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Upper Bound for the Crossing Number of $Q_n \times K_3$

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Abstract : In this paper, we describe a method of finding the upper bound for the crossing Number of $Q_n \times K_3$. We construct a drawing of $Q_n \times K_3$, called a $3-axes$ drawing of $Q_n \times K_3$. A $3-axes$ drawing of $Q_n \times K_3$ is a representation of $Q_n \times K_3$ on the plane such that its vertices are placed on 3 straight lines L_i where $i = 1, 2, 3$ with a fixed vertex ordering.

Keywords : graph; hypercube; crossing number; drawing.
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1 Introduction

Let G be a simple connected graph with a vertex set $V(G) = \{v_1, v_2, v_3, ..., v_n\}$ and an edge set $E(G) = \{e_1, e_2, e_3, ..., e_m\}$. The crossing number of a graph G, denoted $cr(G)$, is the minimum number of pairwise intersections of edge crossing on a plane drawing of the graph G. Clearly, $cr(G) = 0$ if and only if G is planar. It is known that the exact crossing numbers of graphs are very difficult to compute. In 1973, Erdös and Guy [1] wrote, *"Almost all questions that one can ask about crossing numbers remain unsolved"*. In fact, Garey and Johnson [2] proved that computing

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the crossing number is NP-complete.

The $n-cube$ or $n-dimensional\ hypercube\ graph$ Q_n is defined recursively in terms of the cartesian products. The one dimension cube Q_1 is simply K_2 where K_2 is a complete graph with 2 vertices. For $n \geq 2$, Q_n is defined recursively as $Q_{n-1} \times K_2$. The order of Q_n is $\mid V(Q_n) \mid = 2^n$ and its size is $\mid E(Q_n) \mid = n2^{n-1}$.

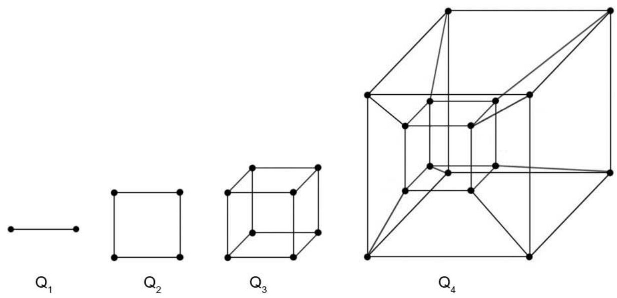

Figure 1: $n - cube$ graphs for $n = 1, 2, 3, 4$

In 1969, Harary [3] mentioned that there does not even exist a conjecture about the crossing number of the hypercube. In 1970, Eggleton and Guy [4] constructed a drawing of Q_n which implies that for $n \geq 3$,

$$cr(Q_n) \leq \frac{5}{32}4^n - \lfloor \frac{n^2+1}{2} \rfloor 2^{n-2}. \tag{1.1}$$

In 2008, Faria, Figueiredo, Sykora and Vrto [5] announced a drawing for which the number of crossings coincides with $\frac{5}{32}4^n - \lfloor \frac{n^2+1}{2} \rfloor 2^{n-2}$ which would imply the inequality above. Yuanshen Yang, Guoqing Wang, Haoli Wang and Yan Zhou [6] presented a new strategy to construct a drawing of Q_n with fewer crossings than the values conjecture by Eggleton and Guy [4], which implies that

$$cr(Q_n) \lneqq \frac{5}{32}4^n - \lfloor \frac{n^2+1}{2} \rfloor 2^{n-2}. \tag{1.2}$$

They prove the following upper bound for the crossing number of hypercube graph,

$$cr(Q_n) \leq \begin{cases} \frac{139}{896}4^n - \lfloor \frac{n^2+1}{2} \rfloor 2^{n-2} + \frac{4}{7} \cdot 2^{3\lfloor \frac{n}{2} \rfloor - n}, & 5 \leq n \leq 10, \\ \frac{26695}{172032}4^n - \lfloor \frac{n^2+1}{2} \rfloor 2^{n-2} - \frac{n^2+2}{3} \cdot 2^{n-2} + \frac{4}{7} \cdot 2^{3\lfloor \frac{n}{2} \rfloor - n}, & n \geq 11. \end{cases} \tag{1.3}$$

The graph $Q_n \times K_3$ is the cartesian product of Q_n and K_3, where K_3 is a complete graph with 3 vertices and Q_n is an $n-$ *dimensional hypercube graph*. The order of $Q_n \times K_3$ is $\mid V(Q_n \times K_3) \mid = 3 \cdot 2^n$ and its size is $\mid E(Q_n \times K_3) \mid = 3[2^n + n2^{n-1}]$.

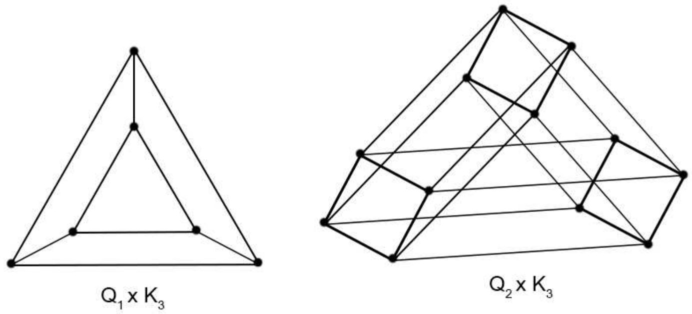

Figure 2: $Q_n \times K_3$ graphs for $n = 1, 2$

As the graph $Q_n \times K_3$, it is formed by 3 copies of Q_n's linked together with multiple edges. As shown in Figure 2, it can be observed that the calculation of crossing number of $Q_n \times K_3$ is quite complicated. We define the crossing number of a graph G to be the minimum number of crosses of a graph isomorphic to G. We have thusly invented another drawing of the graph $Q_n \times K_3$, also known as $3-$ *axes* **drawing of** $Q_n \times K_3$, by taking the formula shown below into account,

$$\begin{aligned} Q_n \times K_3 &= (Q_{n-1} \times K_2) \times K_3 \\ &= (Q_{n-1} \times K_3) \times K_2. \end{aligned} \tag{1.4}$$

We notice that the graph $Q_n \times K_3$ is defined recursively as $(Q_{n-1} \times K_3) \times K_2$.

2 $3-$ *Axes* Drawing of $Q_n \times K_3$

A $3-$ *axes* drawing of $Q_n \times K_3$ is a representation of $Q_n \times K_3$ on the plane such that its vertices are placed on 3 straight lines L_i where $i = 1, 2, 3$ with a fixed vertex ordering.

According to the formula (1.4), the graph $Q_n \times K_3$ is generated from the two copies of $Q_{n-1} \times K_3$ linked together with multiple edges. The next procedure of drawing is to embed one copy of the graph $Q_{n-1} \times K_3$ into the other with the condition that all edges must remain at the same position. Several examples of drawings are illustrated below in Figure 3 and 4.

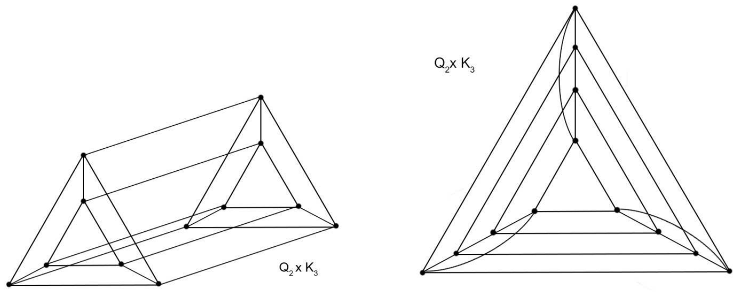

Figure 3: $Q_2 \times K_3$ graph and $Q_2 \times K_3$ which is embeded

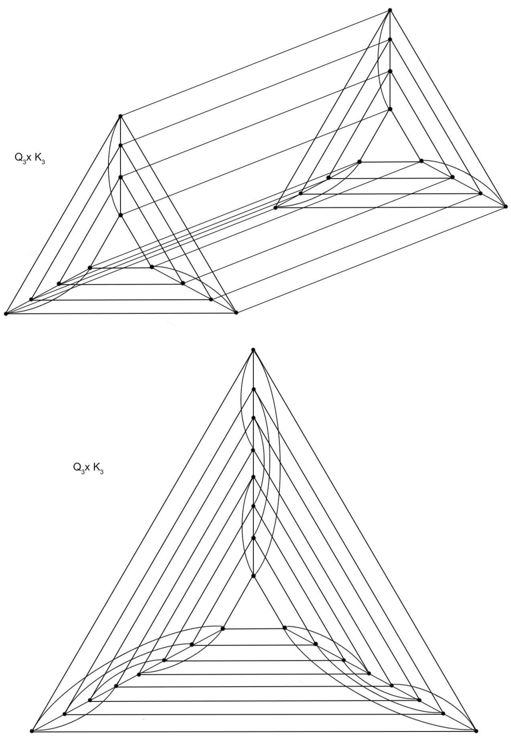

Figure 4: $Q_3 \times K_3$ and $Q_3 \times K_3$ which is embeded

We can observe that this drawing consists of multiple *complete graphs* (K_3). Next, we construct the straight lines L_i where $i = 1, 2, 3$, also known as $3 - axes$. We place L_i where $i = 1, 2, 3$ on the edge which connect all corresponding vertices of K_3. We see that there are some edges of $Q_n \times K_3$ overlapped on $3-axes$. As for the edges on $3-axes$, we redraw the edges need to be drawn in semicircle and never cross L_i. Notice that no edge crosses itself, no adjacent edges cross each other, no two edges cross more than once, and no three edges cross in a point. The $3-axes$ drawing of $Q_n \times K_3$ where $n = 1, 2, 3$ and their crossing numbers can be seen in Figure 5 and 6.

