{2}r_{n_k}+\frac{\beta}{2}\left(\delta_{n_k}+\delta_{m_k}\right)\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad+\frac{\gamma}{2}\left(\delta_{m_k}+\delta_{m_k}+r_{n_{k+1}}\right)\end{array}\right\} \tag{2.42}$$

Taking upper limit when $k\to\infty$ and using (2.29), (2.39) and the fact that $\frac{\alpha}{2}+\frac{\gamma}{2}<1$, we get

$$0<\frac{\alpha}{2}\varepsilon\leq\limsup_{k\to\infty}\Theta(gx_{n_k},gy_{n_k},gx_{n_k-1},gy_{n_k-1})\leq\frac{\alpha}{2}\varepsilon+\frac{\gamma}{2}\varepsilon<\varepsilon,$$

this implies that there exist $\varepsilon_1>0$ and two subsequences $gx_{n_{k_p}}$ and $gy_{n_{k_p}}$such that

$$\lim_{p\to\infty}\Theta(gx_{n_{k_p}},gy_{n_{k_p}},gx_{n_{k_p}-1},gy_{n_{k_p}-1})=\varepsilon_1<\varepsilon. \tag{2.43}$$

By the lower semicontinuity of ψ :

$$\psi(\varepsilon)\leq\liminf_{p\to\infty}\psi\left(\Theta(gx_{n_{k_p}},gy_{n_{k_p}},gx_{n_{k_p}-1},gy_{n_{k_p}-1})\right). \tag{2.44}$$

By the triangular inequality

$$\begin{array}{rcl}r_k&=&d\left(gx_{n(k)},gx_{m(k)}\right)+d\left(gx_{n(k)},gx_{m(k)}\right)\\ &\leq&d\left(gx_{n(k)},gx_{n(k)+1}\right)+d\left(gx_{n(k)+1},gx_{m(k)+1}\right)\\ &&+d(gx_{m(k)},,gx_{m(k)+1})+d\left(gy_{n(k)},gy_{n(k)+1}\right)\\ &&+d\left(gy_{n(k)+1},gy_{m(k)+1}\right)+d(gy_{m(k)},,gy_{m(k)+1})\\ &=&\delta_{n_k}+\delta_{m_k}+d\left(gx_{n(k)+1},gx_{m(k)+1}\right)\\ &&+d\left(gy_{n(k)+1},gy_{m(k)+1}\right).\end{array}$$

Using the property of φ, we have

$$\left.\begin{aligned}\varphi(r_k) &= \varphi\left[\delta_{n_k}+\delta_{m_k}+d\left(gx_{n(k)+1},gx_{m(k)+1}\right)\right.\\ &\quad\left.+d\left(gy_{n(k)+1},gy_{m(k)+1}\right)\right]\\ &\leq \varphi\left(\delta_{n_k}+\delta_{m_k}\right)+\varphi\left(d\left(gx_{n(k)+1},gx_{m(k)+1}\right)\right)\\ &\quad+\varphi\left(d\left(gy_{n(k)+1},gy_{m(k)+1}\right)\right).\end{aligned}\right\} \tag{2.45}$$

Since $n(k) > m(k)$, hence $gx_{n(k)} \succeq gx_{m(k)}$ and $gy_{n(k)} \preceq gy_{m(k)}$, using (2.2) and (2.4), we obtain

$$\left.\begin{aligned}\varphi\left(d\left(gx_{n(k)+1},gx_{m(k)+1}\right)\right) &= \varphi\left(d\left(F(x_{n(k)},y_{n(k)}),F(x_{m(k)},y_{m(k)})\right)\right)\\ &\leq \tfrac{1}{2}\varphi\left(\theta\left(x_{n(k)},y_{n(k)},x_{m(k)},y_{m(k)}\right)\right)\\ &\quad-\psi\left(\theta\left(x_{n(k)},y_{n(k)},x_{m(k)},y_{m(k)}\right)\right).\end{aligned}\right\} \tag{2.46}$$

