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Some Results on Vector-Valued $\mathcal{S}$-Type $|A, p, \prod X_k|$ Operators

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Abstract : This paper deals with the class $\mathscr{A}_{vec}^{(s)} - p$ of vector-valued s-type $|A, p, \prod X_k|$ operators, $0 < p \leq \infty$. It is shown that each component of the class $\mathscr{A}_{vec}^{(s)} - p$ forms a complete linear space. Some inclusion relations are also obtained.

Keywords : s-numbers; approximation numbers; sequence spaces.
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1 Introduction

There has been considerable interest to study s-numbers of operators as they are very powerful tools for estimating eigenvalues of operators in Banach spaces. In 1963, A. Pietsch [1] firstly introduced the approximation numbers of a bounded linear operator in Banach spaces. Subsequently, different s-numbers, namely Kolmogorov numbers, Gel'fand numbers are introduced to the Banach space setting. For the unification of different s-number sequences, A. Pietsch ([2], 1974) developed an axiomatic theory of s-numbers in Banach spaces.

For infinite matrix $A = (a_{nk})$, Rhoades [3] defined $A - p$ space, denoted by

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$|A,p|$ as

$$|A,p| = \begin{cases} x \in w & : \Big(\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty}|a_{nk}x_k|\Big)^p\Big)^{\frac{1}{p}} < \infty \quad \text{for } 0 < p < \infty \\ x \in w & : \sup_{n\geq 1}\Big(\sum_{k=1}^{\infty}|a_{nk}x_k|\Big) < \infty \qquad \text{for } p = \infty, \end{cases}$$

where w is a sequence space of real or complex numbers. Further, Rhoades [4] has shown that if $A = (a_{nk})$ is a triangle, i.e., $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$, then the space $|A,p|$ is separable for $1 < p < \infty$ and complete for $1 < p \leq \infty$. $A-p$ space contains many known sequence spaces as particular case by specifying suitable matrix $A = (a_{nk})$ such as Cesàro sequence space [5] for $1 < p < \infty$, l_p sequence space for $0 < p \leq \infty$, etc.

Let $\mathscr{L}(E,F)$ be the space of all bounded linear operators from a Banach space E to a Banach space F. Pietsch [1] defined an operator $T \in \mathscr{L}(E,F)$ as l^p type operator if $\sum\limits_{n=1}^{\infty}(a_n(T))^p$ is finite for $0 < p < \infty$, where $(a_n(T))$ is the sequence of approximation numbers of the bounded linear operator T. Later on Constantin [6] generalized the class of l_p type operators to the class of $ces-p$ type operators by using the Cesàro sequence space, where an operator $T \in \mathscr{L}(E,F)$ is called $ces-p$ type if $\sum\limits_{n=1}^{\infty}\Big(\frac{1}{n}\sum\limits_{k=1}^{n}a_n(T)\Big)^p$ is finite, $1 < p < \infty$. Rhoades [3] further generalized the class of $ces-p$ type operators to the class of $A-p$ type operators, where $A = (a_{nk})$ is an arbitrary infinite matrix. An operator $T \in \mathscr{L}(E,F)$ is said to be $A-p$ type operator if the sequence of approximation numbers $(a_n(T))$ belongs to $|A,p|$ space, $0 < p \leq \infty$. Let $A = (a_{nk})$ be a fixed matrix satisfying the condition:

$$|a_{n,2k-1}| + |a_{n,2k}| \leq M|a_{nk}| \qquad \text{for each } k \text{ and } n, \tag{1.1}$$

where M is a constant independent of n and k. Rhoades has shown that for $0 < p \leq \infty$ and for each fixed matrix A satisfying the condition (1.1), the set of $A-p$ type operators forms a linear space. Recently, authors [7] have studied some results in the scalar-valued case.

Motivated with the above works, we have studied the $A-p$ type operators in vector-valued case. In fact, this paper deals with the study of a generalized class of operators using the sequence of s-numbers in vector-valued case. We have also shown that each component of the class $\mathscr{A}^{(s)}_{vec}-p$ of vector-valued s-type $|A,p,\prod X_k|$ operators is a complete linear space under certain conditions on the matrix A. Some inclusion relations are also obtained for these spaces.

2 Preliminaries

Throughout this paper we denote E, F as the real or complex Banach spaces and $\mathscr{L}(E,F)$ as the space of all bounded linear operators from E to F. Let $\mathscr{L}$

be the class of all bounded linear operators between arbitrary Banach spaces. We denote $\mathbb{N}$ as the set of all natural numbers and $\mathbb{R}$ as the set of all real numbers.

We now state few results and definitions in scalar case which will be used in the sequel.

Definition 2.1 ([8])**.** A finite rank operator is a bounded linear operator whose dimension of the range space is finite.

Definition 2.2 ([8], [9])**.** A non-negative scalar sequence $(s_n(T))_n$, where $s = (s_n) : \mathscr{L} \to \mathbb{R}^{\mathbb{N}}$ assigning to every operator $T \in \mathscr{L}$, is called an *s-number sequence* if the following conditions are satisfied:

(S1) *monotonicity*: $\|T\| = s_1(T) \geq s_2(T) \geq \cdots \geq 0,\quad$ for $T \in \mathscr{L}(E, F)$

(S2) *additivity*: $s_{m+n-1}(S+T) \leq s_m(S)+s_n(T),\quad$ for $S, T \in \mathscr{L}(E, F)$, $m, n \in \mathbb{N}$

(S3) *ideal property*: $s_n(RST) \leq \|R\| s_n(S) \|T\|,\quad$ for some $R \in \mathscr{L}(F, F_0)$, $S \in \mathscr{L}(E, F)$ and $T \in \mathscr{L}(E_0, E)$, where E_0, F_0 are arbitrary Banach spaces

(S4) *rank property*: If $rank(T) \leq n$ then $s_n(T) = 0$

(S5) *norming property*: $s_n(I : l_2^n \to l_2^n) = 1$, where I denotes the identity operator on the n-dimensional Hilbert space l_2^n.