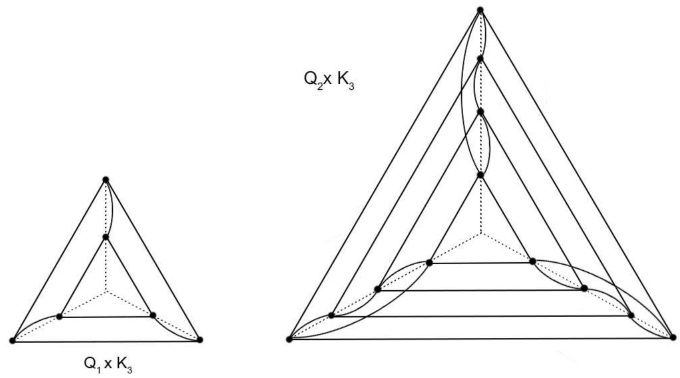

Figure 5: $3-axes$ drawing of $Q_n \times K_3$ where $n = 1, 2$

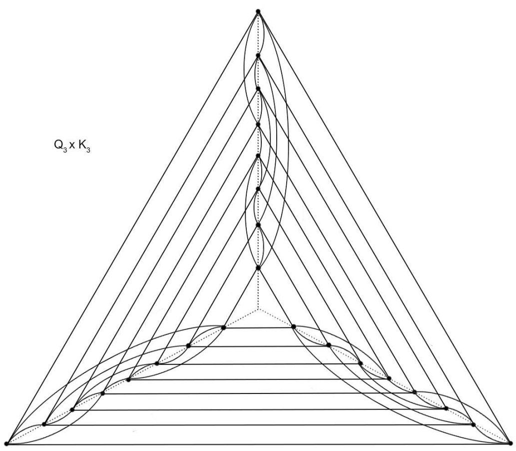

Figure 6: $3-axes$ drawing of $Q_3 \times K_3$

After the semicircles have been drawn, we can define this procedure of drawing as $3-axes$ drawing. We denote without loss of generality that Π_n

is a $3 - axes$ drawing of $Q_n \times K_3$.

3 Construction of Π_n

In this section, we discusss a construction of Π_n. We notice that in the $3 - axes$ drawing of $Q_n \times K_3$, the appearance of each *axis* is identical. This means that we can calculate the crossing number in a more simplistic fashion. That is, by considering from only *one axis*. First, we define Π_n as follow,

$$\Pi_n = \Delta_n \bigcup \Omega_n. \tag{3.1}$$

Given Δ_n is a *perimeter graph* which consists of 2^n *complete graphs* (K_3),

$$\Delta_n = 2^n K_3. \tag{3.2}$$

The vertex set of Δ_n is the same as the vertex set of Π_n and the edge set of Δ_n is a subset of Π_n, where $\mid V(\Delta_n) \mid = \mid E(\Delta_n) \mid = 3 \cdot 2^n$. In Π_n without the set of edges of Δ_n, we see that the graph Π_n remains the set of semicircle edges. Since the appearance of each *axis* is identical, we can only determine from *one axis*. We notice that it is a hypercube graph (Q_n). Therefore we call Π_n without the set of edges of Δ_n as Ω_n and define as follow,

$$\Omega_n = 3Q_n. \tag{3.3}$$

The vertex set of Ω_n is the same as the vertex set of Π_n and the edge set of Ω_n is a subset of Π_n, where $\mid V(\Omega_n) \mid = 3 \cdot 2^n$ and $\mid E(\Omega_n) \mid = 3 \cdot n2^{n-1}$. We notice that the set of all vertices of Δ_n and Ω_n are the same as Π_n.

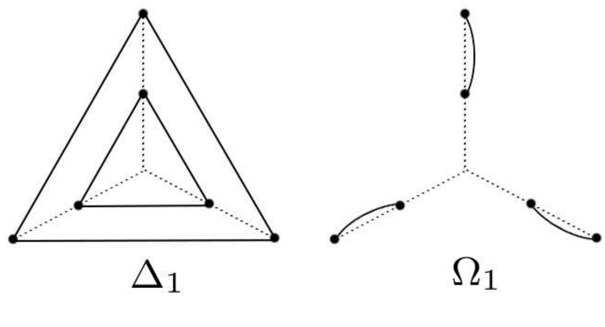

Figure 7: Δ_1 and Ω_1

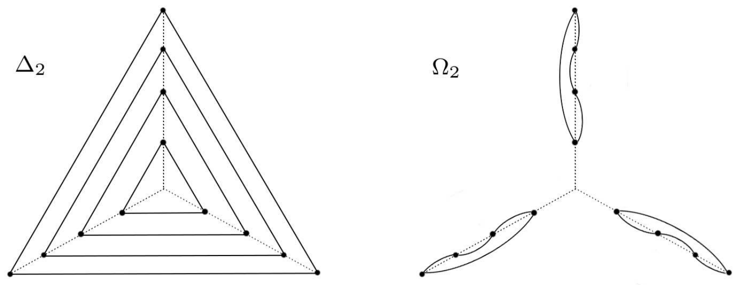

Figure 8: Δ_2 and Ω_2

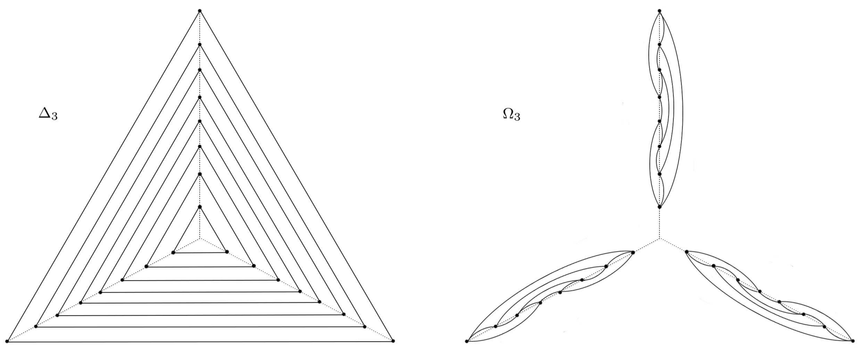

Figure 9: Δ_3 and Ω_3

4 Construction of S_n^i

In this paper, we describe a method of finding the upper bound for the crossing number of $Q_n \times K_3$ by considering from Π_n's drawing. It can be calculated from a cross of Δ_n and Ω_n edges and Ω_n edges crossing themselves. For the cross of Δ_n and Ω_n edges, even if we move the semicircle edges of Ω_n to either way, the number of crossing remains the same. But it is not true for the cross from Ω_n edges crossing themselves. For that, to find the upper bound for crossing number in Π_n, we mention on Ω_n edges crossing themselves.

Next, we introduce a construction of S_n^i for $i = 1, 2, 3$. Given S_n^i where $i = 1, 2, 3$ is a subgraph of Ω_n which is on the $3 - axes$,

$$\Omega_n = S_n^1 \bigcup S_n^2 \bigcup S_n^3. \tag{4.1}$$

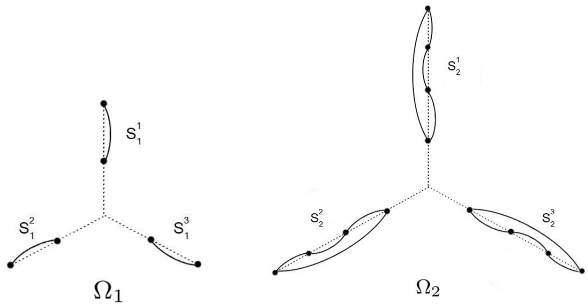

Figure 10: Ω_1 and Ω_2

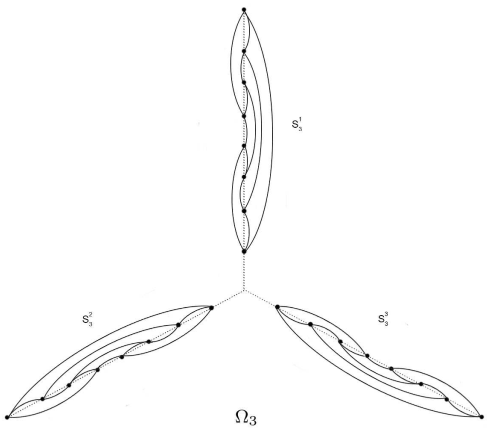

Figure 11: Ω_3

Next, we explain a drawing on graph S_n^i. The vertices lie on $i-axis$ and the edges have to be drawn as semicircles. We begin to explain the drawing of S_n^i by using the examples of S_3^i's drawing.