By the same way, we also have

$$\left.\begin{aligned}\varphi\left(d\left(gy_{n(k)+1},gy_{m(k)+1}\right)\right) &= \varphi\left(d\left(F(y_{n(k)},x_{n(k)}),F(y_{m(k)},x_{m(k)})\right)\right)\\ &\leq \tfrac{1}{2}\varphi\left(\theta\left(y_{n(k)},x_{n(k)},y_{m(k)},x_{m(k)}\right)\right)\\ &\quad-\psi\left(\theta\left(y_{n(k)},x_{n(k)},y_{m(k)},x_{m(k)}\right)\right).\end{aligned}\right\} \tag{2.47}$$

From (2.45), (2.46) and (2.47), we obtain

$$\left.\begin{aligned}\varphi\left(r_{k_p}\right) &\leq \varphi\left(\delta_{n_{k_p}}+\delta_{m_{k_p}}\right)\\ &\quad+\varphi\left(\theta\left(y_{n(k)},x_{n(k)},y_{m(k)},x_{m(k)}\right)\right)\\ &\quad-2\psi\left(\theta\left(y_{n(k)},x_{n(k)},y_{m(k)},x_{m(k)}\right)\right)\end{aligned}\right\} \tag{2.48}$$

then

$$\begin{aligned}\limsup_{p\to\infty}\varphi\left(r_{k_p}\right) &\leq \limsup_{p\to\infty}\varphi\left(\delta_{n_{k_p}}+\delta_{m_{k_p}}\right)\\ &\quad+\limsup_{p\to\infty}\varphi\left(\theta\left(y_{n(k)},x_{n(k)},y_{m(k)},x_{m(k)}\right)\right)\\ &\quad-\liminf_{p\to\infty}2\psi\left(\theta\left(y_{n(k)},x_{n(k)},y_{m(k)},x_{m(k)}\right)\right),\end{aligned}$$

hence

$$\varphi(\varepsilon) \leq \varphi(0)+\varphi(\varepsilon_1)-2\psi(\varepsilon_1) \leq \varphi(\varepsilon)-2\psi(\varepsilon_1).$$

This implies that $\psi(\varepsilon_1)=0$ and so $\varepsilon_1=0$, which is a contradiction. This shows that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences. Since X is a complete metric space, there exist $x, y \in X$ such that

$$\lim_{n\to\infty}F(x_n,y_n)=\lim_{n\to\infty}gx_n=x \tag{2.49}$$

and

$$\lim_{n\to\infty}F(y_n,x_n)=\lim_{n\to\infty}gy_n=y. \tag{2.50}$$

Since F and g are compatible mappings, we have

$$\lim_{n\to\infty} d\left(g(F(x_n,y_n)\right), F(gx_n,gy_n)) = 0 \tag{2.51}$$

$$\lim_{n\to\infty} d\left(g(F(y_n,x_n)\right), F(gy_n,gx_n)) = 0. \tag{2.52}$$

Now, we show that $gx = F(x,y)$ and $gy = F(y,x)$. Suppose that the assumption (a) holds. For all $n \geq 0$, we have

$$d(gx, F(gx_n,gy_n)) \leq d(gx, g\left(F(x_n,y_n)\right) + d\left(g\left(F(x_n,y_n)\right), F(gx_n,gy_n)\right).$$

Taking the limit as $n\to\infty$ and using (2.4), (2.49), (2.51) and by using the fact that F and g are continuous, we obtain $d(gx,F(x,y)) = 0$, hence $gx = F(x,y)$. With the same way, we obtain $d(gy,F(y,x)) = 0$, hence $gy = F(y,x)$. Combining the two results above, we get

$$gx = F(x,y) \text{ and } gy = F(y,x).$$

Finally, we suppose (b) holds. By (2.9) and from $(2.49), (2.50)$, we have $\{gx_n\}$ is a nondecreasing sequence, $gx_n \to x$ and $\{gy_n\}$ is a nondecreasing sequence, $gy_n \to y$ as $n\to\infty$. Hence, by assumption (b), we have for all $n\geq 0$:

$$gx_n \preceq x \text{ and } gy_n \succeq y. \tag{2.53}$$

Since F and g are compatible mappings and g is continuous, by (2.51) and (2.52), we have

$$\lim_{n\to\infty} g(gx_n) = gx = \lim_{n\to\infty} g(F(x_n,y_n)) = \lim_{n\to\infty} F(gx_n,gy_n) \tag{2.54}$$

and

$$\lim_{n\to\infty} g(gy_n) = gy = \lim_{n\to\infty} g(F(y_n,x_n)) = \lim_{n\to\infty} F(gy_n,gx_n). \tag{2.55}$$

