Thai Journal of Mathematics
Volume 15 (2017) Number 1 : 261–276

http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Best Proximity Point Theorems for G-Proximal Generalized Contraction in Complete Metric Spaces endowed with Graphs

Chalongchai Klanarong and Suthep Suantai[1]

Department of Mathematics, Faculty of Science, Chiang Mai University
Chiang Mai 50200, Thailand
e-mail : chalongchai1001@gmail.com (C. Klanarong)
suthep.s@cmu.ac.th (S. Suantai)

Abstract : In this work, we present the notion of a G-proximal generalized contraction which is a development of well known mappings by Banach, Kannan, Chatterjea from self mappings to non-self mappings and we prove best proximity point theorems for this mapping in a complete metric space endowed with a directed graph. Moreover, we apply our main theorems to prove an existence of coupled best proximity points in a complete metric space endowed with a directed graph.

Keywords : best proximity point; proximally G-edge preserving; G-proximal generalized contraction.
2010 Mathematics Subject Classification : 47H10; 46N10; 30L99.

1 Introduction and Preliminaries

In 1922, Stefan Banach [1] presented the notion of contractions and established the famous theorem which is called a *Banach contraction principle* or a *Banach fixed point theorem.*

Theorem 1.1. [1] *Let* (X, d) *be a complete metric space and a self mapping* $S : X \to X$ *be a contraction, that is, there exists a nonnegative real number* $k < 1$

[1]Corresponding author.

such that

$$d(Sx, Sy) \leq kd(x, y), \text{ for all } x, y \in X. \tag{1.1}$$

Then S has a unique fixed point in X, i.e., there exists $x \in X$ such that $Sx = x$.

The Banach contraction principle has been applied for solving the existence of solutions of various equations in many fields of analysis such as Applied Mathematics, Applied Sciences, Physics, Economics, etc. In fact, if S satisfies (1.1), then it is always forced continuity.

In 1968, Kannan [2] introduced the concept of a Kannan mapping which is another notion of contraction that need not be continuous as follows:

A mapping $S : X \to X$ is called a *Kannan mapping* if there exists a nonnegative real number $a < \frac{1}{2}$ such that

$$d(Sx, Sy) \leq a\big[d(x, Sx) + d(y, Sy)\big], \text{ for all } x, y \in X. \tag{1.2}$$

He proved the existence of a fixed point of the Kannan mapping in a complete metric space. Based on the condition (1.2), Chatterjea [3] introduced the concept of a *C-contraction mapping* as follows:

$$d(Sx, Sy) \leq a\big[d(x, Sy) + d(y, Sx)\big], \text{ for all } x, y \in X. \tag{1.3}$$

He proved that every mappings satisfy the condition (1.3) in a complete metric space have a unique fixed point. It can be seen in [4] that the conditions (1.1) and (1.2) are independent. Similarly, (1.1) and (1.3) are also independent. Some generalizations of Banach, Kannan, C-contractions were studied in [4–7].

The study and inspiration of literatures mentioned above, the purpose of this article is to study the best proximity point theorems of non-self mappings which is general than the mappings above. Let W and V be two nonempty subsets of a metric space (X, d) and let $S : W \to V$ a non-self mapping. Observe that the equation $Sx = x$ may not have a solution, if $W \cap V$ is nonempty. So, it is natural to ask that how far is the distance between x and Sx ? Therefore, the study of a best proximity point has played an important role and it is a problem of global optimization for determining the minimum valued of the distance $d(x, Sx) = \min\{d(x, y) : x \in W \text{ and } y \in V\}$.

In 1969, Fan [8] presented the first result concerning best proximity point theorems. He proved that if $S : W \to X$ is a continuous non-self mapping, where W is a nonempty compact convex subset in a normed vector space X, then there exists $w \in W$ such that $\|w - Sw\| = d(Sw, W)$ where $d(Sw, W) := \min\{\|Sw - a\| : a \in W\}$. Following the Fan's Theorem, best proximity point theorems of non-self mappings get a lot of attention and have been studied by many researchers. For more details about best proximity point theorems, see Kirk et al. [9], Reich [10], Polla [11], Sehgal and Singh [12, 13], Vetrivel et al. [14], Anuradha and Veeramani [15], Basha [16, 17], Basha and Veeramani [18], Eldred et al. [19], Eldred and Veeramani [20], Raj [21], Abkar and Gabeleh [22], and Gabeleh [23].