Step 1. We start drawing S_1^i. S_1^i is a graph with 2 vertices and only one edge. First, we locate the vertices on the $i-axis$ by fixed vertex ordering. Then the edge (semicircle) that connects between 2 vertices is drawn on the right side of $i-axis$.

Step 2. We draw S_2^i. S_2^i is a graph that consists of 2 copies of S_1^i. We put the second copy of S_1^i above the first one. The edges connecting the copies are on the left side of $i-axis$.

Step 3. Finally S_3^i's drawing is similary to Step 2. We locate the second copy of S_2^i above the first one but the edges connecting between 2 copies are on the right.

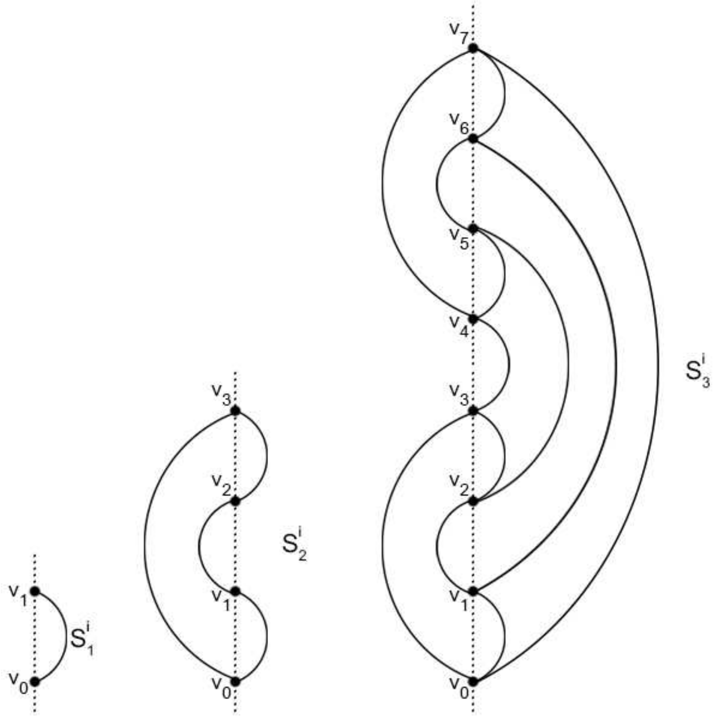

Figure 12: The construction of S_n^i for $n = 1, 2, 3$

Definition 4.1. A set of edges connecting of 2 copies of S_{n-1}^i is called *graph* Φ_n where $n = 1, 2, 3, ...$ The order of Φ_n is $\mid V(\Phi_n) \mid = 2^n$ and its size is $\mid E(\Phi_n) \mid = 2^{n-1}$.

We define the edges connecting of 2 copies of S_{n-1}^i when n is *odd* as Φ_{2j-1} and the edges connecting of 2 copies of S_{n-1}^i when n is *even* as Φ_{2j} for $j = 1, 2, ...$ In Figure 12, we observe that the edges connecting of 2 copies of S_1^i are located on the right and the edges connecting of 2 copies of S_2^i are located on the left. Also the edges connecting of 2 copies of S_3^i are located on the right. We conclude that Φ_{2j-1} edges have to be located on the opposite side of Φ_{2j} for $j = 1, 2, ...$

Definition 4.2. For the S_n^i's drawing, the location of the edges of Φ_{2j-1} which locate on the opposite side of Φ_{2j} is called *correct side*, apart from that is called *wrong side*.

5 Good Drawing of Ω_n

In this section, we want to show that the construction of Ω_n is a drawing which the minimum number of a cross in Π_n. We called this drawing **Good Drawing of** Ω_n. Since $\Omega_n = S_n^1 \bigcup S_n^2 \bigcup S_n^3$ and S_n^1, S_n^2, and S_n^3 drawings are similar, a consideration of any part is optional. From Figure 12, S_1^i, S_2^i, S_3^i where $i = 1, 2, 3$ have no crossing number.

Since graph S_n^i is drawn from 2 copies of graph S_{n-1}^i linked together with graph Φ_n, that is

$$S_n^i \quad = \quad 2S_{n-1}^i \bigcup \Phi_n. \tag{5.1}$$

Next, we order the vertices of S_n^i where $\mid V(S_n^i) \mid = 2^n$ so as to prove that Ω_n drawing is good. We define the first copy of S_n^i as *lower of* S_n^i, while the other copy, so called *upper of* S_n^i denoted by $\mathbf{L}S_n^i$ and $\mathbf{U}S_n^i$ respectively. Then we place $\mathbf{L}S_n^i$ next to origin of $i - axis$ and place $\mathbf{U}S_n^i$ next to $\mathbf{L}S_n^i$ copy to positive axis. For the plotting order, we start ordering from *lower of* S_n^i to *upper of* S_n^i that is

$$\begin{aligned} V(\mathbf{L}S_n^i) &= \{v_0, v_1, ..., v_{2^{n-1}-1}\}, \\ V(\mathbf{U}S_n^i) &= \{v_{2^{n-1}}, v_{2^{n-1}+1}, ..., v_{2^n-1}\}. \end{aligned} \tag{5.2}$$

For multiple edges, e_{ij}'s $\in \Phi_n$, which link between $\mathbf{L}S_n^i$ and $\mathbf{U}S_n^i$, they have to follow to the condition below,

$$i + j = 2^n - 1. \tag{5.3}$$

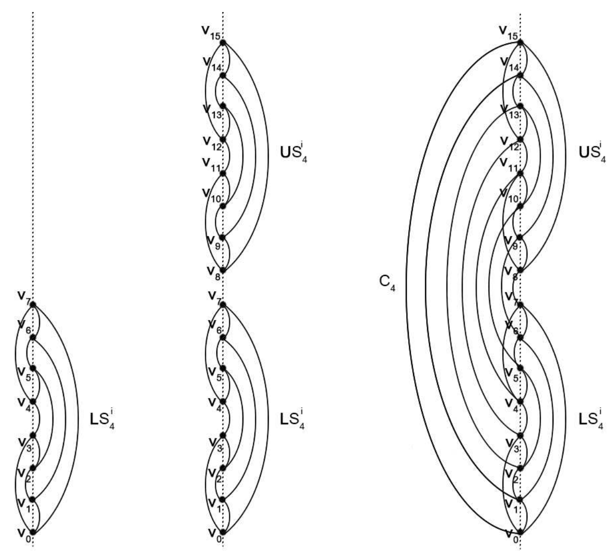

Figure 13: Construction of S_4^i

It is noticable that in graph S_n^i is composed of graph Φ_j where $j = 1, 2, 3, ..., n$ and number of Φ_j in S_n^i equals 2^{n-j} copies. This fact follows the fact that Φ_{2j-1} must be on the opposite side of Φ_{2j}.

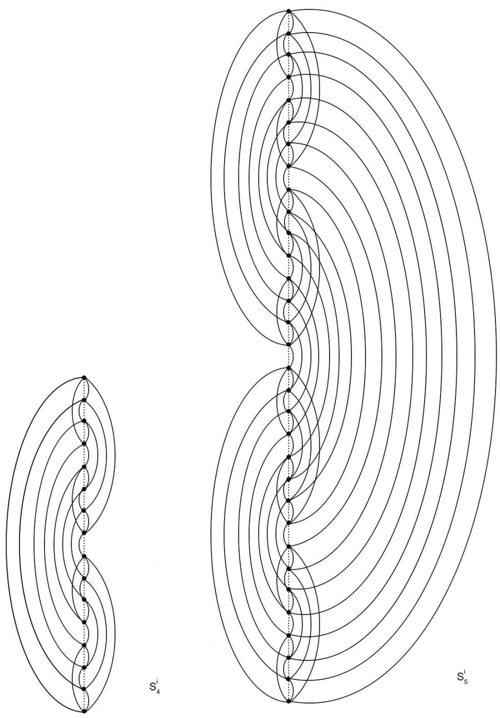

Figure 14: S_4^i and S_5^i

Lemma 5.1. *The graph S_{n-k}^i is a subgraph of S_n^i. In particular, there are 2^k copies of S_{n-k}^i, for $k = 1, 2, 3, ..., n-1$.*

Proof. From (5.1), graph S_n^i is made up of 2 copies of graph S_{n-1}^i,

graph S_{n-1}^i is made up of 2 copies of graph S_{n-2}^i,

$\vdots$

graph S_{n-k+1}^i is made up of 2 copies of graph S_{n-k}^i.