Now, by triangular inequality, we have

$$d\left(gx, F(x,y)\right) \leq d\left(gx, g(gx_{n+1})\right) + d\left(g(gx_{n+1}), F(x,y)\right),$$

taking $n\to\infty$ in the above inequality, using (2.4) and (2.48) we have

$$\left.\begin{aligned} d\left(gx, F(x,y)\right) &\leq \lim_{n\to\infty} d\left(gx, g(gx_{n+1})\right) + \lim_{n\to\infty} d\left(g(gx_{n+1}), F(x,y)\right) \\ &\leq \lim_{n\to\infty} d\left(F(gx_n,gy_n), F(x,y)\right) \end{aligned}\right\} \tag{2.56}$$

Using the property of φ, we get

$$\varphi\left(d\left(gx, F(x,y)\right)\right) \leq \lim_{n\to\infty} \varphi\left(d\left(F(gx_n,gy_n), F(x,y)\right)\right).$$

Since the mapping g is monotone and increasing, using $(2.2), (2.53)$ and (2.56), we have, for all $n\geq 0$:

$$\begin{aligned} \varphi\left(d\left(gx, F(x,y)\right)\right) \quad \leq \quad & \limsup_{n\to\infty} \frac{1}{2}\varphi\left(\Theta(g\left(gx_n\right), g\left(gy_n\right), gx, gy)\right) \\ & -\liminf_{n\to\infty} \psi\left(\Theta(g\left(gx_n\right), g\left(gy_n\right), gx, gy)\right), \end{aligned}$$

$$\limsup_{n\to\infty}\frac{1}{2}\varphi\left(\Theta(g\left(gx_n\right),g\left(gy_n\right),gx,gy)\right)=\frac{1}{2}\varphi\left(\Theta(gx,gy,gx,gy)\right)$$

and

$$\psi\left(\Theta(gx,gy,gx,gy)\right)\leq\liminf_{n\to\infty}\psi\left(\Theta(g\left(gx_n\right),g\left(gy_n\right),gx,gy)\right),$$

then

$$\left.\begin{array}{rcl}\varphi\left(d\left(gx,F(x,y)\right)\right) & \leq & \frac{1}{2}\varphi\left(\Theta(gx,gy,gx,gy)\right)\\ & & -\psi\left(\Theta(gx,gy,gx,gy)\right)\end{array}\right\}\tag{2.57}$$

With the same way and by using (2.16), we prove that

$$\left.\begin{array}{rcl}\varphi\left(d\left(gy,F(y,x)\right)\right) & \leq & \frac{1}{2}\varphi\left(\Theta(gx,gy,gx,gy)\right)\\ & & -\psi\left(\Theta(gx,gy,gx,gy)\right)\end{array}\right\}\tag{2.58}$$

$$\left.\begin{array}{rcl}\Theta(gx,gy,gx,gy) & = & (\beta+\gamma)\left(d\left(gx,F(x,y)\right)+d\left(gy,F(y,x)\right)\right)\\ & \leq & d\left(gx,F(x,y)\right)+d\left(gy,F(y,x)\right)\end{array}\right\}\tag{2.59}$$

and φ is nondecreasing, then from (2.57) , (2.58),(2.59) and the property of φ, we get

$$\begin{array}{rcl}\varphi\left(d\left(gx,F(x,y)\right)+d\left(gy,F(y,x)\right)\right) & \leq & \varphi\left(d\left(gx,F(x,y)\right)+d\left(gy,F(y,x)\right)\right)\\ & & -2\psi\left(\Theta(gx,gy,gx,gy)\right)\end{array}$$

Then

$$\Theta(gx,gy,gx,gy)=0=(\beta+\gamma)\left(d\left(gx,F(x,y)\right)+d\left(gy,F(y,x)\right)\right).$$

