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A Common Fixed Point Theorem for Strongly Tangential and Weakly Compatible Mappings Satisfying Implicit Relations

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Abstract : The aim of this paper is to prove a common fixed point theorem for multivalued mappings satisfying an implicit relation and the strongly tangential property, our results improve those of Sedghi *et al.* [1].

Keywords : common fixed point; weakly compatible; strongly tangential property; implicit relation.
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1 Introduction

Let (X, d) be a metric space, $B(X)$ is the set of all non-empty bounded subsets of X. For all $A, B \in B(X)$ we define the two functions:$D, \delta : B(X) \times B(X) \to \mathbb{R}_+$ such that

$$D(A, B) = \inf\{d(a, b); a \in A, b \in B\},$$

$$\delta(A, B) = \sup\{d(a, b); a \in A, b \in B\}.$$

If A consists of a single point a, we write $\delta(A, B) = \delta(a, B)$ and $D(A, B) = D(a, B)$, also if $B = \{b\}$ is a singleton we write

$$\delta(A, B) = D(A, B) = d(a, b).$$

Clearly that δ satisfies the following properties:

$$\delta(A,B)=\delta(B,A)\geq 0,$$

$$\delta(A,B)\leq \delta(A,C)+\delta(C,B),$$

$$\delta(A,A)=\text{diam}A,$$

$$\delta(A,B)=0 \text{ implies } A=B=\{a\},$$

for all $A,B,C\in B(X)$,where diam$A$ is the diameter of the set A .
Notice that for all $a\in A$ and $b\in B$ we have

$$D(A,B)\leq d(a,b)\leq \delta(A,B),$$

where $A,B\in B(X)$.
Sessa [2] defined two mappings $S:X\to B(X)$ and $f:X\to X$ are to be weakly commuting on X if $fSx\in B(X)$ and for all $x\in X$:

$$\delta(Sfx,fSx)\leq \max\{\delta(fx,Sx),diam(fSx)\}.$$

2 Preliminaries

Liu and Li-Shan[3] introduced the following definition:

Definition 2.1. Two mappings $f:X\to X$ and $S:X\to B(X)$ on metric space (X,d) are said to be $\delta-$*compatible* if

$$\lim_{n\to\infty}\delta(Sfx_n,fSx_n)=0,$$

whenever $\{x_n\}$ is a sequence in X such that $fSx_n\in B(X)$, $\lim_{n\to\infty}Sx_n=\{z\}$, and $\lim_{n\to\infty}fx_n=z$ for some $z\in X$.

Jungck and Rhoades [4] generalized the concept of $\delta-$compatible as follows:

Definition 2.2. Let $f:X\to X$ a single mapping of space (X,d) into itself and $S:X\to B(X)$, the pair $\{f,S\}$ is *weakly compatible* if they commute at their coincidence point, i.e if $fu\in Su$ for some $u\in X$, then $fSu=Sfu$.

Example 2.3. Let $X=[0,2]$ with the euclidian metric, we define two mappings f,S as follows:

$$fx=\begin{cases}2-x, & 0\leq x\leq 1\\ 0, & 1<x\leq 1\end{cases}\qquad Sx=\begin{cases}\{1\}, & 0\leq x\leq 1\\ [1,x], & 1<x\leq 1\end{cases}$$

1 is the unique coincidence point, we have $f(1)=1\in S(1)$ and $fS(1)=Sf(1)=\{1\}$, then f and S are weakly compatible.

Recently, Pathak and Shahzad [5] introduced the concept of tangential mappings as follows:
Let $f, g : X \to X$ two mappings,a point $z \in X$ is said to be a weak tangent point to (f, g) if there exist two sequences $\{x_n\}, \{y_n\}$ in X such

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = z,$$

for some $z \in X$.
In 2011, Sintunavarat and Kumam [6] extended the last notion for single and multi valued maps:

Definition 2.4. Let $f, g : X \to X$ be single mappings and $S, T : X \to B(X)$ two multi-valued mappings on metric space (X, d), the pair $\{f, g\}$ is said to be *tangential with respect to* $\{S, T\}$ if there exists two sequences $\{x_n\}, \{y_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = A,$$

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = z \in A.$$

In their recent paper[7], Chauhan, Imdad, Karapinar and Fisher modified the last definition by adding another condition to introduce a new notion:

Definition 2.5. Let $f, g : X \to X$ be single mappings and $S, T : X \to B(X)$ two multi-valued mappings on metric space (X, d),we said $\{f, g\}$ is *strongly tangential with respect to* $\{S, T\}$ if there exists two sequences $\{x_n\}, \{y_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = A, \qquad \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = z \in A$$

and $z \in f(X) \cap g(X)$.