Throughout this article, we denote W and V are nonempty subsets of a metric space (X, d) and we also need the following notions:

$$d(W, V) := \inf\{d(x, y) \ : \ x \in W \text{ and } y \in V\},$$

$$W_0 := \{x \in W \ : \ d(x, y) = d(W, V) \text{ for some } y \in V\},$$

$$V_0 := \{y \in V \ : \ d(x, y) = d(W, V) \text{ for some } x \in W\}.$$

In 2011, Basha [16] gave the following definition of a proximal contraction for non-self mappings in a metric space:

Definition 1.2. [16] Let $S : W \to V$ be a non-self mapping. Then S is called a *a proximal contraction* if there exists $k \in [0, 1)$ and for every $u_1, u_2, x, y \in W$,

$$\left.\begin{aligned} d(u_1, Sx) = d(W, V) \\ d(u_2, Sy) = d(W, V) \end{aligned}\right\} \Longrightarrow d(u_1, u_2) \leq k d(x, y). \tag{1.4}$$

Inspired and motivated by the above works, in this article, we introduce the new concept of a G-proximal generalized contraction for non-self mappings and establish best proximity point theorems for a G-proximal generalized contraction in a complete metric space endowed with a directed graph. Moreover, we can apply our main results to prove an existence of coupled best proximity point in a complete metric space endowed with a directed graph. An example to support and explain our main result is also presented.

Next, we recall some mappings and notions regarding a graph.

Let (X, d) be a metric space and $G = (V(G), E(G))$ a directed graph which has no parallel edges such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges is a subset of $X \times X$. The conversion of a graph G denoted by G^{-1} *i.e.*,

$$E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}.$$

We start with the following definition:

Definition 1.3. Let (X, d) be a metric space and $G = (V(G), E(G))$ a directed graph such that $V(G) = X$. A non-self mapping $S : W \to V$ is called a *G-proximal Kannan mapping* if there exists $b \in [0, \frac{1}{2})$ such that

$$\left.\begin{aligned} &(x, y) \in E(G) \\ &d(u, Sx) = d(W, V) \\ &d(v, Sy) = d(W, V) \end{aligned}\right\} \Longrightarrow d(u, v) \leq b\big[d(x, v) + d(y, u)\big], \tag{1.5}$$

where $x, y, u, v \in W$.

Definition 1.4. Let (X, d) be a metric space and $G = (V(G), E(G))$ a directed graph such that $V(G) = X$. A non-self mapping $S : W \to V$ is said to be (*i*) *proximally G-edge-preserving* if for each $x, y, u, v \in W$,

$$\left.\begin{array}{l}(x,y)\in E(G)\\ d(u,Sx)=d(W,V)\\ d(v,Sy)=d(W,V)\end{array}\right\}\Longrightarrow (u,v)\in E(G);$$

(*ii*) *G-proximal generalized contraction* if there exists $k \in [0, 1)$ and each x, y, $u, v \in W$ such that

$$\left.\begin{array}{l}(x,y)\in E(G)\\ d(u,Sx)=d(W,V)\\ d(v,Sy)=d(W,V)\end{array}\right\}\Longrightarrow d(u,v)\le kM(x,y), \tag{1.6}$$

where $M(x, y) = \max\left\{d(x, y), d(x, u), d(y, v), \frac{d(x,v)+d(y,u)}{2}\right\}$.

From the Definition 1.4(*ii*), we observe that (1) S is said to be a *G-proximal contraction*, if $M(x, y) = d(x, y)$, and
(2) S is said to be a *G-proximal C-contraction*, if $M(x, y) = \frac{d(x,v)+d(y,u)}{2}$.