This implies that

$$d\left(gx,F(x,y)\right)=0 \text{ and } d\left(gy,F(y,x)\right)=0.$$

Hence $gx=F(x,y)$ and $gy=F(y,x)$. Thus we proved that F and g have a coupled coincidence point. □

3 Uniqueness of Coupled Coincidence Point.

In this section, we will prove the uniqueness of the coupled coincidence point. Note that if $(X,\preceq)$ is a partially ordered set, then we endow the product $X\times X$, with the following partial order relation, for all (x,y), $(u,v)\in X\times X$:

$$(x,y)\preceq(u,v) \text{ if, and only if } x\preceq u \text{ and } y\succeq v.$$

Theorem 3.1. *In addition to hypotheses of theorem 2.1, Suppose that for every (x,y), (z,t) in $X\times X$, there exist a element (u,v) in $X\times X$ that is comparable to (x,y) and (z,t), then F has a coupled coincidence point.*

Proof. Suppose (x,y) and (z,t) are coupled coincidence points of F and g, that is $gx=F(x,y), gy=F(y,x), gz=F(z,t)$ and $gt=F(t,z)$. We are going to show that $gx=gz$ and $gy=gt$. By assumption, there exists $(u,v)\in X\times X$, comparable to (x,y) and (z,t). We define sequences $\{gx_n\}$ and $\{gy_n\}$ as follows:

$$u_0=u, v_0=v \quad gu_{n+1}=F(u_n,v_n) \text{ and } gv_{n+1}=F(v_n,u_n) \quad \text{for all } n\geq 1.$$

Since (u,v) is comparable with (x,y), we may assume that

$$(x,y)\succeq(u,v)=(u_0,v_0).$$

Using the mathematical induction, it is easy to prove that

$$(x,y)\succeq(u_n,v_n) \text{ for all } n\geq 1. \tag{3.1}$$

Using (2.2) and (3.1), we have

$$\left.\begin{array}{rcl}\varphi\left(d\left(gx,gu_{n+1}\right)\right) &=& \varphi\left(d\left(F(x,y),F\left(u_n,v_n\right)\right)\right)\\ &\leq& \frac{1}{2}\varphi\left(\theta\left(gx,gy,gu_n,gv_n\right)\right)\\ && -\psi\left(\theta\left(gx,gy,gu_n,gv_n\right)\right)\end{array}\right\} \tag{3.2}$$

We have

$$\theta\left(gx,gy,gu_n,gv_n\right)=\theta\left(gv_n,gu_n,gy,gx\right). \tag{3.3}$$

Indeed

$$\begin{array}{rcl}\theta\left(gx,gy,gu_n,gv_n\right) &=& \dfrac{\alpha}{2}\left[d(gx,gu_n)+d(gy,gv_n)\right]\\ && +\dfrac{\beta}{2}\left[d(gu_n,gu_{n+1})+d(gv_n,gv_{n+1})\right]\\ && +\dfrac{\gamma}{2}\left[d(gx,gu_{n+1})+d(gy,gv_{n+1})\right.\\ && \left.+d(gx,gu_n)+d(gy,gv_n)\right]\end{array}$$

and

$$\begin{array}{rcl}\theta\left(gv_n,gu_n,gy,gx\right) &=& \dfrac{\alpha}{2}\left[d(gu_n,gx)+d(gv_n,gy)\right]\\ && +\dfrac{\beta}{2}\left[d(gv_n,gv_{n+1})+d(gu_n,gu_{n+1})\right]\\ && +\dfrac{\gamma}{2}\left[d(gv_n,gy)+d(gu_n,gx)\right.\\ && \left.+d(gy,gv_{n+1})+d(gx,gu_{n+1})\right],\end{array}$$

then

$$\left.\begin{array}{rcl}\varphi\left(d\left(gv_{n+1},gy\right)\right) &=& \varphi\left(d\left(F(v_n,u_n),F\left(y,x\right)\right)\right)\\ &\leq& \frac{1}{2}\varphi\left(\theta\left(gx,gy,gu_n,gv_n\right)\right)\\ && -\psi\left(\theta\left(gx,gy,gu_n,gv_n\right)\right)\end{array}\right\} \tag{3.4}$$