The n-th s-number of the operator T is denoted by $s_n(T)$. Various results on s-number sequence can be viewed in ([1], [10], [11], [12]). It can be easily shown that the following numbers are s-number sequence. Let $T \in \mathscr{L}(E, F)$ and $n \in \mathbb{N}$.

1. The *n-th approximation number*, denoted by $a_n(T)$, is defined as

$$a_n(T) = \inf \Big\{ \|T - L\| : \quad L \in \mathscr{L}(E, F),\ \text{rank(L)} < \text{n} \Big\}.$$

2. The *n-th Gel'fand number*, denoted by $c_n(T)$, is defined as

$$c_n(T) = \inf \Big\{ \|T J_M\| : \quad M \subset E,\ \text{codim(M)} < \text{n} \Big\},$$

where $J_M : M \to E$ be the natural embedding from subspace M of E into E.

3. The *n-th Kolmogorov number*, denoted by $d_n(T)$, is defined as

$$d_n(T) = \inf \Big\{ \|Q_N(T)\| : \quad N \subset F,\ \text{dim(N)} < \text{n} \Big\},$$

where $Q_N : E \to E/N$ be the quotient map from E onto E/N.

4. The *n-th Weyl number*, denoted by $x_n(T)$, is defined as

$$x_n(T) = \inf \Big\{ a_n(TA) : \|A : l_2 \to E\| \leq 1 \Big\},$$

where $a_n(TA)$ is an n-th approximation number of the operator TA.

5. The *n-th Chang number*, denoted by $y_n(T)$, is defined as

$$y_n(T) = \inf\Big\{a_n(BT) : \|B : F \to l_2\| \leq 1\Big\},$$

where $a_n(BT)$ is an n-th approximation number of the operator BT.

6. The *n-th Hilbert number*, denoted by $h_n(T)$, is defined as

$$h_n(T) = \sup\Big\{a_n(BTA) : \|B : F \to l_2\| \leq 1,\ \|A : l_2 \to E\| \leq 1\Big\}.$$

Remark 2.3 ([8])**.** *Among all the s-number sequences defined above, it is easy to verify that the approximation number, $a_n(T)$ is the largest and the Hilbert number, $h_n(T)$ is the smallest s-number sequence i.e., $h_n(T) \leq s_n(T) \leq a_n(T)$ for any bounded linear operator T. If T is defined on a Hilbert space then all the s-numbers coincide with the singular values of T i.e., the eigenvalues of $|T|$, where $|T| = (T^*T)^{\frac{1}{2}}$.*

Proposition 2.4 ([8], p.115)**.** *Let $T \in \mathscr{L}(E, F)$. Then*

$$h_n(T) \leq x_n(T) \leq c_n(T) \leq a_n(T) \quad \textit{and} \quad h_n(T) \leq y_n(T) \leq d_n(T) \leq a_n(T).$$

Definition 2.5 ([8], p.81)**.** An s-number sequence is called *multiplicative* if

$$s_{m+n-1}(ST) \leq s_m(S)s_n(T)$$

for $T \in \mathscr{L}(E, F)$, $S \in \mathscr{L}(F, F_0)$ and $m, n \in \mathbb{N}$.

Lemma 2.6 ([2])**.** *Let $S, T \in \mathscr{L}(E, F)$, then $|s_n(T) - s_n(S)| \leq \|T - S\|$ for $n = 1, 2, \cdots$.*

Lemma 2.7 ([8], p. 107)**.** *Let $s = (s_n)$ be any s-number sequence and $D_{(\tau_n)}$ be any diagonal operator from the sequence space l_2 to itself with $\tau_1 \geq \tau_2 \geq \ldots \geq 0$. Then $s_n(D_{(\tau_n)} : l_2 \to l_2) = \tau_n$ for all n.*

3 Vector-Valued $\mathcal{S}$-Type $|A, p, \prod X_k|$ Operators

Let $(E_k, \|.\|_{E_k})$ be a sequence of Banach spaces. It is easy to show that $\prod_{k=1}^{\infty} E_k$ is a Banach space with respect to the norm $\|.\|_p$ for $1 \leq p \leq \infty$, where

$$\|x\|_p = \Big(\sum_{k=1}^{\infty} \|x_k\|_{E_k}^p\Big)^{\frac{1}{p}} \qquad \text{for } \ 1 \leq p < \infty$$

$$\|x\|_\infty = \sup_{k\geq 1} \|x_k\|_{E_k} \qquad \text{for } \ p = \infty.$$

Throughout the paper we shall write $\prod E_k$ instead of $\prod_{k=1}^{\infty} E_k$.

Let $(F_k, \|.\|_{F_k})$ be another sequence of Banach spaces. A linear operator $T : \prod E_k \to \prod F_k$ is defined by

$$T(x) = T((x_1, x_2, \cdots, x_k, \cdots)) = (T_1x_1, T_2x_2, \cdots, T_kx_k, \cdots),$$

where $k \in \mathbb{N}$, $x = (x_k) \in \prod X_k$ and $T_k \in \mathscr{L}(E_k, F_k)$. It can be shown that T is a bounded linear operator if and only if $\sup_{k\geq 1} \|T_k\| < \infty$ and the norm $\|T\| = \sup_{k\geq 1} \|T_k\|$.