Therefore, S_n^i graph contains S_{n-k}^i graph where $k = 1, 2, 3, ..., n-1$, equaling number of $2 \cdot 2 \cdot 2 \cdots 2 = 2^k$ copies. □

Lemma 5.2. *The graph Φ_j is a subgraph of S_n^i. In particular, there are 2^{n-j} copies of Φ_j for $j = 1, 2, 3, ..., n$.*

Proof. From Lemma 5.1 and (5.1), graph S_n^i contains 2^k copies of S_{n-k}^i and S_{n-k}^i contains 1 of graph Φ_{n-k}. So S_n^i consists of Φ_{n-k} equal to number of S_{n-k}^i in S_n^i, that is 2^k. We let $j = n-k$ so $k = n-j$, therefore graph S_n^i consists of Φ_j where $j = 1, 2, 3, ..., n$ equaling number of 2^{n-j} copies. □

Definition 5.1. For graph S_n^i, we define $\underline{pc}(\Phi_k, \Phi_l)$ as the number of potential crossing which is created from crossing between graph Φ_k and graph Φ_l by considering only graph Φ_l which in S_n^i where $k = 2, 3, 4, ..., n$ and $l < k$.

Lemma 5.3. *For any integer* $2 \le k \le n$ *and* $l < k$*,*

$$\underline{pc}(\Phi_k, \Phi_l) = (2^{l-1} - 1)(2^{l-1}). \tag{5.4}$$

From (5.4), we see that $\underline{pc}(\Phi_k, \Phi_l)$ only depends on Φ_l, so

$$\underline{pc}(\Phi_{k_1}, \Phi_l) = \underline{pc}(\Phi_{k_2}, \Phi_l), \tag{5.5}$$

where $l < \min\{k_1, k_2\}$.

Proof. We let e_{mn} and e_{op} be edges of Φ_k and Φ_l respectively. By ordering vertices of graph S_n^i and $\mid V(\Phi_n) \mid = 2^n$, $\mid E(\Phi_n) \mid = 2^{n-1}$, we have

$$\begin{array}{rcl} V(\mathbf{L}\Phi_k) & = & \{v_0, v_1, ..., v_{2^{k-1}-1}\}, \\ V(\mathbf{U}\Phi_k) & = & \{v_{2^{k-1}}, v_{2^{k-1}+1}, ..., v_{2^k-1}\}, \\ V(\mathbf{L}\Phi_l) & = & \{v_0, v_1, ..., v_{2^{l-1}-1}\}, \\ V(\mathbf{U}\Phi_l) & = & \{v_{2^{l-1}}, v_{2^{l-1}+1}, ..., v_{2^l-1}\}. \end{array} \tag{5.6}$$

We can see that $m + n = 2^k - 1$ and $o + p = 2^l - 1$ where $m \in V(\mathbf{L}\Phi_k)$, $n \in V(\mathbf{U}\Phi_k)$, $o \in V(\mathbf{L}\Phi_l)$ and $p \in V(\mathbf{U}\Phi_l)$. The edges e_{mn} and e_{op} are potential crossing if and only if $v_m < v_o < v_n < v_p$ or $v_o < v_m < v_p < v_n$.

Next, we consider the number of potential crossing between Φ_k and Φ_l from all of edges in Φ_l.

The edge $e_{0,2^l-1}$ crosses with some edges in Φ_k when $0 < m < 2^l - 1 < n$. We find the number of m that correspond to the condition $0 < m < 2^l - 1 < n$. We let $M_0 = \{m \mid 0 < m < 2^l - 1 < n\}$ and we see that m can be $1, 2, 3, ..., 2^l - 2$, that is

$$\mid M_0 \mid = 2^l - 2. \tag{5.7}$$

The edge $e_{1,2^l-2}$ crosses with edges in Φ_k when $1 < m < 2^l - 2 < n$. We find the number of m that correspond to $1 < m < 2^l - 2 < n$. We let

$M_1 = \{m \mid 1 < m < 2^l - 2 < n\}$ and we see that m can be $2, 3, 4, ..., 2^l - 3$, that is

$$\mid M_1 \mid = 2^l - 4. \tag{5.8}$$

The edge $e_{2,2^l-3}$ crosses with edges in Φ_k when $2 < m < 2^l - 3 < n$. We find the number of m that correspond to $2 < m < 2^l - 3 < n$. We let $M_2 = \{m \mid 2 < m < 2^l - 3 < n\}$ and we see that m can be $3, 4, 5, ..., 2^l - 4$, that is

$$\mid M_2 \mid = 2^l - 6. \tag{5.9}$$

$$\vdots$$

Similarly to the edge $e_{2^{l-1}-2,2^{l-1}+1}$ crosses with edges in Φ_k when $2^{l-1} - 2 < m < 2^{l-1}+1 < n$. We let $M_{2^{l-1}-2} = \{m \mid 2^{l-1}-2 < m < 2^{l-1}+1 < n\}$. We see that m can be $2^{l-1} - 1$ and 2^{l-1}, that is

$$\mid M_{2^{l-1}-2} \mid = 2. \tag{5.10}$$

Finally, for the edge $e_{2^{l-1}-1,2^{l-1}}$, we can see that this edge can not cross with edges in Φ_k, So

$$\mid M_{2^{l-1}-1} \mid = 0. \tag{5.11}$$

From (5.7)-(5.11), that is all a number of potential crossing between Φ_k and Φ_l, so

$$\begin{aligned} \underline{pc}(\Phi_k, \Phi_l) &= |M_0 \mid + \mid M_1 \mid + \mid M_2 \mid + ... + \mid M_{2^{l-1}-2} \mid + \mid M_{2^{l-1}-1} \mid \\ &= (2^l - 2) + (2^l - 4) + (2^l - 6) + ... + 2 + 0 \\ &= \frac{(2^{l-1})}{2}(0 + 2^l - 2) \qquad (5.12) \\ &= (2^{l-1} - 1)(2^{l-1}). \end{aligned}$$

□

Definition 5.2. For graph S_n^i, we define $\overline{pc}(\Phi_k, \Phi_l)$ as the number of potential crossing which is created from crossing between graph Φ_k and graph Φ_l by considering from the whole of graph Φ_k which in S_n^i where $k = 2, 3, 4, ..., n$ and $l < k$. That is a crossing of a creation of every copy of graph Φ_l in graph Φ_k.

Lemma 5.4. *For any integer $2 \leq k \leq n$ and $l < k$,*

$$\overline{pc}(\Phi_k, \Phi_l) = 2^{k-1}(2^{l-1} - 1). \tag{5.13}$$

Proof. The number of potential crossing between graph Φ_k and graph Φ_l when we fix k depends on the number of all subgraph of Φ_l in Φ_k, that equal to 2^{k-l} copy. That is

$$\begin{aligned}\overline{pc}(\Phi_k,\Phi_l) &= 2^{k-l}\cdot \underline{pc}(\Phi_k,\Phi_l)\\ &= 2^{k-l}(2^{l-l}-1)(2^{l-l}) \qquad (5.14)\\ &= 2^{k-1}(2^{l-l}-1).\end{aligned}$$ □

By Lemma 5.4, it is notable that

$$\overline{pc}(\Phi_n,\Phi_l) > \overline{pc}(\Phi_{n-1},\Phi_l) > \overline{pc}(\Phi_{n-2},\Phi_l) > ... > \overline{pc}(\Phi_3,\Phi_l) > \overline{pc}(\Phi_2,\Phi_l). \qquad (5.15)$$

Definition 5.3. For graph S_n^i, we define $pc^{(n)}(\Phi_k,\Phi_l)$ as the number of potential crossing which is created from crossing between graph Φ_k and graph Φ_l by considering from the whole of graph S_n^i which in S_n^i where $k = 2,3,4,...,n$ and $l < k$. That is a crossing of a creation of every copy of graph Φ_l in graph Φ_k and every copy of graph Φ_k in graph S_n^i.

Lemma 5.5. *For any integer $2 \le k \le n$ and $l < k$,*

$$pc^{(n)}(\Phi_k,\Phi_l) = 2^{n-1}(2^{l-l}-1). \qquad (5.16)$$

Proof. Consider,

$$pc^{(n)}(\Phi_k,\Phi_l) = 2^{n-k}\cdot\overline{pc}(\Phi_k,\Phi_l) = 2^{n-k}(2^{k-1})(2^{l-l}-1) = 2^{n-1}(2^{l-l}-1).$$ □

Lemma 5.6. *For $n = 3,4,5,...$ and $j < n-1$, we have*

$$pc^{(n)}(\Phi_n,\Phi_{j+1}) > pc^{(n)}(\Phi_n,\Phi_j). \qquad (5.17)$$

Proof. Consider,

$$\begin{aligned}pc^{(n)}(\Phi_n,\Phi_{j+1}) - pc^{(n)}(\Phi_n,\Phi_j) &= [2^{n-(j+1)}\cdot\underline{pc}(\Phi_n,\Phi_{j+1})] - [2^{n-j}\cdot\underline{pc}(\Phi_n,\Phi_j)]\\ &= [2^{n-(j+1)}(2^{(j+1)-1}-1)(2^{(j+1)-1})]\\ &\quad - [2^{n-j}(2^{j-1}-1)(2^{j-1})]\\ &= [2^{n-(j+1)+(j+1)-1}(2^{(j+1)-1}-1)]\\ &\quad - [2^{n-j+j-1}(2^{j-1}-1)]\\ &= [2^{n-1}(2^j-1)] - [2^{n-1}(2^{j-1}-1)]\\ &= (2^j-1) - (2^{j-1}-1)\\ &= 2^j - 2^{j-1} > 0.\end{aligned}$$

Thus, $pc^{(n)}(\Phi_n, \Phi_j) - pc^{(n)}(\Phi_n, \Phi_{j+1}) > 0$ is true for all $n = 3, 4, 5, ...$ □

Definition 5.4. For graph Π_n, we define $pc^{(n)}(\Delta_n, \Phi_m)$ as the number of potential crossing which is created from crossing between graph Δ_n and graph Φ_m, $m = 1, 2, ..., n$, by considering from the whole of graph Π_n.