Using (3.2), (3.4) and the property of φ, we get

$$\begin{array}{rcl}\varphi(d\left(gx,gu_{n+1}\right)+d\left(gv_{n+1},gy\right)) &\leq& \varphi(d\left(gx,gu_{n+1}\right))+\varphi(d\left(gv_{n+1},gy\right))\\ &\leq& \varphi\left(\theta\left(gx,gy,gu_n,gv_n\right)\right)\\ && -2\psi\left(\theta\left(gx,gy,gu_n,gv_n\right)\right).\end{array}$$

Since ψ is nonnegative, we obtain

$$\varphi(d\left(gx, gu_{n+1}\right) + d\left(gv_{n+1}, gy\right)) \leq \varphi\left(\theta\left(gx, gy, gu_n, gv_n\right)\right).$$

Thus, since φ is nondecreasing, it follows

$$d\left(gx, gu_{n+1}\right) + d\left(gv_{n+1}, gy\right) \leq \theta\left(gx, gu_n, gy, gv_n\right),$$

hence

$$\left.\begin{aligned} d\left(gx, gu_{n+1}\right) + d\left(gv_{n+1}, gy\right) \leq\ & \frac{\alpha}{2}\left[d(gx, gu_n) + d(gy, gv_n)\right] \\ & + \frac{\beta}{2}\left[d(gu_n, gu_{n+1}) + d(gv_n, gv_{n+1})\right] \\ & + \frac{\gamma}{2}\left[d(gx, gu_{n+1}) + d(gy, gv_{n+1})\right. \\ & \left. + d(gx, gu_n) + d(gy, gv_n)\right] \end{aligned}\right\} \tag{3.5}$$

By using the triangular inequality

$$d(gu_n, gu_{n+1}) \leq d(gu_n, gx) + d(gx, gu_{n+1}),$$

$$d(gv_n, gv_{n+1}) \leq d(gv_n, gy) + d(gy, gv_{n+1}),$$

we get from (3.5)

$$d\left(gx, gu_{n+1}\right) + d\left(gv_{n+1}, gy\right) \leq \left(\frac{\alpha + \beta + \gamma}{1 - \beta - \gamma}\right)\left[d(gx, gu_n) + d(gy, gv_n)\right] \tag{3.6}$$

By iteration, we get

$$d\left(gx, gu_{n+1}\right) + d\left(gv_{n+1}, gy\right) \leq \left(\frac{\alpha + \beta + \gamma}{1 - \beta - \gamma}\right)^{n+1}\left[d(gx, gu_0) + d(gy, gv_0)\right] \tag{3.7}$$

since $\frac{\alpha + \beta + \gamma}{1 - \beta - \gamma} < 1$, (3.7) implies that

$$\lim_{n\to\infty}\left[d\left(gx, gu_{n+1}\right) + d\left(gv_{n+1}, gy\right)\right] = 0 \tag{3.8}$$

Similarly, we show that

$$\lim_{n\to\infty}\left[d\left(gz, gu_{n+1}\right) + d\left(gv_{n+1}, gt\right)\right] = 0. \tag{3.9}$$

Using (3.8) and (3.9), we have $gx = gz$ and $gy = gt$. □

4 Application.

Set $Y = \{\psi : \mathbb{R}^+ \to \mathbb{R}^+, \psi$ is a Lebesgue integrable which is nonnegative, satisfies $\int_0^{\varepsilon} \psi(t)\mathrm{dt} > 0$, for each $\varepsilon > 0$ and subadditive, that is: $\int_0^{\varepsilon+\mu} \psi(t)\mathrm{dt} \leq \int_0^{\varepsilon} \psi(t)\mathrm{dt} + \int_0^{\mu} \psi(t)\mathrm{dt}$ for all $\varepsilon > 0$ and $\mu > 0\}$.