Let $0 < p \leq \infty$ and $(X_k, \|.\|_{X_k})$ be a sequence of Banach spaces. For a fixed matrix $A = (a_{nk})$, we define vector-valued $A - p$ space, denoted by $|A, p, \prod X_k|$ as

$$|A, p, \prod X_k| = \begin{cases} x = (x_k) \in \prod X_k & : \Big(\sum_{n=1}^{\infty} \Big(\sum_{k=1}^{\infty} |a_{nk}| \|x_k\|_{X_k}\Big)^p\Big)^{\frac{1}{p}} < \infty, \ 0 < p < \infty \\ x = (x_k) \in \prod X_k & : \sup_{n\geq 1} \Big(\sum_{k=1}^{\infty} |a_{nk}| \|x_k\|_{X_k}\Big) < \infty, \qquad p = \infty. \end{cases}$$

Particular examples:

There are many examples of vector-valued $A - p$ space with the particular choice of the matrix A, e.g.,

1. Choose A as an identity matrix and $1 \leq p < \infty$ and $X_k = X$, a Banach space for all k, then the space $|A, p, \prod X_k|$ reduces to $l_p(X)$ (see, [13], p. 33), where $l_p(X)$ is the set of all X-valued sequences $x = (x_k)$ such that $\Big(\sum_{n=1}^{\infty} \|x_n\|^p\Big)^{\frac{1}{p}} < \infty$.

2. Choose A as a Cesàro matrix of order 1 and $X_k = X$, a Banach space for all k. Then the space $|A, p, \prod X_k|$, $1 < p < \infty$ becomes X-valued Cesàor sequence space $Ces_p(X)$ (see [14]), where $Ces_p(X)$ is the set of all X-valued sequences $x = (x_k)$ such that $\Big(\sum_{n=1}^{\infty} \Big(\frac{1}{n} \sum_{k=1}^{n} \|x_n\|\Big)^p\Big)^{\frac{1}{p}} < \infty$.

If an operator $T \in \mathscr{L}(\prod E_k, \prod F_k)$ satisfying the conditions

$$\left.\begin{aligned} &\sup_{k\geq 1} \Big(\sum_{n=1}^{\infty} \Big(\sum_{l=1}^{\infty} |a_{nl}| s_l(T_k)\Big)^p\Big)^{\frac{1}{p}} < \infty, \quad 0 < p < \infty \\ &\sup_{k\geq 1} \sup_{n\geq 1} \Big(\sum_{l=1}^{\infty} |a_{nl}| s_l(T_k)\Big) < \infty, \qquad p = \infty, \end{aligned}\right\} \tag{3.1}$$

then we call T as a **vector-valued s-type $|A, p, \prod X_k|$ operator**. In particular if A is a Cesàro matrix of order 1, then (3.1) reduces to

$$\left.\begin{aligned} &\sup_{k\geq 1} \Big(\sum_{n=1}^{\infty} \Big(\frac{1}{n} \sum_{l=1}^{n} s_l(T_k)\Big)^p\Big)^{\frac{1}{p}} < \infty, \quad 0 < p < \infty \\ &\sup_{k\geq 1} \sup_{n\geq 1} \Big(\frac{1}{n} \sum_{l=1}^{n} s_l(T_k)\Big) < \infty, \qquad p = \infty. \end{aligned}\right\} \tag{3.2}$$

For $A = I$, an identity matrix, then (3.1) reduces to

$$\left.\begin{array}{ll} \sup\limits_{k\geq 1}\Big(\sum\limits_{n=1}^{\infty}\Big(s_n(T_k)\Big)^p\Big)^{\frac{1}{p}} < \infty, & 0 < p < \infty \\ \sup\limits_{k\geq 1}\sup\limits_{n\geq 1}\Big(s_n(T_k)\Big) < \infty, & p = \infty. \end{array}\right\} \tag{3.3}$$

We shall call an operator $T \in \mathscr{L}(\prod E_k, \prod F_k)$ as a vector-valued s-type ces_p operator and vector-valued s-type l_p operator if the conditions (3.2) and (3.3) hold respectively.

Particular examples:

Here we shall give some vector-valued s-type operators.

1. For $A = I$, an infinite identity matrix and choose a particular vector-valued $A-p$ space $l_p(l_p)$. Let $T : l_p(l_p) \to l_p(l_p)$, where $T = (T_k)_{k\geq 1}$ and $T_k : l_p \to l_p$ for all k such that for some $n_0 \in \mathbb{N}$ all T_k are finite rank operator for $1 \leq k \leq n_0$ and for $k > n_0$, T_k are zero operator, i.e., $T_k u = 0$ for all $u \in l_p$. Then by the property of s-number (see, Definition 2.2, $(S4)$), there exists some $n_1 \in \mathbb{N}$ such that $s_n(T_k) = 0$ for all $n > n_1$, $1 \leq k \leq n_0$ and $s_n(T_k) = 0$ for all $k > n_0$ and for all n. Thus $\sup\limits_{k\geq 1}\Big(\sum\limits_{n=1}^{\infty}(s_n(T_k))^p\Big)^{\frac{1}{p}} = \sup\limits_{1\leq k\leq n_0}\Big(\sum\limits_{n=1}^{n_1}(s_n(T_k))^p\Big)^{\frac{1}{p}} < \infty$. Hence $T = (T_k)_{k\geq 1}$ is a vector-valued s-type operators.
2. Consider $T : l_p(l_2) \to l_p(l_2)$ for $1 \leq p < \infty$, where $T = (T_k)_{k\geq 1}$ and each $T_k : l_2 \to l_2$ is a diagonal operator is defined as $T_k(y) = (y_1, \frac{1}{2}y_2, \frac{1}{3}y_3, \ldots)$ for $y = (y_n) \in l_p$ for all $k \geq 1$. Then each $T_k \in \mathscr{L}(l_2, l_2)$ and $\|T_k\| = 1$ for all k. Therefore $T \in \mathscr{L}(l_p(l_2), l_p(l_2))$. Also by using Lemma 2.7, we have $s_n(T_k : l_2 \to l_2) = \frac{1}{n}$ for all k. Thus $\sup\limits_{k\geq 1}\Big(\sum\limits_{n=1}^{\infty}(s_n(T_k))^p\Big)^{\frac{1}{p}} = \sup\limits_{k\geq 1}\Big(\sum\limits_{n=1}^{\infty}\frac{1}{n^p}\Big)^{\frac{1}{p}} < \infty$. Hence $T = (T_k)_{k\geq 1}$ is a vector-valued s-type operator.