2 Main Results

In this section, we will prove best proximity point theorems for a G-proximal generalized contraction in a complete metric space endowed with a directed graph.

Theorem 2.1. *Let (X, d) be a complete metric space, $G = (V(G), E(G))$ a directed graph such that $V(G) = X$. Let W and V be nonempty closed subsets of X with W_0 is nonempty and let $S : W \to V$ be a non-self mapping which satisfies the following properties:*

(*i*) *S is proximally G-edge-preserving, continuous and G-proximal generalized contraction such that $S(W_0) \subseteq V_0$;*

(*ii*) *there exist $x_0, x_1 \in W_0$ such that*

$$d(x_1, Sx_0) = d(W, V) \text{ and } (x_0, x_1) \in E(G).$$

Then S has a best proximity point in W, that is, there exists an element $w \in W$ such that $d(w, Sw) = d(W, V)$.
Further, the sequence $\{x_n\}$, defined by

$$d(x_n, Sx_{n-1}) = d(W, V), \text{ for all } n \in \mathbb{N},$$

converges to the element w.

Proof. From the condition (ii), there exist $x_0, x_1 \in W_0$ such that

$$d(x_1, Sx_0) = d(W,V) \text{ and } (x_0, x_1) \in E(G). \tag{2.1}$$

Since $S(W_0) \subseteq V_0$, we have $Sx_1 \in V_0$ and hence there exits $x_2 \in W_0$ such that

$$d(x_2, Sx_1) = d(W,V). \tag{2.2}$$

By the proximally G-edge preserving of S and using both (2.1) and (2.2), we get $(x_1, x_2) \in E(G)$. By continuing this process, we can form the sequence $\{x_n\}$ in W_0 such that

$$d(x_n, Sx_{n-1}) = d(W,V) \text{ with } (x_{n-1}, x_n) \in E(G), \text{ for all } n \in \mathbb{N}. \tag{2.3}$$

Next, we will show that S has a best proximity point in W. Suppose that there exists $n_0 \in \mathbb{N}$, such that $x_{n_0} = x_{n_0+1}$. By using (2.3), we obtain that $d(x_{n_0}, Sx_{n_0}) = d(x_{n_0+1}, Sx_{n_0}) = d(W,V)$ and so x_{n_0} is a best proximity point of S. Now, Suppose that $x_{n-1} \neq x_n$, for all $n \in \mathbb{N}$. We show that $\{x_n\}$ is a Cauchy sequence in W. As S is G-proximal generalized contraction and for each $n \in \mathbb{N}$,

$$\left.\begin{array}{l} (x_{n-1}, x_n) \in E(G) \\ d(x_n, Sx_{n-1}) = d(W,V) \\ d(x_{n+1}, Sx_n) = d(W,V). \end{array}\right\}$$

Thus we have

$$d(x_n, x_{n+1}) \leq kM(x_{n-1}, x_n), \tag{2.4}$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\left\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \tfrac{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)}{2}\right\} \\ &= \max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1})}{2}\right\} \end{aligned} \tag{2.5}$$

Case 1. If $M(x_{n-1}, x_n) = d(x_{n-1}, x_n)$, for all $n \in \mathbb{N}$, then by (2.4) we obtain

$$d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n), \text{ for all } n \in \mathbb{N}.$$

By above inequality, we have

$$d(x_1, x_2) \leq kd(x_0, x_1),$$

and hence

$$d(x_2, x_3) \leq k^2 d(x_0, x_1).$$

By induction, we can conclude that

$$d(x_n, x_{n+1}) \leq k^n d(x_0, x_1), \text{ for all } n \in \mathbb{N}. \tag{2.6}$$

From (2.6), for each $m, n \in \mathbb{N}$ with $m > n$,

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{m-1}, x_m) \\ &\leq k^n d(x_0, x_1) + k^{n+1} d(x_0, x_1) + ... + k^{m-1} d(x_0, x_1) \\ &= d(x_0, x_1) \sum_{i=n}^{m-1} k^i \\ &\leq \frac{k^n}{1-k} d(x_0, x_1). \end{aligned}$$