Example 4.1. *We consider* $\psi(x) = \dfrac{1}{1+x}, \psi : \mathbb{R}^+ \to \mathbb{R}^+, \psi$ *is a Lebesgue integrable which is nonnegative, satisfies* $\int_0^{\varepsilon} \psi(t)\mathrm{dt} = \ln(1+\varepsilon) > 0$*, for each* $\varepsilon > 0$*.* ψ *is subadditive, indeed: since* $1+\varepsilon+\mu \leq 1+\varepsilon+\mu+\epsilon\mu$*, then*

$$\begin{array}{rcl} \int_0^{\varepsilon+\mu} \psi(t)\mathrm{dt} = \ln(1+\varepsilon+\mu) & \leq & \ln(1+\varepsilon)(1+\mu) \\ & = & \ln(1+\varepsilon) + \ln(1+\mu) \\ & = & \int_0^{\varepsilon} \psi(t)\mathrm{dt} + \int_0^{\mu} \psi(t)\mathrm{dt}. \end{array}$$

This shows that ψ *is an example of subadditive, non-negative, Lebesgue integrable function.*

Theorem 4.2. *Let* $(X, \preceq)$ *be a partially ordered set and suppose there exists a metric* d *on* X *such that* (X, d) *is a complete metric space. Let* $F : X \times X \to X$ *be a continuous mapping having the mixed monotone property on* X *such that there exist two elements* $x_0, y_0 \in X$ *with*

$$gx_0 \preceq F(x_0, y_0) \quad \text{and} \quad gy_0 \succeq F(y_0, x_0).$$

Suppose there exist $\varphi \in \Phi$ *and* $\psi \in \Psi_1$ *such that*

$$\int_0^{\varphi(d(F(x,y),F(u,v)))} \chi(t)\mathrm{dt} \leq \int_0^{\frac{1}{2}\varphi(\Theta(gx,gy,gu,gv))} \chi(t)\mathrm{dt} - \int_0^{\psi(\Theta(gx,gy,gu,gv))} \chi(t)\mathrm{dt}$$

for all $gx \succeq gu$ *and* $gy \preceq gv$*.*

$$\begin{array}{rcl} \Theta(gx, gy, gu, gv) & = & \frac{\alpha}{2}\left[d(gx, gu) + d(gy, gv)\right] \\ & & +\frac{\beta}{2}\left[d(gx, F(x, y)) + d(gy, F(y, x))\right. \\ & & \left.+d(gu, F(u, v)) + d(gv, F(v, u))\right] \\ & & +\frac{\gamma}{2}\left[d(gx, F(u, v)) + d(gy, F(v, u))\right. \\ & & \left.+d(gu, F(x, y)) + d(gv, F(y, x))\right] \end{array}$$

where $\alpha > 0$ *and* $\beta, \gamma \geq 0$ *such that* $\alpha + 2\beta + 2\gamma < 2$*. Suppose* $F(X \times X) \subset g(X)$*,* g *continuous and compatible with* F *also suppose either*

(a) F *is continuous or*

(b) X *has the following property*

  (i) *If a nondecreasing sequence* $\{x_n\} \to x$*, then* $x_n \preceq x$*, for all* $n \geq 1$*,*

  (ii) *If a non-increasing sequence* $\{y_n\} \to y$*. then* $y_n \succeq y$*, for all* $n > 1$*,*

then there exist $x, y \in X$ *such that* $gx = F(x, y)$ *and* $gy = F(y, x)$*. That is* F *and* g *have a coupled coincidence point in* X*.*

Proof. Define $\Lambda:\mathbb{R}^+\to\mathbb{R}^+$ by $\Lambda(\varepsilon)=\int_0^\varepsilon\psi(t)\mathrm{d}t$, where $\psi\in Y$. Thus Λ is continuous and nondecreasing with $\Lambda(0)=0$. Then 3.1 becomes

$$\begin{aligned}\Lambda(\varphi\left(d\left(F(x,y),F(u,v)\right)\right) &\leq \frac{1}{2}\Lambda(\varphi\left(\Theta(gx,gy,gu,gv)\right)\\ &\quad -\Lambda(\psi\left(\Theta(gx,gy,gu,gv)\right)\end{aligned}$$

which further can be written as

$$\varphi_1\left(d\left(F(x,y),F(u,v)\right)\right)\leq\frac{1}{2}\varphi_1\left(\Theta(gx,gy,gu,gv)\right)-\psi_1\left(\Theta(gx,gy,gu,gv)\right)$$

where $\varphi_1=\Lambda\circ\varphi$ and $\psi_1=\Lambda\circ\psi$, hence by Theorem 2.1, we have the results. □

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