We denote the set of all vector-valued s-type $|A, p, \prod X_k|$ operators between any two arbitrary countably infinite product of Banach spaces by $\mathscr{A}_{vec}^{(s)} - p$ and the set of all vector-valued s-type $|A, p, \prod X_k|$ operators from $\prod E_k$ to $\prod F_k$ by $\mathscr{A}^{(s)}_{(\prod E_k \to \prod F_k)} - p$ for $0 < p \leq \infty$. We say $\mathscr{A}^{(s)}_{(\prod E_k \to \prod F_k)} - p$ is a component of the class $\mathscr{A}_{vec}^{(s)} - p$. To study the class $\mathscr{A}_{vec}^{(s)} - p$, we will actually study each component of this class.

Proposition 3.1. *Let $A = (a_{nk})$ be an infinite matrix, where $a_{nk} = 0$ for $k > n$ and satisfies $\sum\limits_{k=1}^{n} |a_{nk}| \geq \lambda > 0$ for all n. If $T = (T_k)_{k\geq 1}$ is a vector-valued s-type $|A, p, \prod X_k|$ operator, then T is a vector-valued s-type l_p operator for $0 < p \leq \infty$.*

Proof. Let $0 < p < \infty$ and $T = (T_k)_{k\geq 1}$ be a vector-valued s-type $|A, p, \prod X_k|$

operator. Consider

$$\sum_{n=1}^{\infty}\Big(\sum_{l=1}^{n}|a_{nl}|s_l(T_k)\Big)^p \geq \sum_{n=1}^{\infty}\Big(s_n(T_k)\sum_{l=1}^{n}|a_{nl}|\Big)^p$$
$$\geq \lambda^p \sum_{n=1}^{\infty}\Big(s_n(T_k)\Big)^p,$$

which gives

$$\lambda\Big(\sup_{k\geq 1}\Big(\sum_{n=1}^{\infty}\Big(s_n(T_k)\Big)^p\Big)^{\frac{1}{p}}\Big) \leq \sup_{k\geq 1}\Big(\sum_{n=1}^{\infty}\Big(\sum_{l=1}^{n}|a_{nl}|s_l(T_k)\Big)^p\Big)^{\frac{1}{p}} < \infty.$$

Thus T is a vector-valued s-type l_p operator.
Similarly for $p = \infty$, it can be shown easily. Hence the proof is complete. □

Theorem 3.2. *Let $0 < p \leq \infty$. For fixed infinite matrix $A = (a_{nk})$ satisfying (1.1), each component of the class $\mathscr{A}_{vec}^{(s)} - p$ is a linear space.*

Proof. Let $0 < p < \infty$. Let $\mathscr{A}^{(s)}_{(\prod E_k \to \prod F_k)} - p$ be any one of the component of the class $\mathscr{A}_{vec}^{(s)} - p$. Let $S, T \in \mathscr{A}^{(s)}_{(\prod E_k \to \prod F_k)} - p$. Consider

$$\begin{aligned}\sum_{l=1}^{\infty}|a_{nl}s_l(T_k+S_k)| &= \sum_{l=1}^{\infty}|a_{n,2l-1}s_{2l-1}(T_k+S_k)| + \sum_{l=1}^{\infty}|a_{n,2l}s_{2l}(T_k+S_k)|\\ &\leq \sum_{l=1}^{\infty}(|a_{n,2l-1}|+|a_{n,2l}|)s_{2l-1}(T_k+S_k)\\ &\leq M\Big(\sum_{l=1}^{\infty}|a_{nl}|s_l(T_k)+\sum_{l=1}^{\infty}|a_{nl}|s_l(S_k)\Big). \qquad (3.4)\end{aligned}$$

Case I: $0 < p < 1$.
For $0 < p < 1$ and $a, b > 0$, $\Big(a+b\Big)^p \leq \Big(a^p + b^p\Big)$, $\Big(a+b\Big)^{\frac{1}{p}} \leq C\Big(a^{\frac{1}{p}} + b^{\frac{1}{p}}\Big)$, where $C \geq 1$ is a constant. So, from (3.4) we have

$$\begin{aligned}\Big(\sum_{n=1}^{\infty}\Big(\sum_{l=1}^{\infty}|a_{nl}|s_l(T_k+S_k)\Big)^p\Big)^{\frac{1}{p}} &\leq M\Big(\sum_{n=1}^{\infty}\Big(\sum_{l=1}^{\infty}|a_{nl}|s_l(T_k)+\sum_{l=1}^{\infty}|a_{nl}|s_l(S_k)\Big)^p\Big)^{\frac{1}{p}}\\ &\leq CM\Big[\Big(\sum_{n=1}^{\infty}\Big(\sum_{l=1}^{\infty}|a_{nl}|s_l(T_k)\Big)^p\Big)^{\frac{1}{p}}\\ &\quad+\Big(\sum_{n=1}^{\infty}\Big(\sum_{l=1}^{\infty}|a_{nl}|s_l(S_k)\Big)^p\Big)^{\frac{1}{p}}\Big],\end{aligned}$$

where $C \geq 1$ is a constant. Therefore,

$$\sup_{k\geq 1}\Big(\sum_{n=1}^{\infty}\Big(\sum_{l=1}^{\infty}|a_{nl}|s_l(T_k+S_k)\Big)^p\Big)^{\frac{1}{p}} \leq CM\Big[\sup_{k\geq 1}\Big(\sum_{n=1}^{\infty}\Big(\sum_{l=1}^{\infty}|a_{nl}|s_l(T_k)\Big)^p\Big)^{\frac{1}{p}} + \sup_{k\geq 1}\Big(\sum_{n=1}^{\infty}\Big(\sum_{l=1}^{\infty}|a_{nl}|s_l(S_k)\Big)^p\Big)^{\frac{1}{p}}\Big] < \infty.$$