Lemma 5.7. *For any integer* $1 \leq m \leq n$*,*

$$pc^{(n)}(\Delta_n, \Phi_m) = 3(2^{n-1})(2^{m-1} - 1). \tag{5.18}$$

Proof. We let $e_{i,j}$ and $e_{k,*}$ be edges of Φ_m and Δ_n respectively. We note that $*$ refers to the vertex on other axis. By ordering vertices of graph S_n^i in (5.6) and $\mid V(\Phi_m) \mid = 2^m$, $\mid E(\Phi_m) \mid = 2^{m-1}$, we have the edge set of graph Φ_m,

$$E(\Phi_m) = \{e_{0,2^m-1}, e_{1,2^m-2}, e_{2,2^m-3}, ..., e_{2^{m-1}-2,2^{m-1}+1}, e_{2^{m-1}-1,2^{m-1}}\}. \tag{5.19}$$

The edge set of graph Δ_n,

$$E(\Delta_n) = \{e_{0,*}, e_{1,*}, e_{2,*}, ..., e_{2^{n-1}-2,*}, e_{2^{n-1}-1,*}, ..., e_{2^n-2,*}, e_{2^n-1,*}\}. \tag{5.20}$$

The edges e_{ij} and e_{kl} are potential crossing if and only if $v_i < v_k < v_j < v_l$ or $v_k < v_i < v_l < v_j$. Next, we consider the number of potential crossing between Δ_n and Φ_m.

The edge $e_{0,2^m-1}$ in Φ_m crosses with some edges in Δ_n when $0 < k < 2^m - 1 < *$. We find the number of k that correspond to the condition $0 < k < 2^m - 1 < *$. We let $K_0 = \{k \mid 0 < k < 2^m - 1 < *\}$ and we see that k can be $1, 2, 3, ..., 2^m - 2$, that is

$$\mid K_0 \mid = 3(2^m - 2). \tag{5.21}$$

The edge $e_{1,2^m-2}$ in Φ_m crosses with some edges in Δ_n when $1 < k < 2^m - 2 < *$. We find the number of k that correspond to the condition $1 < k < 2^m - 2 < *$. We let $K_1 = \{k \mid 1 < k < 2^m - 2 < *\}$ and we see that k can be $2, 3, 4, ..., 2^m - 3$, that is

$$\mid K_1 \mid = 3(2^m - 4). \tag{5.22}$$

The edge $e_{2,2^m-3}$ in Φ_m crosses with some edges in Δ_n when $2 < k < 2^m - 3 < *$. We find the number of k that correspond to the condition $2 < k < 2^m - 3 < *$. We let $K_2 = \{k \mid 2 < k < 2^m - 3 < *\}$ and we see that k can be $3, 4, 5, ..., 2^m - 4$, that is

$$\mid K_2 \mid = 3(2^m - 6). \tag{5.23}$$

$\vdots$

Similarly to the edge $e_{2^{m-1}-2,2^{m-1}+1}$ in Φ_m crosses with edges in Δ_n when $2^{m-1}-2<k<2^{m-1}+1<*$. We let $K_{2^{m-1}-2}=\{k\mid 2^{m-1}-2<k<2^{m-1}+1<*\}$. We see that k can be $2^{m-1}-1$ and 2^{m-1}, that is

$$\mid K_{2^{m-1}-2}\mid = 3(2). \tag{5.24}$$

Finally, for the edge $e_{2^{m-1}-1,2^{m-1}}$ in Φ_m, we can see that this edge can not cross with edges in Δ_n, So

$$\mid K_{2^{m-1}-1}\mid = 0. \tag{5.25}$$

From (5.21)-(5.25) and Lemma 5.2, we see that the number of each subgraph Φ_m of S_n^i, that equal to 2^{n-m}. That is the number of potential crossing between Φ_m and Δ_n depends on the number of subgraph Φ_m, so

$$\begin{aligned}
pc^{(n)}(\Delta_n,\Phi_m) &= 2^{n-m}(\mid K_0\mid+\mid K_1\mid+\mid K_2\mid+...+\mid K_{2^{m-1}-2}\mid+\mid K_{2^{m-1}-1}\mid)\\
&= 2^{n-m}(3(2^m-2)+3(2^m-4)+3(2^m-6)+...+3(2)+0)\\
&= 2^{n-m}(3[2^m-2+2^m-4+2^m-6+...+2+0])\\
&= 2^{n-m}(3\frac{(2^{m-1})}{2}(0+2^m-2)) \qquad (5.26)\\
&= 2^{n-m}(3(2^{m-1}-1)(2^{m-1}))\\
&= (2^{n-m+m-1})(3(2^{m-1}-1))\\
&= 3(2^{n-1})(2^{m-1}-1).
\end{aligned}$$

□

Theorem 5.8. *A graph Ω_n is called* ***Good Drawing of*** *Ω_n with the minimum number of a cross in Π_n under the following condition:*
(A) *If moving every edges in Φ_n to wrong side,i.e., moving Φ_n to Φ_{n-1} side, the number of a cross will increase.*
(B) *If moving every edges in Φ_{n-1} to wrong side,i.e., moving Φ_{n-1} to Φ_n side, the number of a cross will increase.*
(C) *If moving every edges in Φ_j where $2\le j\le n-2$ to wrong side, the number of a cross will increase.*

Proof. In order to prove, the approch that we use is we move all edges in Φ_j where $j=2,3,...,n$ to the *wrong side*. Then the number of a cross between Φ_j and other Φ's in the *correct side* will disappear. However, the number of a cross between Φ_j and other Φ's in the *wrong side* will appear, that is the number of a cross of graph S_n^i will decrease and increase. That is, we are proving that the decreased number of a cross of graph S_n^i is less than the increased number of a cross of graph S_n^i.

According to **(A)**, we can prove in 2 cases; n is even and n is odd. We prove when n is even. After all edges in Φ_n have been moved to the *wrong side*, we can observe that the number of a cross between Φ_n and Φ_{2j} has disappeared. Nevertheless, the number of a cross between Φ_n and Φ_{2j-1} will appear. Next we define Φ_{2j} as Φ_{n-2k} and Φ_{2j-1} as $\Phi_{n-(2k-1)}$ where n is even and $k = 1, 2, ..., \lfloor \frac{n-1}{2} \rfloor$. That is, it is sufficient to show that,

$$\sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} pc^{(n)}(\Phi_n, \Phi_{n-(2k-1)}) \quad > \quad \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} pc^{(n)}(\Phi_n, \Phi_{n-2k}).$$

We consider,

$$\begin{aligned}
&\sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} pc^{(n)}(\Phi_n, \Phi_{n-(2k-1)}) - \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} pc^{(n)}(\Phi_n, \Phi_{n-2k}) \\
&= pc^{(n)}(\Phi_n, \Phi_{n-1}) + pc^{(n)}(\Phi_n, \Phi_{n-3}) + ... + pc^{(n)}(\Phi_n, \Phi_{n-(2k-1)}) + ... \\
&\quad + pc^{(n)}(\Phi_n, \Phi_6) + pc^{(n)}(\Phi_n, \Phi_4) + pc^{(n)}(\Phi_n, \Phi_2) - pc^{(n)}(\Phi_n, \Phi_{n-2}) \\
&\quad - pc^{(n)}(\Phi_n, \Phi_{n-4}) - ... - pc^{(n)}(\Phi_n, \Phi_{n-2k}) - ... - pc^{(n)}(\Phi_n, \Phi_5) \\
&\quad - pc^{(n)}(\Phi_n, \Phi_3) - pc^{(n)}(\Phi_n, \Phi_1) \\
&= [pc^{(n)}(\Phi_n, \Phi_{n-1}) - pc^{(n)}(\Phi_n, \Phi_{n-2})] + [pc^{(n)}(\Phi_n, \Phi_{n-3}) - pc^{(n)}(\Phi_n, \Phi_{n-4})] \\
&\quad + ... + [pc^{(n)}(\Phi_n, \Phi_{n-(2k-1)}) - pc^{(n)}(\Phi_n, \Phi_{n-2k})] + ... \\
&\quad + [pc^{(n)}(\Phi_n, \Phi_6) - pc^{(n)}(\Phi_n, \Phi_5)] + [pc^{(n)}(\Phi_n, \Phi_4) - pc^{(n)}(\Phi_n, \Phi_3)] \\
&+ [pc^{(n)}(\Phi_n, \Phi_2) - pc^{(n)}(\Phi_n, \Phi_1)] \\
&> 0
\end{aligned}$$

by grouping and from Lemma 5.6.