Example 2.6. Let $([0, \infty), |.|)$, we define f, g, S and T by:

$$fx = x + 2, \quad gx = 2x,$$

$$Sx = [x, x + 2] \quad Tx = [x, 3x],$$

we have $f(X) \cap g(X) = [2, \infty)$.
We consider two sequences $\{x_n\}, \{y_n\}$ which defined for all $n \geq 1$ by:

$$x_n = \frac{1}{n}, \quad y_n = 1 + \frac{1}{n}.$$

Clearly that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} T(y_n = [0, 3]$ and $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} gy_n = 2 \in [0, 3]$ also $2 \in [2, \infty) = f(X) \cap g(X)$, then $\{f, g\}$ is strongly tangential with respect to $\{S, T\}$.

For $S = T$ and $f = g$ the last definition becomes:

Definition 2.7. Let $f : X \to X$ and $S : X \to B(X)$ two mappings on metric space (X, d), f is said to be *strongly tangential with respect to* S if

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Sy_n = A,$$

and $z \in f(X)$, whenever $\{x_n\}, \{y_n\}$ two sequences in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} fy_n = z \in A.$$

Example 2.8. Let $([0, 2], |.|)$, f and S defined by:

$$fx = \begin{cases} 0, & 0 \leq x < 1 \\ \frac{x+1}{2}, & 1 \leq x \leq 2 \end{cases} \qquad Sx = \begin{cases} \{\frac{1}{2}\}, & 0 \leq x < 1 \\ [1, x] & 1 \leq x \leq 2 \end{cases}$$

$$f(X) = \{0\} \cup [1, \frac{3}{2}].$$

Consider two sequence $\{x_n\}, \{y_n\}$ which defined by:

$$x_n = 1 + \frac{1}{n}, \quad \text{for all } n \geq 1,$$

$$x_n = 1 + e^{-n}, \quad \text{for all } n \geq 1,$$

we have:

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} fy_n = 1$$
$$\lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Sx_n = \{1\},$$

also $1 \in f(X)$ then f is strongly tangential to respect S.

Let $\mathcal{F}$ be the set of all continuous functions $F : \mathbb{R}^5_+ \to \mathbb{R}$ satisfying:

(F_1) : F is non decreasing in t_1 and non increasing in t_2, t_3, t_4, t_5.

(F_2) : For all $u > 0$ we have:

$$F(u, 0, 0, u, u) > 0, F(u, 0, u, 0, u) > 0, F(u, 0, u, 0, 2u) > 0.$$

The aim of this paper is to establish some common fixed point theorem for single and set valued mappings in metric spaces, which satisfying a contractive condition of integral type by using the strongly tangential property with weak compatibility.

3 Main Results

Theorem 3.1. *Let $f, g : X \to X$, be single valued mappings and $S, T : X \to B(X)$ multi-valued mappings of metric space (X, d) such for all x, y in X we have:*

$$F\Big(\int_0^{\delta(Sx,Ty)} \varphi(t)dt, \int_0^{d(fx,gy)} \varphi(t)dt, \int_0^{D(fx,Sx)} \varphi(t)dt,$$

$$\int_0^{D(gy,Ty)} \varphi(t)dt, \int_0^{D(fx,Ty)+D(gy,Sx)} \varphi(t)dt\Big) \leq 0, \quad (3.1)$$

where $F \in \mathcal{F}$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue-integrable function which is summable on each compact subset of $\mathbb{R}^+$, non-negative, and such that for each $\varepsilon > 0, \int_0^\varepsilon \varphi(t)dt > 0$, and satisfying for all $a > 0, b > 0$:

$$\int_0^{a+b} \varphi(t)dt \leq \int_0^{a} \varphi(t)dt + \int_0^{b} \varphi(t)dt. \quad (3.2)$$

Suppose that the pair $\{f, g\}$ is strongly tangential with respect to $\{S, T\}$ and the pairs $\{f, S\}, \{g, T\}$ are weakly compatible, then f, g, S and T have a unique common fixed point in X.