Case II: $1 \leq p < \infty$.
Using Minkowski inequality for $1 \leq p < \infty$, we have from (3.4)

$$\Big(\sum_{n=1}^{\infty}\Big(\sum_{l=1}^{\infty}|a_{nl}|s_l(T_k+S_k)\Big)^p\Big)^{\frac{1}{p}} \leq M\Big[\sum_{n=1}^{\infty}\Big(\sum_{l=1}^{\infty}|a_{nl}|s_l(T_k)+\sum_{l=1}^{\infty}|a_{nl}|s_l(S_k)\Big)^p\Big]^{\frac{1}{p}}$$
$$\leq M\Big[\Big(\sum_{n=1}^{\infty}\Big(\sum_{l=1}^{\infty}|a_{nl}|s_l(T_k)\Big)^p\Big)^{\frac{1}{p}} + \Big(\sum_{n=1}^{\infty}\Big(\sum_{l=1}^{\infty}|a_{nl}|s_l(S_k)\Big)^p\Big)^{\frac{1}{p}}\Big].$$

Therefore

$$\sup_{k\geq 1}\Big(\sum_{n=1}^{\infty}\Big(\sum_{l=1}^{\infty}|a_{nl}|s_l(T_k+S_k)\Big)^p\Big)^{\frac{1}{p}} \leq M\Big[\sup_{k\geq 1}\Big(\sum_{n=1}^{\infty}\Big(\sum_{l=1}^{\infty}|a_{nl}|s_l(T_k)\Big)^p\Big)^{\frac{1}{p}} + \sup_{k\geq 1}\Big(\sum_{n=1}^{\infty}\Big(\sum_{l=1}^{\infty}|a_{nl}|s_l(S_k)\Big)^p\Big)^{\frac{1}{p}}\Big] < \infty.$$

Thus $S+T \in \mathscr{A}^{(s)}_{(\prod E_k \to \prod F_k)} - p$.

If $T \in \mathscr{A}^{(s)}_{(\prod E_k \to \prod F_k)} - p$ and λ be any scalar then it is easy to see that $\lambda T \in \mathscr{A}^{(s)}_{(\prod E_k \to \prod F_k)} - p$. Hence $\mathscr{A}^{(s)}_{(\prod E_k \to \prod F_k)} - p$ is a linear space. Similarly for $p = \infty$, it can be shown that $\mathscr{A}^{(s)}_{(\prod E_k \to \prod F_k)} - \infty$ is a linear space. This completes the proof. $\square$

Remark 3.3. *The condition* (1.1) *on the matrix A is no longer necessary for the set of vector-valued s-type* $|A, p, \prod X_k|$ *operators from* $\prod E_k$ *to* $\prod F_k$ *be a linear space. Justification is given as below.*

Let $0 < p < \infty$ *and* $A = (a_{nk})$ *be an infinite identity matrix. Clearly identity matrix does not satisfy the condition* (1.1) *but the triangle inequality can be proved as follows. Let* $S = (S_k)_{k\geq 1}$, $T = (T_k)_{k\geq 1} \in \mathscr{L}(\prod E_k, \prod F_k)$ *be any two vector-*

valued s-type $|A, p, \prod X_k|$ *operators. Then*

$$\begin{aligned}\sum_{n=1}^{\infty}\Big(\sum_{l=1}^{\infty}|a_{nl}s_l(T_k+S_k)|\Big)^p &= \sum_{n=1}^{\infty}\Big(s_n(T_k+S_k)\Big)^p \\ &= \sum_{n=1}^{\infty}\Big(s_{2n-1}(T_k+S_k)\Big)^p + \sum_{n=1}^{\infty}\Big(s_{2n}(T_k+S_k)\Big)^p \\ &\le 2\sum_{n=1}^{\infty}\Big(s_{2n-1}(T_k+S_k)\Big)^p \\ &\le 2\Big(\sum_{n=1}^{\infty}\Big(s_n(T_k)+s_n(S_k)\Big)^p\Big).\end{aligned}$$

Thus

$$\Big(\sum_{n=1}^{\infty}\Big(\sum_{l=1}^{\infty}|a_{nl}s_l(T_k+S_k)|\Big)^p\Big)^{\frac{1}{p}} \le 2^{\frac{1}{p}}.C\Big[\Big(\sum_{n=1}^{\infty}\Big(s_n(T_k)\Big)^p\Big)^{\frac{1}{p}}+\Big(\sum_{n=1}^{\infty}\Big(s_n(S_k)\Big)^p\Big)^{\frac{1}{p}}\Big],$$

where $C \ge 1$ *is a constant. Therefore,*

$$\sup_{k\ge 1}\Big(\sum_{n=1}^{\infty}\Big(\sum_{l=1}^{\infty}|a_{nl}s_l(T_k+S_k)|\Big)^p\Big)^{\frac{1}{p}} < \infty.$$

Hence $S+T$ *belongs to the set of vector-valued s-type* $|A, p, \prod X_k|$ *operators from* $\prod E_k$ *to* $\prod F_k$*. Clearly* λT *belongs to the set of vector-valued s-type* $|A, p, \prod X_k|$ *operators, where* λ *be any scalar. Thus the condition* (1.1) *on the matrix* $A = (a_{nk})$ *is not necessary to form a linear space.*

Proposition 3.4. *For* $1 \le p < q \le \infty$*, we have* $\mathscr{A}_{vec}^{(s)} - p \subseteq \mathscr{A}_{vec}^{(s)} - q$.

Proof. We omit the proof as it is trivial. □

Let $\mathscr{A}^{(s)}_{(\prod E_k \to \prod F_k)} - p$ for $0 < p < \infty$ be a linear space. Define $\bar{\beta}^{(s)}_{A,p}$: $\mathscr{A}^{(s)}_{(\prod E_k \to \prod F_k)} - p \to \mathbb{R}$ as

$$\bar{\beta}^{(s)}_{A,p}(T) = \sup_{k\ge 1}\Big(\sum_{n=1}^{\infty}\Big(\sum_{l=1}^{\infty}|a_{nl}s_l(T_k)|\Big)^p\Big)^{\frac{1}{p}},$$

where $T = (T_k)_{k\ge 1} \in \mathscr{A}^{(s)}_{(\prod E_k \to \prod F_k)} - p$. It can be shown that $\bar{\beta}^{(s)}_{A,p}$ is a quasi-norm on this linear space.