Therefore, if we move Φ_n where n is even to the *wrong side*, the number of a cross which is in S_n^i will increase. In the case of proving n is odd, the method is similar.

According to **(B)**, the approch is similar to **(A)**. We only prove in case n is even. After all edges in Φ_{n-1} have been moved to the *wrong side*, we can observe that the number of a cross between Φ_n and Φ_{2j-1} has disappeared. Nevertheless, the number of a cross between Φ_{n-1} and Φ_{2j} will appear. Next we define Φ_{2j-1} as $\Phi_{n-(2k+1)}$ and Φ_{2j} as Φ_{n-2k} where n is even and

$k = 1, 2, ..., \lfloor \frac{n-2}{2} \rfloor$. That is, it is sufficient to show that,

$$\sum_{k=1}^{\lfloor \frac{n-2}{2} \rfloor} pc^{(n)}(\Phi_{n-1}, \Phi_{n-2k}) \quad > \quad \sum_{k=1}^{\lfloor \frac{n-2}{2} \rfloor} pc^{(n)}(\Phi_{n-1}, \Phi_{n-(2k+1)}).$$

We consider,

$$\begin{aligned}
&\sum_{k=1}^{\lfloor \frac{n-2}{2} \rfloor} pc^{(n)}(\Phi_{n-1}, \Phi_{n-2k}) - \sum_{k=1}^{\lfloor \frac{n-2}{2} \rfloor} pc^{(n)}(\Phi_{n-1}, \Phi_{n-(2k+1)}) \\
&= pc^{(n)}(\Phi_n, \Phi_{n-1}) + pc^{(n)}(\Phi_{n-1}, \Phi_{n-2}) + pc^{(n)}(\Phi_{n-1}, \Phi_{n-4}) + ... \\
&\quad + pc^{(n)}(\Phi_{n-1}, \Phi_{n-2k}) + ... + pc^{(n)}(\Phi_{n-1}, \Phi_6) + pc^{(n)}(\Phi_{n-1}, \Phi_4) \\
&\quad + pc^{(n)}(\Phi_{n-1}, \Phi_2) - pc^{(n)}(\Phi_{n-1}, \Phi_{n-3}) - pc^{(n)}(\Phi_{n-1}, \Phi_{n-5}) - ... \\
&\quad - pc^{(n)}(\Phi_{n-1}, \Phi_{n-(2k+1)}) - ... - pc^{(n)}(\Phi_{n-1}, \Phi_5) - pc^{(n)}(\Phi_{n-1}, \Phi_3) \\
&\quad - pc^{(n)}(\Phi_{n-1}, \Phi_1) \\
&= pc^{(n)}(\Phi_n, \Phi_{n-1}) + [pc^{(n)}(\Phi_{n-1}, \Phi_{n-2}) - pc^{(n)}(\Phi_{n-1}, \Phi_{n-3})] \\
&\quad + [pc^{(n)}(\Phi_{n-1}, \Phi_{n-4}) - pc^{(n)}(\Phi_{n-1}, \Phi_{n-5})] + ... + [pc^{(n)}(\Phi_{n-1}, \Phi_{n-2k}) \\
&\quad - pc^{(n)}(\Phi_{n-1}, \Phi_{n-(2k+1)})] + ... + [pc^{(n)}(\Phi_{n-1}, \Phi_6) - pc^{(n)}(\Phi_{n-1}, \Phi_5)] \\
&\quad + [pc^{(n)}(\Phi_{n-1}, \Phi_4) - pc^{(n)}(\Phi_{n-1}, \Phi_3)] + [pc^{(n)}(\Phi_{n-1}, \Phi_2) - pc^{(n)}(\Phi_{n-1}, \Phi_1)] \\
&> 0
\end{aligned}$$

by grouping and from Lemma 5.6.

Therefore, if we move Φ_{n-1} where n is even to the *wrong side*, the number of a cross which is in S_n^i will increase. In the case of proving n is odd, the method is similar.

Finally, for **(C)**, 4 cases can be proved as follows;

(i) n is odd and j is odd,
(ii) n is odd and j is even,
(iii) n is even and j is odd,
(iv) n is even and j is even.

First, we prove when n and j are odd. After all edges in Φ_j where j is odd and $1 < j < n$ have been moved to the *wrong side*, we can observe that the number of a cross between Φ_j and Φ_{2j-1} has disappeared. Nevertheless, the number of a cross between Φ_j and Φ_{2j} will appear.

That is, we are going to show that the inequality below is true.

$$\sum_{i=1,i-odd}^{n-j-1} pc^{(n)}(\Phi_{j+i},\Phi_j) + \sum_{i=1,i-odd}^{j-2} pc^{(n)}(\Phi_j,\Phi_{j-i})$$
$$> \sum_{i=2,i-even}^{n-j} pc^{(n)}(\Phi_{j+i},\Phi_j) + \sum_{i=2,i-even}^{j-1} pc^{(n)}(\Phi_j,\Phi_{j-i}).$$

We will consider,

$$\sum_{i=1,i-odd}^{n-j-1} pc^{(n)}(\Phi_{j+i},\Phi_j) + \sum_{i=1,i-odd}^{j-2} pc^{(n)}(\Phi_j,\Phi_{j-i})$$
$$- \sum_{i=2,i-even}^{n-j} pc^{(n)}(\Phi_{j+i},\Phi_j) - \sum_{i=2,i-even}^{j-1} pc^{(n)}(\Phi_j,\Phi_{j-i})$$
$$= pc^{(n)}(\Phi_{j+1},\Phi_j) + pc^{(n)}(\Phi_{j+3},\Phi_j) + pc^{(n)}(\Phi_{j+5},\Phi_j) + ... + pc^{(n)}(\Phi_{n-3},\Phi_j)$$
$$+ pc^{(n)}(\Phi_{n-1},\Phi_j) + pc^{(n)}(\Phi_j,\Phi_{j-1}) + pc^{(n)}(\Phi_j,\Phi_{j-3}) + pc^{(n)}(\Phi_j,\Phi_{j-5})$$
$$+ ... + pc^{(n)}(\Phi_j,\Phi_4) + pc^{(n)}(\Phi_j,\Phi_2) - pc^{(n)}(\Phi_{j+2},\Phi_j) - pc^{(n)}(\Phi_{j+4},\Phi_j)$$
$$- pc^{(n)}(\Phi_{j+6},\Phi_j) - ... - pc^{(n)}(\Phi_{n-2},\Phi_j) - pc^{(n)}(\Phi_n,\Phi_j) - pc^{(n)}(\Phi_j,\Phi_{j-2})$$
$$- pc^{(n)}(\Phi_j,\Phi_{j-4}) - pc^{(n)}(\Phi_j,\Phi_{j-6}) - ... - pc^{(n)}(\Phi_j,\Phi_3) - pc^{(n)}(\Phi_j,\Phi_1)$$
$$= pc^{(n)}(\Phi_{j+1},\Phi_j) + pc^{(n)}(\Phi_{j+3},\Phi_j) + pc^{(n)}(\Phi_{j+5},\Phi_j) + ... + pc^{(n)}(\Phi_{n-3},\Phi_j)$$
$$+ pc^{(n)}(\Phi_{n-1},\Phi_j) - pc^{(n)}(\Phi_{j+2},\Phi_j) - pc^{(n)}(\Phi_{j+4},\Phi_j) - pc^{(n)}(\Phi_{j+6},\Phi_j)$$
$$- ... - pc^{(n)}(\Phi_{n-2},\Phi_j) - pc^{(n)}(\Phi_n,\Phi_j) + [pc^{(n)}(\Phi_j,\Phi_{j-1}) - pc^{(n)}(\Phi_j,\Phi_{j-2})]$$
$$+ [pc^{(n)}(\Phi_j,\Phi_{j-3}) - pc^{(n)}(\Phi_j,\Phi_{j-4})] + [pc^{(n)}(\Phi_j,\Phi_{j-5}) - pc^{(n)}(\Phi_j,\Phi_{j-6})]$$
$$+ ... + [pc^{(n)}(\Phi_j,\Phi_4) - pc^{(n)}(\Phi_j,\Phi_3)] + [pc^{(n)}(\Phi_j,\Phi_2) - pc^{(n)}(\Phi_j,\Phi_1)]$$
$$> 0.$$

Next, we are going to case(ii), n is odd and j is even. After all edges in Φ_j where j is even and $1 < j < n-1$ have been moved to the *wrong side*, we can observe that the number of a cross between Φ_j and Φ_{2j} has disappeared. Nevertheless, the number of a cross between Φ_j and Φ_{2j-1} will appear.