Proof. Suppose $\{f, g\}$ is strongly tangential with respect to $\{S, T\}$, then there exists two sequences $\{x_n\}, \{y_n\}$ such

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = M, \text{ and } \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = z \in M,$$

and $z \in f(X) \cap g(X)$, which implies there exists $u, v \in X$ such $z = fu = gv$.
Firstly we claim $z \in Su$, if not by using (3.1) we get

$$F\Big(\int_0^{\delta(Su,Ty_n)} \varphi(t)dt, \int_0^{d(fu,gy_n)} \varphi(t)dt, \int_0^{D(fu,Su)} \varphi(t)dt, \int_0^{D(gy_n,Ty_n)} \varphi(t)dt,$$
$$\int_0^{D(fu,Ty_n)+D(gy_n,Su)} \varphi(t)dt\Big) \leq 0,$$

letting $n \to \infty$, since $z \in M$, $D(M, Su) \leq \delta(Su, M)$ and from (F_1), we get

$$F\Big(\int_0^{\delta(Su,fu)} \varphi(t)dt, 0, \int_0^{\delta(fu,Su)} \varphi(t)dt, 0, \int_0^{\delta(fu,Su)} \varphi(t)dt\Big) \leq$$

$$F\Big(\int_0^{\delta(Su,M)} \varphi(t)dt, 0, \int_0^{D(fu,Su)} \varphi(t)dt, 0, \int_0^{D(fu,Su)} \varphi(t)dt\Big) \leq 0,$$

which contradicts (F_2), then $\delta(Su, fu) = 0$, which implies $Su = \{fu\}$ and u is a strict coincidence point for f and S.
Now we prove $z = gv \in Tv$, if not by using (3.1) we get

$$F\Big(\int_0^{\delta(Sx_n,Tv)} \varphi(t)dt, \int_0^{d(fx_n,gv)} \varphi(t)dt, \int_0^{D(fx_n,Sx_n)} \varphi(t)dt, \int_0^{D(gv,Tv)} \varphi(t)dt,$$
$$\int_0^{D(fx_n,Tv)+D(gv,Sx_n)} \varphi(t)dt\Big) \leq 0,$$

letting $n \to \infty$, since $z \in M$, $D(M, Tv) \leq \delta(Tv, gv)$ and from (F_1), we get

$$F\Big(\int_0^{\delta(gv,Tv)} \varphi(t)dt, 0, 0, \int_0^{\delta(gv,Tv)} \varphi(t)dt, 0, \int_0^{\delta(gv,Tv)} \varphi(t)dt\Big) \leq$$

$$F\Big(\int_0^{\delta(Su,M)} \varphi(t)dt, 0, 0, \int_0^{D(gv,M)} \varphi(t)dt, 0, \int_0^{D(gv,M)} \varphi(t)dt\Big) \leq 0,$$

which is a contradiction with (F_2), and so $\delta(gv, Tv) = 0$, then $Tv = \{gv\}$.
The pair $\{f, S\}$ is weakly compatible and so $fSu = Sfu$, then $Sz = \{fz\}$, as well as $\{g, T\}$ since they are weakly compatible and the point v is a strict coincidence point, we obtain $Tz = \{gz\}$.
Nextly we prove $z = fz$, if not and using (3.1) we get:

$$F\Big(\int_0^{\delta(Sz,Tv)} \varphi(t)dt, \int_0^{d(fz,gv)} \varphi(t)dt, \int_0^{D(fz,Sz)} \varphi(t)dt,$$
$$\int_0^{D(gv,Tv)} \varphi(t)dt, \int_0^{D(fz,Tv)+D(gv,Sz)} \varphi(t)dt\Big) \leq 0,$$

by using (3.2) we get:

$$F\Big(\int_0^{d(fz,z)} \varphi(t)dt, \int_0^{d(fz,z)} \varphi(t)dt, 0, 0, 2\int_0^{d(fz,z)} \varphi(t)dt\Big) \leq$$
$$F\Big(\int_0^{\delta(Sz,Tv)} \varphi(t)dt, \int_0^{d(fz,gv)} \varphi(t)dt, \int_0^{D(fz,Sz)} \varphi(t)dt,$$
$$\int_0^{D(gv,Tv)} \varphi(t)dt, \int_0^{D(fz,Tv)+D(gv,Sz)} \varphi(t)dt\Big) \leq 0,$$

which is a contradiction, then $Sz = \{fz\} = \{z\}$.
Similarly we claim $z = gz$, if not by using (3.1) we get:

$$F\Big(\int_0^{\delta(Sz,Tz)} \varphi(t)dt, \int_0^{d(fz,gz)} \varphi(t)dt, \int_0^{D(fz,Sz)} \varphi(t)dt,$$
$$\int_0^{D(gz,Tz)} \varphi(t)dt, \int_0^{D(fz,Tz)+D(gz,Sz)} \varphi(t)dt\Big) \leq 0,$$

by using (3.1), we get:

$$F\Big(\int_0^{d(gz,z)} \varphi(t)dt, \int_0^{d(gz,z)} \varphi(t)dt, 0, 0, 2\int_0^{d(gz,z)} \varphi(t)dt\Big) \leq$$
$$F\Big(\int_0^{\delta(Sz,Tv)} \varphi(t)dt, \int_0^{d(fz,gv)} \varphi(t)dt, \int_0^{D(fz,Sz)} \varphi(t)dt,$$

$$\int_0^{D(gv,Tv)}\varphi(t)dt, \int_0^{D(fz,Tv)+D(gv,Sz)}\varphi(t)dt\Big) \leq 0,$$

which is a contradiction, then $Tz = \{gz\} = \{z\}$ and z is a common fixed point for f, g, S and T, it is strict for S and T .
For the uniqueness, suppose there is another fixed point w and using (3.1) we get:

$$F\Big(\int_0^{d(z,w)}\varphi(t)dt, \int_0^{d(z,w)}\varphi(t)dt, 0, 0, 2\int_0^{D(z,w)}\varphi(t)dt\Big) \leq,$$

$$F\Big(\int_0^{\delta(Sz,Tw)}\varphi(t)dt, \int_0^{d(fz,gw)}\varphi(t)dt, \int_0^{D(fz,Sz)}\varphi(t)dt,$$

$$\int_0^{D(gw,Tw)}\varphi(t)dt, \int_0^{D(fz,Tw)+D(gw,Sz)}\varphi(t)dt\Big) \leq 0,$$

which contradicts (F_2), then z is unique. □

Remark 3.2. *Theorem 3.1 improves and generalizes Theorem 1 of Sedghi, Altun and Shobe in their paper [1] and Theorem 2 in paper [8] to the setting single and multivalued mappings.*

If $f = g$ and $S = T$ we obtain the following corollary:

Corollary 3.3.

$$F\Big(\int_0^{\delta(Sx,Sy)}\varphi(t)dt, \int_0^{d(fx,fy)}\varphi(t)dt, \int_0^{D(fx,Sx)}\varphi(t)dt,$$

$$\int_0^{D(fy,Sy)}\varphi(t)dt, \int_0^{D(fx,Sy)+D(fy,Sx)}\varphi(t)dt\Big) \leq 0,$$

where $F \in \mathcal{F}$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue-integrable function which is summable on each compact subset of $\mathbb{R}^+$, non-negative, and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon}\varphi(t)dt > 0$, and satisfying (3.2).
Suppose that f is strongly tangential with respect to S and the pair $\{f, S\}$ is weakly compatible, then f and S have a unique common fixed point in X.

Corollary 3.4. *Let $f, g : X \to X$, and $S, T : X \to B(X)$ be single and set valued mappings of metric space (X, d) such:*

$$\int_0^{\delta(Sx,Ty)}\varphi(t)dt \leq \alpha\int_0^{d(fx,fy)}\varphi(t)dt + \beta\int_0^{D(fx,Sx)}\varphi(t)dt + \gamma\int_0^{D(gy,Ty)}\varphi(t)d)$$

where α, β, γ are non negative real numbers satisfying $2\alpha+2\beta+\delta < 1, \alpha+\beta-\gamma \geq 0$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue-integrable function which is summable on each compact subset of $\mathbb{R}^+$, non-negative, satisfying (3.2) and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon}\varphi(t)dt > 0$, if $\{f, g\}$ is strongly tangential with respect to $\{S, T\}$, and the two pairs $\{f, S\}$ and $\{g, T\}$ are weakly compatible, then f, g, S and T have a unique common fixed point.