Remark 3.5. *For* $p = \infty$*, we define* $\bar{\beta}^{(s)}_{A,\infty}(T) = \sup_{k\ge 1}\sup_{n\ge 1}\Big(\sum_{l=1}^{\infty}|a_{nl}s_l(T_k)|\Big)$

Theorem 3.6. *Let $0 < p < \infty$. For fixed nonzero matrix $A = (a_{nk})$ satisfying the condition (1.1) and $\sum\limits_{n=1}^{\infty} |a_{n1}|^p < \infty$, each component of the class $\mathscr{A}_{vec}^{(s)} - p$ is complete under the normalized quasi-norm $\hat{\bar{\beta}}_{A,p}^{(s)}$, where*

$$\hat{\bar{\beta}}_{A,p}^{(s)}(.) = \frac{\bar{\beta}_{A,p}^{(s)}(.)}{\Big(\sum\limits_{n=1}^{\infty} |a_{n1}|^p\Big)^{\frac{1}{p}}}.$$

Proof. Let $\mathscr{A}^{(s)}_{(\prod E_k \to \prod F_k)} - p$ be any one of the component of the class $\mathscr{A}_{vec}^{(s)} - p$ for $0 < p < \infty$. We consider

$$\begin{aligned} \bar{\beta}_{A,p}^{(s)}(T) &= \sup_{k \geq 1} \Big(\sum_{n=1}^{\infty} \Big(\sum_{l=1}^{\infty} |a_{nl} s_l(T_k)| \Big)^p \Big)^{\frac{1}{p}} \\ &\geq \sup_{k \geq 1} \Big(\sum_{n=1}^{\infty} \Big(|a_{n1} s_1(T_k)| \Big)^p \Big)^{\frac{1}{p}} \\ &= \sup_{k \geq 1} \|T_k\| \Big(\sum_{n=1}^{\infty} |a_{n1}|^p \Big)^{\frac{1}{p}}. \end{aligned}$$

$$\Rightarrow \|T\| \leq \hat{\bar{\beta}}_{A,p}^{(s)}(T) \quad \text{for } T \in \mathscr{A}^{(s)}_{(\prod E_k \to \prod F_k)} - p. \tag{3.5}$$

Let (T^m) be a Cauchy sequence in $\mathscr{A}^{(s)}_{(\prod E_k \to \prod F_k)} - p$. Then for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\hat{\bar{\beta}}_{A,p}^{(s)}(T^m - T^r) < \epsilon, \quad \forall\, m, r \geq N. \tag{3.6}$$

Now from (3.5), we have

$$\|T^m - T^r\| \leq \hat{\bar{\beta}}_{A,p}^{(s)}(T^m - T^r).$$

Using (3.6), we get

$$\|T^m - T^r\| \leq \hat{\bar{\beta}}_{A,p}^{(s)}(T^m - T^r) < \epsilon \quad \forall\, m, r \geq N.$$

Hence (T^m) is a Cauchy sequence in $\mathscr{L}(\prod E_k, \prod F_k)$. Since each F_k is a Banach space, $\mathscr{L}(\prod E_k, \prod F_k)$ is also a Banach space. Therefore $T^m \to T$ as $m \to \infty$ in $\mathscr{L}(\prod E_k, \prod F_k)$. We shall now show that $T^m \to T$ as $m \to \infty$ in $\mathscr{A}^{(s)}_{(\prod E_k \to \prod F_k)} - p$. Using Lemma 2.6, we have for each $k \in \mathbb{N}$

$$|s_n(T_k^r - T_k^m) - s_n(T_k - T_k^m)| \leq \|T_k^r - T_k\|.$$

Letting $r \to \infty$,

$$s_n(T_k^r - T_k^m) \to s_n(T_k - T_k^m). \tag{3.7}$$

From (3.6), we get

$$\sup_{k\geq 1}\Big(\sum_{n=1}^{\infty}\Big(\sum_{l=1}^{\infty}|a_{nl}s_l(T_k^r - T_k^m)|\Big)^p\Big)^{\frac{1}{p}} < \epsilon\Big(\sum_{n=1}^{\infty}|a_{n1}|^p\Big)^{\frac{1}{p}}, \quad \forall\, m, r \geq N.$$

Using (3.7), it can be shown that as $r \to \infty$ (keeping $m \geq N$ fixed)

$$\sup_{k\geq 1}\Big(\sum_{n=1}^{\infty}\Big(\sum_{l=1}^{\infty}|a_{nl}s_l(T_k - T_k^m)|\Big)^p\Big)^{\frac{1}{p}} \leq \epsilon\Big(\sum_{n=1}^{\infty}|a_{n1}|^p\Big)^{\frac{1}{p}}$$
$$\Rightarrow \hat{\hat{\beta}}^{(s)}_{A,p}(T - T^m) \leq \epsilon \quad \forall\, m \geq N.$$

This implies that $T^m \to T$ under the quasi-norm $\hat{\hat{\beta}}^{(s)}_{A,p}$.
Next we show that $T \in \mathscr{A}^{(s)}_{(\prod E_k \to \prod F_k)} - p$. Consider

$$\sum_{l=1}^{\infty}|a_{nl}s_l(T_k)| = \sum_{l=1}^{\infty}|a_{n,2l-1}s_{2l-1}(T_k)| + \sum_{l=1}^{\infty}|a_{n,2l}s_{2l}(T_k)|$$
$$\leq \sum_{l=1}^{\infty}\Big(|a_{n,2l-1}| + |a_{n,2l}|\Big)s_{2l-1}(T_k),$$

since $0 \leq s_{n+1}(T_k) \leq s_n(T_k)$ for all n. Using the inequality (1.1), we have

$$\sum_{l=1}^{\infty}|a_{nl}s_l(T_k)| \leq M\Big(\sum_{l=1}^{\infty}|a_{nl}|s_l(T_k - T_k^m) + \sum_{l=1}^{\infty}|a_{nl}|s_l(T_k^m)\Big).$$