That is, we are going to show that the inequality below is true.

$$\sum_{i=1,i-odd}^{n-j} pc^{(n)}(\Phi_{j+i},\Phi_j) + \sum_{i=1,i-odd}^{j-1} pc^{(n)}(\Phi_j,\Phi_{j-i})$$
$$> \sum_{i=2,i-even}^{n-j-1} pc^{(n)}(\Phi_{j+i},\Phi_j) + \sum_{i=2,i-even}^{j-2} pc^{(n)}(\Phi_j,\Phi_{j-i}).$$

We will consider,

$$\sum_{i=1,i-odd}^{n-j} pc^{(n)}(\Phi_{j+i},\Phi_j)+\sum_{i=1,i-odd}^{j-1} pc^{(n)}(\Phi_j,\Phi_{j-i})$$

$$-\sum_{i=2,i-even}^{n-j-1} pc^{(n)}(\Phi_{j+i},\Phi_j)-\sum_{i=2,i-even}^{j-2} pc^{(n)}(\Phi_j,\Phi_{j-i})$$

$$\begin{aligned}
&= pc^{(n)}(\Phi_{j+1},\Phi_j)+pc^{(n)}(\Phi_{j+3},\Phi_j)+pc^{(n)}(\Phi_{j+5},\Phi_j)+...+pc^{(n)}(\Phi_{n-2},\Phi_j)\\
&\quad +pc^{(n)}(\Phi_n,\Phi_j)+pc^{(n)}(\Phi_j,\Phi_{j-1})+pc^{(n)}(\Phi_j,\Phi_{j-3})+pc^{(n)}(\Phi_j,\Phi_{j-5})\\
&\quad +...+pc^{(n)}(\Phi_j,\Phi_5)+pc^{(n)}(\Phi_j,\Phi_3)+pc^{(n)}(\Phi_j,\Phi_1)-pc^{(n)}(\Phi_{j+2},\Phi_j)\\
&\quad -pc^{(n)}(\Phi_{j+4},\Phi_j)-pc^{(n)}(\Phi_{j+6},\Phi_j)-...-pc^{(n)}(\Phi_{n-3},\Phi_j)-pc^{(n)}(\Phi_{n-1},\Phi_j)\\
&\quad -pc^{(n)}(\Phi_j,\Phi_{j-2})-pc^{(n)}(\Phi_j,\Phi_{j-4})-pc^{(n)}(\Phi_j,\Phi_{j-6})-...-pc^{(n)}(\Phi_j,\Phi_4)\\
&\quad -pc^{(n)}(\Phi_j,\Phi_2)\\
&= pc^{(n)}(\Phi_{j+1},\Phi_j)+pc^{(n)}(\Phi_{j+3},\Phi_j)+pc^{(n)}(\Phi_{j+5},\Phi_j)+...+pc^{(n)}(\Phi_{n-2},\Phi_j)\\
&\quad +pc^{(n)}(\Phi_n,\Phi_j)-pc^{(n)}(\Phi_{j+2},\Phi_j)-pc^{(n)}(\Phi_{j+4},\Phi_j)-pc^{(n)}(\Phi_{j+6},\Phi_j)\\
&\quad -...-pc^{(n)}(\Phi_{n-3},\Phi_j)-pc^{(n)}(\Phi_{n-1},\Phi_j)+[pc^{(n)}(\Phi_j,\Phi_{j-1})-pc^{(n)}(\Phi_j,\Phi_{j-2})]\\
&\quad +[pc^{(n)}(\Phi_j,\Phi_{j-3})-pc^{(n)}(\Phi_j,\Phi_{j-4})]+[pc^{(n)}(\Phi_j,\Phi_{j-5})-pc^{(n)}(\Phi_j,\Phi_{j-6})]\\
&\quad +...+[pc^{(n)}(\Phi_j,\Phi_5)-pc^{(n)}(\Phi_j,\Phi_4)]+[pc^{(n)}(\Phi_j,\Phi_3)-pc^{(n)}(\Phi_j,\Phi_2)]\\
&\quad +pc^{(n)}(\Phi_j,\Phi_1)\\
&> 0.
\end{aligned}$$

□

6 Calculation of the Upper Bound for Crossing Number in Π_n

The reason why we mention to draw in $3-axes$ form is that we are able to notice that we have the same drawing from each *axis*. Therefore, the calculation of the upper bound for crossing number in Π_n is easier. That is, the result can be calculated from just one *axis*.

The upper bound for crossing number in Π_n can be calculated from a cross of graph Δ_n and graph Ω_n edges and graph Ω_n edges crossing themselves. We define $\mid \Delta_n cr\Omega_n \mid$ is the number of crosses for graph Δ_n cross graph Ω_n and $\mid cr\Omega_n \mid$ is the number of a cross in graph Ω_n.

Theorem 6.1. *For any integer* $n \geq 1$,

$$| \Delta_n cr\Omega_n | = 3(2^{n-1})(2^n - n - 1), \tag{6.1}$$

where $| \Delta_n cr\Omega_n |$ *is the number of crosses for graph* Δ_n *cross graph* Ω_n.

Proof. We prove by mathematical induction that, for all $n \in \mathbb{I}^+$. We precise by induction on n. For $n = 1, 2$, it can be easily seen $| \Delta_1 cr\Omega_1 | = 0$ and $| \Delta_2 cr\Omega_2 | = 6$ by directly counting. Assume $| \Delta_n cr\Omega_n |$ holds true. Now we consider $| \Delta_{n+1} cr\Omega_{n+1} |$ which is the number of crosses for graph Δ_{n+1} cross graph Ω_{n+1}.

The number of crosses for graph Δ_{n+1} cross graph Ω_{n+1} is calculated from the number of potential crossing between graph Δ_{n+1} and all subgraphs Φ_m in Ω_{n+1}, where $m = 1, 2, ..., n+1$. By Lemma 5.7,

$$\begin{aligned}
| \Delta_{n+1} cr\Omega_{n+1} | &= \sum_{m=1}^{n+1} pc^{(n+1)}(\Delta_{n+1}, \Phi_m) \\
&= pc^{(n+1)}(\Delta_{n+1}, \Phi_1) + pc^{(n+1)}(\Delta_{n+1}, \Phi_2) + ... \\
&\quad + pc^{(n+1)}(\Delta_{n+1}, \Phi_{n+1}) \\
&= 3(2^{(n+1)-1})(2^{1-1} - 1) + 3(2^{(n+1)-1})(2^{2-1} - 1) + ... \\
&\quad + 3(2^{(n+1)-1})(2^{n-1} - 1) + 3(2^{(n+1)-1})(2^{(n+1)-1} - 1) \\
&= 3(2^n)(2^0 - 1) + 3(2^n)(2^1 - 1) + ... + 3(2^n)(2^n - 1) \\
&= 3(2^n)[2^0 + 2^1 + 2^2 + ... + 2^{n-1} + 2^n - 1 - 1 - ... - 1] \\
&= 3(2^n)[\frac{2^0(2^{n+1} - 1)}{2 - 1} - (n+1)] \\
&= 3(2^{(n+1)-1})[2^{n+1} - (n+1) - 1].
\end{aligned}$$

□

Theorem 6.2. *For any integer* $n \geq 4$,

$$|cr\Omega_n| = \begin{cases} 3(2^{n-1})[\frac{16}{9}(2^{n-4} - 1) + \frac{8}{9}(2^{n-2} - 1) - \frac{3n-10}{3} - \frac{(n-2)^2}{4}], & n\text{-even}, \\ 3(2^{n-1})[\frac{24}{9}(2^{n-3} - 1) - n + 3 - \frac{(n-3)(n-1)}{4}], & n\text{-odd}, \end{cases} \tag{6.2}$$

where $| cr\Omega_n |$ *is the number of crosses in graph* Ω_n.

Proof. We start proving for case n is even, we let $n = 2k$ where $k \in \mathbb{I}^+$. We prove by mathematical induction on k, for all $k \in \mathbb{I}^+$. If $k = 1$, it can be easily seen $\mid cr\Omega_2 \mid = 0$. For $k = 2$, $\mid cr\Omega_4 \mid = 24$ by directly counting. Assume $\mid cr\Omega_{2k} \mid$ holds true, that is

$$|cr\Omega_{2k}| = 3(2^{2k-1})[\frac{16}{9}(2^{2k-4}-1)+\frac{8}{9}(2^{2k-2}-1)-\frac{3(2k)-10}{3}-\frac{(2k-2)^2}{4}].$$