Proof. It is clear that the function:$F:\mathbb{R}_+^5\to\mathbb{R}$ which defined by

$$F(t_1,t_2,t_3,t_4,t_5)=t_1-(\alpha t_2+\beta t_3+\gamma t_4)$$

where $\alpha,\beta,\gamma\geq 0, 2\alpha+2\beta+\delta<1$ and $\alpha+\beta-\gamma\geq 0$, satisfies (F_1) and (F_2), so $F\in\mathcal{F}$. □

Corollary 3.5. *Let $f,g:X\to X$, and $S,T:X\to B(X)$ be single and set valued mappings of metric space (X,d) such:*

$$\int_0^{\delta(Sx,Ty)}\varphi(t)dt\leq\alpha\max(\int_0^{d(fx,fy)}\varphi(t)dt,\int_0^{D(fx,Sx)}\varphi(t)dt,\int_0^{D(gy,Sy)}\varphi(t)d)$$

$$+\beta\int_0^{D(fx,Ty)+D(gy,Sx)}\varphi(t)dt,$$

where $\alpha+2\beta<1$ and $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ is a Lebesgue-integrable function which is summable on each compact subset of $\mathbb{R}^+$, non-negative, satisfying (3.2) such that for each $\varepsilon>0$, $\int_0^\varepsilon\varphi(t)dt>0$, if $\{f,g\}$ tangential with respect to $\{S,T\}$, and the two pairs $\{f,S\}$ and $\{g,T\}$ are weakly compatible, then f,g,S and T have a unique common fixed point.

Proof. It suffices to show that the function $F:\mathbb{R}_+^5\to\mathbb{R}$ such

$$F(t_1,t_2,t_3,t_4,t_5)=t_1-\alpha\max(t_2,t_3,t_4)-\beta t_5,$$

where $\alpha+2\beta<1$ satisfies (F_1) and (F_2).
(F_1): Obviously.
(F_2): For all $u>0$, we have $F(u,0,0,u,u)=F(u,0,u,0,u)=(1-\alpha-\beta)u>0$, and $F(u,0,u,0,2u)=(1-\alpha-2\beta)u>0$, consequently $F\in\mathcal{F}$. □

Corollary 3.5 generalizes Corollary 2 in paper [1].

Example 3.6. Let $X=[0,4]$, $d(x,y)=|x-y|$ and f,g,S and T defined by

$$fx=\begin{cases}\frac{x+2}{2}, & 0\leq x\leq 2\\ 1, & 1<x\leq 2\end{cases}\qquad gx=\begin{cases}4-x, & 0\leq x\leq 2\\ 0, & 2<x\leq 4\end{cases}$$

$$Sx=\begin{cases}\{2\}, & 0\leq x\leq 2\\ (2,\frac{9}{4}], & 1<x\leq 4\end{cases}\qquad Tx=\{2\}$$

Consider the two sequence: for all $n\geq 1, x_n=2-\frac{1}{n}, y_n=2-e^{-n}$, it is clear that

$$\lim_{n\to\infty}fx_n=\lim_{n\to\infty}gy_n=2,$$

$$\lim_{n\to\infty}Sx_n=\lim_{n\to\infty}Ty_n=\{2\},$$

which implies that the pair $\{f,g\}$ is strongly tangential with respect $\{S,T\}$ The point $u=2$ satisfy $f(2)=2\in S(2), g(2)=2\in T(1)$ and $fS(2)=\{2\}=Sf(2)$, and so f and S are weakly compatible, as well as g and T, because $gT(1) = \{1\}$.

We define some

$$F(t_1, t_2, t_3, t_4, t_5) = t_1 - \frac{2}{3}\max(t_2, t_3, t_4, \frac{1}{2}t_5)$$

For all $t \geq 0$, we define $\varphi(t) = 1$,
For $x, y \in [0, 1[$, we have:

$$\delta(Sx, Ty) = 0 \leq \frac{1}{3}|x - 1| = \frac{2}{3}D(fx, Sx).$$

For $x \in [0, 2)$ and $y \in (1, 2]$, we have:

$$\delta(Sx, Ty) = 0 \leq \frac{4}{3} = \frac{2}{3}D(gy, Ty).$$

For $x, y \in]1, 2]$, we have

$$\delta(Sx, Ty) = \frac{1}{4} \leq \frac{2}{3} = \frac{2}{3}d(fx, gy).$$

For $x \in]1, 2]$ and $y \in [0, 1[$ we have

$$\delta(Sx, Ty) = \frac{1}{4} \leq \frac{2}{3} = \frac{2}{3}D(fx, Sx),$$

then f, g, S and T satisfying (3.1).
Therefore $S(2) = T(2) = \{f(2)\} = \{g(2)\} = \{2\}$, then 2 is the unique fixed point of f, g, S and T and it is strict fixed point for S and T.

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