Therefore

$$\Big(\sum_{n=1}^{\infty}\Big(\sum_{l=1}^{\infty}|a_{nl}s_l(T_k)|\Big)^p\Big)^{\frac{1}{p}} \leq C.M\Big[\Big(\sum_{n=1}^{\infty}\Big(\sum_{l=1}^{\infty}|a_{nl}s_l(T_k - T_k^m)|\Big)^p\Big)^{\frac{1}{p}}$$
$$+ \Big(\sum_{n=1}^{\infty}\Big(\sum_{l=1}^{\infty}|a_{nl}s_l(T_k^m)|\Big)^p\Big)^{\frac{1}{p}}\Big],$$

where $C \geq 1$ is a constant. Thus

$$\sup_{k\geq 1}\Big(\sum_{n=1}^{\infty}\Big(\sum_{l=1}^{\infty}|a_{nl}s_l(T_k)|\Big)^p\Big)^{\frac{1}{p}} < \infty,$$

which follows from the fact that $\hat{\hat{\beta}}^{(s)}_{A,p}(T - T^m) \to 0$ as $m \to \infty$ and $(T^m) \in \mathscr{A}^{(s)}_{(\prod E_k \to \prod F_k)} - p$. Hence $T \in \mathscr{A}^{(s)}_{(\prod E_k \to \prod F_k)} - p$. This completes the proof. □

Corollary 3.7. *Let $A = (a_{nk})$ be a nonzero infinite matrix satisfying the condition* (1.1) *and* $\sup_{n\geq 1}|a_{n1}| < \infty$*, then each component of the class* $\mathscr{A}^{(s)}_{vec} - \infty$ *is complete* *under the normalized quasi-norm* $\hat{\bar{\beta}}^{(s)}_{A,\infty}$, *where*

$$\hat{\bar{\beta}}^{(s)}_{A,\infty}(.) = \frac{\bar{\beta}^{(s)}_{A,\infty}(.)}{\sup\limits_{n\geq 1}|a_{n1}|}.$$

Proposition 3.8. *If* $R \in \mathscr{L}(\prod F_k, \prod H_k)$ *and* $S \in \mathscr{A}^{(s)}_{(\prod E_k \to \prod F_k)} - p$, *then* $RS \in \mathscr{A}^{(s)}_{(\prod E_k \to \prod H_k)} - p$ *and* $\hat{\bar{\beta}}^{(s)}_{A,p}(RS) \leq \|R\| \hat{\bar{\beta}}^{(s)}_{A,p}(S)$. *Also if* $T \in \mathscr{L}(\prod G_k, \prod E_k)$ *and* $S \in \mathscr{A}^{(s)}_{(\prod E_k \to \prod F_k)} - p$, *then* $ST \in \mathscr{A}^{(s)}_{(\prod G_k \to \prod F_k)} - p$ *and* $\hat{\bar{\beta}}^{(s)}_{A,p}(ST) \leq \|T\| \hat{\bar{\beta}}^{(s)}_{A,p}(S)$.

Proof. We omit the proof. □

Next we derive some inclusion relations.

Theorem 3.9. *Let* $0 < p \leq \infty$. *Then*

(I) $\mathscr{A}^{(a)}_{vec} - p \subseteq \mathscr{A}^{(c)}_{vec} - p \subseteq \mathscr{A}^{(x)}_{vec} - p \subseteq \mathscr{A}^{(h)}_{vec} - p$ *and*

(II) $\mathscr{A}^{(a)}_{vec} - p \subseteq \mathscr{A}^{(d)}_{vec} - p \subseteq \mathscr{A}^{(y)}_{vec} - p \subseteq \mathscr{A}^{(h)}_{vec} - p$.

Proof. Let $0 < p < \infty$. Suppose that $T = (T_k)_{k\geq 1}$ belongs to any one of the component of the class $\mathscr{A}^{(a)}_{vec} - p$. Then

$$\sup_{k\geq 1} \Big(\sum_{n=1}^{\infty} \Big(\sum_{l=1}^{\infty} |a_{nl} a_l(T_k)| \Big)^p \Big)^{\frac{1}{p}} < \infty.$$

Using Proposition 2.4, we have

$$\begin{aligned} \sum_{n=1}^{\infty} \Big(\sum_{l=1}^{\infty} |a_{nl} h_l(T_k)| \Big)^p \leq \sum_{n=1}^{\infty} \Big(\sum_{l=1}^{\infty} |a_{nl} x_l(T_k)| \Big)^p &\leq \sum_{n=1}^{\infty} \Big(\sum_{l=1}^{\infty} |a_{nl} c_l(T_k)| \Big)^p \\ &\leq \sum_{n=1}^{\infty} \Big(\sum_{l=1}^{\infty} |a_{nl} a_l(T_k)| \Big)^p. \end{aligned}$$

Hence the proof of (I) follows for $0 < p < \infty$. It is trivial to check for $p = \infty$. We omit the proof of (II) as it is similar to the previous one. □

There are some converse estimates among s-number sequences as given below.

Lemma 3.10 ([15], p.165)**.** *Let* $T \in \mathscr{L}(E, F)$. *Then* $a_n(T) \leq 2n^{\frac{1}{2}} c_n(T)$ *and* $a_n(T) \leq 2n^{\frac{1}{2}} d_n(T)$.

We now define the class $\mathscr{L}^{(s)}_{vec} - (r, p)$ of vector-valued s-type $l_{r,p}$ operators as follows:

$$\mathscr{L}^{(s)}_{vec} - (r,p) = \{T = (T_k)_{k\geq 1} \in \mathscr{L} : \sup_{k\geq 1} \Big(\sum_{n=1}^{\infty} \Big(n^{\frac{1}{r} - \frac{1}{p}} s_n(T_k) \Big)^p \Big)^{\frac{1}{p}} < \infty\}$$

for $0 < r, p < \infty$.