We must show that $\mid cr\Omega_{2k} \mid$ implies $\mid cr\Omega_{2(k+1)} \mid$. Now we consider $\mid cr\Omega_{2(k+1)} \mid$ which is the number of crosses in graph $\Omega_{2(k+1)}$. $\mid cr\Omega_{2(k+1)} \mid$ is a result from the number of potential crossing which is created from crossing between all subgraphs Φ_i and Φ_j in $\Omega_{2(k+1)}$, where both of i and j are odd, including in case both of i and j are even. That is

$$\begin{aligned}
&|cr\Omega_{2(k+1)}| \\
&= 3[pc^{(2k+2)}(\Phi_3,\Phi_1) + pc^{(2k+2)}(\Phi_5,\Phi_1) + ... + pc^{(2k+2)}(\Phi_{2k+1},\Phi_1)] \\
&+ 3[pc^{(2k+2)}(\Phi_5,\Phi_3) + pc^{(2k+2)}(\Phi_7,\Phi_3) + ... + pc^{(2k+2)}(\Phi_{2k+1},\Phi_3)] \\
&+ 3[pc^{(2k+2)}(\Phi_7,\Phi_5) + pc^{(2k+2)}(\Phi_9,\Phi_5) + ... + pc^{(2k+2)}(\Phi_{2k+1},\Phi_5)] + ... \\
&+ 3[pc^{(2k+2)}(\Phi_{2k-1},\Phi_{2k-3}) + pc^{(2k+2)}(\Phi_{2k+1},\Phi_{2k-3}) + pc^{(2k+2)}(\Phi_{2k+1},\Phi_{2k-1})] \\
&+ 3[pc^{(2k+2)}(\Phi_4,\Phi_2) + pc^{(2k+2)}(\Phi_6,\Phi_2) + ... + pc^{(2k+2)}(\Phi_{2k+2},\Phi_2)] \\
&+ 3[pc^{(2k+2)}(\Phi_6,\Phi_4) + pc^{(2k+2)}(\Phi_8,\Phi_4) + ... + pc^{(2k+2)}(\Phi_{2k+2},\Phi_4)] \\
&+ 3[pc^{(2k+2)}(\Phi_8,\Phi_6) + pc^{(2k+2)}(\Phi_{10},\Phi_6) + ... + pc^{(2k+2)}(\Phi_{2k+2},\Phi_6)] + ... \\
&+ 3[pc^{(2k+2)}(\Phi_{2k},\Phi_{2k-2}) + pc^{(2k+2)}(\Phi_{2k+2},\Phi_{2k-2}) + pc^{(2k+2)}(\Phi_{2k+2},\Phi_{2k})] \\
&= 3(2^{2k+1})[(2^{1-1}-1)+(2^{1-1}-1)+(2^{1-1}-1)+...+(2^{1-1}-1)+(2^{1-1}-1)] \\
&+ 3(2^{2k+1})[(2^{3-1}-1)+(2^{3-1}-1)+(2^{3-1}-1)+...+(2^{3-1}-1)+(2^{3-1}-1)] \\
&+ 3(2^{2k+1})[(2^{5-1}-1)+(2^{5-1}-1)+(2^{5-1}-1)+...+(2^{5-1}-1)+(2^{5-1}-1)] \\
&+ ... + 3(2^{2k+1})[(2^{(2k-3)-1}-1)+(2^{(2k-3)-1}-1)+(2^{(2k-1)-1}-1)] \\
&+ 3(2^{2k+1})[(2^{2-1}-1)+(2^{2-1}-1)+(2^{2-1}-1)+...+(2^{2-1}-1)+(2^{2-1}-1)] \\
&+ 3(2^{2k+1})[(2^{4-1}-1)+(2^{4-1}-1)+(2^{4-1}-1)+...+(2^{4-1}-1)+(2^{4-1}-1)] \\
&+ 3(2^{2k+1})[(2^{6-1}-1)+(2^{6-1}-1)+(2^{6-1}-1)+...+(2^{6-1}-1)+(2^{6-1}-1)] \\
&+ ... + 3(2^{2k+1})[(2^{(2k-2)-1}-1)+(2^{(2k-2)-1}-1)] + 3(2^{2k+1})[(2^{(2k)-1}-1)] \\
&= 3(2^{2k+1})[0+0+...+0] \\
&+ 3(2^{2k+1})[(2^2+2^2+...+2^2)-(1+1+...+1)]
\end{aligned}$$

$+3(2^{2k+1})[(2^4+2^4+...+2^4)-(1+1+...+1)]$
$+...+3(2^{2k+1})[(2^{2k-4}+2^{2k-4})-(1+1)]+3(2^{2k+1})[(2^{2k-2}-1)]$
$+3(2^{2k+1})[(2^1+2^1+...+2^1)-(1+1+...+1)]$
$+3(2^{2k+1})[(2^3+2^3+...+2^3)-(1+1+...+1)]$
$+3(2^{2k+1})[(2^5+2^5+...+2^5)-(1+1+...+1)]$
$+...+3(2^{2k+1})[(2^{2k-3}+2^{2k-3})-(1+1)]+3(2^{2k+1})[(2^{2k-1}-1)]$

$=3(2^{2k+1})[2^2(k-1)-(k-1)]+3(2^{2k+1})[2^4(k-2)-(k-2)]$
$+...+3(2^{2k+1})[2^{2k-4}(2)-2]+3(2^{2k+1})[(2^{2k-2}(1)-1)]$
$+3(2^{2k+1})[2^1(k)-(k)]+3(2^{2k+1})[2^3(k-1)-(k-1)]$
$+...+3(2^{2k+1})[2^{2k-3}(2)-2]+3(2^{2k+1})[(2^{2k-1}(1)-1)]$

$=3(2^{2k+1})[(2^2(k-1)+2^4(k-2)+...+2^{2k-4}(2)+2^{2k-2}(1))$
$-((k-1)+(k-2)+...+2+1)]$
$+3(2^{2k+1})[(2^1(k)+2^3(k-1)+...+2^{2k-3}(2)+2^{2k-1}(1))$
$-((k)+(k-1)+...+2+1)]$

$=3(2^{2k+1})[\frac{16}{9}(2^{2k-2}-1)-\frac{4}{3}(k-1)-\frac{(k-1)(k)}{2}]$
$+3(2^{2k+1})[\frac{8}{9}(2^{2k}-1)-\frac{2}{3}(k)-\frac{(k)(k+1)}{2}]$

$=3(2^{2k+1})[\frac{16}{9}(2^{2k-2}-1)+\frac{8}{9}(2^{2k}-1)-(\frac{4}{3}(k-1)+\frac{2}{3}k)-(\frac{(k-1)k}{2}+\frac{k(k+1)}{2})]$

$=3(2^{2k+1})[\frac{16}{9}(2^{2k-2}-1)+\frac{8}{9}(2^{2k}-1)-\frac{4k-4+2k}{3}-\frac{k^2-k+k^2+k}{2}]$

$=3(2^{2k+1})[\frac{16}{9}(2^{2k-2}-1)+\frac{8}{9}(2^{2k}-1)-\frac{6k-4}{3}-k^2]$

$=3(2^{2k+1+1-1})[\frac{16}{9}(2^{2k-2+4-4}-1)+\frac{8}{9}(2^{2k+2-2}-1)-\frac{6k-4+10-10}{3}-(\frac{2k+2-2}{2})^2]$

$=3(2^{(2k+2)-1})[\frac{16}{9}(2^{(2k+2)-4}-1)+\frac{8}{9}(2^{(2k+2)-2}-1)-\frac{(6k+6)-10}{3}-(\frac{(2k+2)-2}{2})^2]$

$=3(2^{2(k+1)-1})[\frac{16}{9}(2^{2(k+1)-4}-1)+\frac{8}{9}(2^{2(k+1)-2}-1)-\frac{3(2(k+1))-10}{3}-\frac{(2(k+1)-2)^2}{4}].$

Hence the prove is complete. □

The result of the upper bound for crossing number in Π_n can be calculated from the Theorem 6.1 and 6.2. We present in the next theorem.

Theorem 6.3. *For any integer $n \geq 4$, the upper bound of crossing number of $Q_n \times K_3$ satisfies the inequality*

$$cr(Q_n \times K_3) \leq \begin{cases} 3(2^{n-1})[2^n + \frac{16}{9}(2^{n-4}-1) + \frac{8}{9}(2^{n-2}-1) - \frac{6n-7}{3} - \frac{(n-2)^2}{4}], & n\text{-even}, \\ 3(2^{n-1})[2^n + \frac{24}{9}(2^{n-3}-1) - 2n + 2 - \frac{(n-3)(n-1)}{4}], & n\text{-odd}. \end{cases} \tag{6.3}$$

7 Numerical Result

In this section, we consider the cartesian product of Q_n and K_3 for $n = 3, 4, ..., 12$. Then we give some results for the upper bound of crossing number of $Q_n \times K_3$ in the form (6.3) in the Table below.

n	$\mid \Delta_n cr\Omega_n \mid$	$\mid cr\Omega_n \mid$	Upper bound of $cr(Q_n \times K_3)$
3	48	0	48
4	264	24	288
5	1,248	192	1,440
6	5,472	1,152	6,624
7	23,040	5,760	28,800
8	94,848	26,496	121,344
9	385,536	115,200	500,736
10	1,555,968	485,376	2,041,344
11	6,254,592	2,003,200	8,257,792
12	25,085,952	8,165,376	33,251,328

Table 1: The upper bound for crossing number of $Q_n \times K_3$ for $n = 3, ..., 12$

8 Concluding Remarks

In this paper, we consider the cartesian product of Q_n and K_3. We present a drawing of $Q_n \times K_3$ called, a $3 - axes$ drawing of $Q_n \times K_3$ for finding the upper bound for the crossing number of $Q_n \times K_3$. Then we construct a graph Π_n and prove that Π_n is a good drawing which the minimum number of a cross in Π_n.

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