Theorem 3.11. *Let $0 < r, p < \infty$ and $A = (a_{nl})$ be a diagonal matrix, where*

$$a_{nl} = \begin{cases} n^{\frac{1}{r}-\frac{1}{p}-\frac{1}{2}} & : \quad l = n \\ 0 & : \quad l \neq n. \end{cases}$$

If a bounded linear operator T belongs to $\mathscr{L}^{(c)}_{(\prod E_k \to \prod F_k)} - (r, p)$, then T belongs to $\mathscr{A}^{(a)}_{(\prod E_k \to \prod F_k)} - p$.

Proof. For $0 < p < \infty$, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\Big(\sum_{l=1}^{\infty}|a_{nl}a_l(T_k)|\Big)^p &= \sum_{n=1}^{\infty}\Big(n^{\frac{1}{r}-\frac{1}{p}-\frac{1}{2}}a_n(T_k)\Big)^p \\ &\leq \sum_{n=1}^{\infty}\Big(n^{\frac{1}{r}-\frac{1}{p}-\frac{1}{2}}.2n^{\frac{1}{2}}c_n(T_k)\Big)^p \qquad \text{(Using Lemma 3.10.)} \\ &= 2^p\sum_{n=1}^{\infty}\Big(n^{\frac{1}{r}-\frac{1}{p}}c_n(T_k)\Big)^p.\end{aligned}$$

Thus $\sup_{k\geq 1}\Big(\sum_{n=1}^{\infty}\Big(\sum_{l=1}^{\infty}|a_{nl}a_l(T_k)|\Big)^p\Big)^{\frac{1}{p}} \leq 2\sup_{k\geq 1}\Big(\sum_{n=1}^{\infty}\Big(n^{\frac{1}{r}-\frac{1}{p}}c_n(T_k)\Big)^p\Big)^{\frac{1}{p}} < \infty$.
Hence the result follows. □

Theorem 3.12. *Let $0 < r, p < \infty$ and $A = (a_{nl})$ be a diagonal matrix, where*

$$a_{nl} = \begin{cases} n^{\frac{1}{r}-\frac{1}{p}-\frac{1}{2}} & : \quad l = n \\ 0 & : \quad l \neq n. \end{cases}$$

If a bounded linear operator T belongs to $\mathscr{L}^{(d)}_{(\prod E_k \to \prod F_k)} - (r, p)$, then T belongs to $\mathscr{A}^{(a)}_{(\prod E_k \to \prod F_k)} - p$.

Proof. The proof is similar to the preceding Theorem 3.11. □

Theorem 3.13. *If $T = (T_k) \in \mathscr{A}^{(s)}_{(\prod E_k \to \prod F_k)} - p$ and $S = (S_k) \in \mathscr{A}^{(s)}_{(\prod F_k \to \prod H_k)} - q$, then $ST = (S_kT_k)_{k\geq 1} \in \mathscr{A}^{(s)}_{(\prod E_k \to \prod H_k)} - r$, where*

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

Proof. Here we use the generalized Hölder inequality, i.e., if $x \in l_p$ and $y \in l_q$ then

$$\Big\{\sum_{n=1}^{\infty}|x_ny_n|^r\Big\}^{\frac{1}{r}} \leq \Big\{\sum_{n=1}^{\infty}|x_n|^p\Big\}^{\frac{1}{p}}\Big\{\sum_{n=1}^{\infty}|y_n|^q\Big\}^{\frac{1}{q}}.$$

Now we have

$$\begin{aligned}\sum_{l=1}^{\infty}|a_{nl}s_l(S_kT_k)| &\leq \sum_{l=1}^{\infty}\Big(|a_{n,2l-1}|+|a_{n,2l}|\Big)s_{2l-1}(S_kT_k)\\ &\leq M\sum_{l=1}^{\infty}|a_{nl}|s_{2l-1}(S_kT_k)\\ &\leq M\sum_{l=1}^{\infty}|a_{nl}|s_l(S_k)s_l(T_k) \qquad \text{(Using Definition 2.5).}\end{aligned}$$

Therefore

$$\begin{aligned}\Big\{\sum_{n=1}^{\infty}\Big(\sum_{l=1}^{\infty}|a_{nl}s_l(S_kT_k)|\Big)^r\Big\}^{\frac{1}{r}} &\leq M\Big\{\sum_{n=1}^{\infty}\Big(\sum_{l=1}^{\infty}|a_{nl}|s_l(S_k)s_l(T_k)\Big)^r\Big\}^{\frac{1}{r}}\\ &\leq M\Big\{\sum_{n=1}^{\infty}\Big(\sum_{l=1}^{\infty}|a_{nl}|s_l(T_k)\Big)^p\Big\}^{\frac{1}{p}}\Big\{\sum_{n=1}^{\infty}\Big(\sum_{l=1}^{\infty}|a_{nl}|s_l(S_k)\Big)^q\Big\}^{\frac{1}{q}}.\end{aligned}$$

Thus

$$\sup_{k\geq 1}\Big\{\sum_{n=1}^{\infty}\Big(\sum_{l=1}^{\infty}|a_{nl}s_l(S_kT_k)|\Big)^r\Big\}^{\frac{1}{r}}$$

$$\leq M\sup_{k\geq 1}\Big\{\sum_{n=1}^{\infty}\Big(\sum_{l=1}^{\infty}|a_{nl}s_l(T_k)|\Big)^p\Big\}^{\frac{1}{p}}\sup_{k\geq 1}\Big\{\sum_{n=1}^{\infty}\Big(\sum_{l=1}^{\infty}|a_{nl}s_l(S_k)|\Big)^q\Big\}^{\frac{1}{q}} < \infty.$$

This completes the proof. □

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