Since $0 \leq k < 1$, it follows that $\{x_n\}$ is a Cauchy sequence in W.
Case 2. If $M(x_{n-1}, x_n) = d(x_n, x_{n+1})$, for all $n \in \mathbb{N}$, then by using (2.4), we have $d(x_n, x_{n+1}) \leq kd(x_n, x_{n+1})$, and hence $k = 1$ which is a contradiction.
Case 3. If $M(x_{n-1}, x_n) = \frac{d(x_{n-1}, x_{n+1})}{2}$, for all $n \in \mathbb{N}$, then by using (2.4), we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \frac{k}{2} d(x_{n-1}, x_{n+1}) \\ &\leq \frac{k}{2} \left[d(x_{n-1}, x_n) + d(x_n, x_{n+1}) \right]. \end{aligned}$$

It implies that

$$d(x_n, x_{n+1}) \leq \frac{k}{2-k} d(x_{n-1}, x_n). \tag{2.7}$$

By using the same method as in the Case 1 and $0 < \frac{k}{2-k} < 1$, we obtain that $\{x_n\}$ is a Cauchy sequence in W. Therefore, $\{x_n\}$ is a Cauchy sequence in W. Since W is closed, there exists $w \in W$ such that $x_n \to w$. By the continuing of S, we have $Sx_n \to Sw$ as $n \to \infty$. As the metric function is continuous, we obtain

$$d(x_{n+1}, Sx_n) \to d(w, Sw) \text{ as } n \to \infty.$$

Similarly, By (2.3) we can conclude that

$$d(w, Sw) = d(W, V).$$

This implies that $w \in W$ is a best proximity point of S.
Indeed, the sequence $\{x_n\}$ defined by

$$d(x_{n+1}, Sx_n) = d(W, V), \ n \in \mathbb{N},$$

converges to an element w. The proof is completed. □

Example 2.2. Let $X = \mathbb{R}^2$ equipped with the metric d given by

$$d((x, y), (u, v)) = \sqrt{(x-u)^2 + (y-v)^2}.$$

Let $W = \{(x,1) : 0 \leq x \leq 1\}$ and $V = \{(x,-1) : 0 \leq x \leq 1\} \cup \{(0,y) : -2 \leq y \leq -1\}$. It is easy to see that $d(W,V) = 2$, $W_0 = W$, $V_0 = \{(x,-1) : 0 \leq x \leq 1\}$, and W,V are closed subsets of X. Define a directed graph $G = (V(G), E(G))$ by $V(G) = X$ and

$$E(G) = \{((x,y),(u,v)) \in \mathbb{R}^2 \times \mathbb{R}^2 : x \leq u \text{ and } |y-v| \leq \frac{1}{2}\}.$$

Let $S : W \to V$ be a mapping defined by

$$S(x,1) = (\frac{x}{2}, -1), \text{ for all } (x,1) \in W.$$

Then S is continuous and $S(W_0) \subseteq V_0$.

We will show that S is both proximally G-edge preserving and G-proximal generalized contraction. Let $(x,1),(y,1) \in W$ such that

$$((x,1),(y,1)) \in E(G),\ d((u,1),S(x,1)) = d(W,V) = d((v,1),S(y,1)),$$

where $(u,1),(v,1) \in W$. Then

$$x \leq y,\ d((u,1),(\frac{x}{2},-1)) = 2 = d((v,1),(\frac{y}{2},-1)).$$

This implies that $u = \frac{x}{2}$ and $v = \frac{y}{2}$. Since $x \leq y$, it follows that. Thus $((u,1),(v,1)) \in E(G)$. We also note that for all $k \in [\frac{1}{2}, 1)$, we have

$$\begin{aligned}
d((u,1),(v,1)) &= \frac{1}{2}|x-y| \\
&\leq k|x-y| \\
&= kd((x,1),(y,1)) \\
&\leq k \max\Big\{ d((x,1),(y,1)),\ d((x,1),(u,1)),\ d((y,1),(v,1)), \\
&\qquad \frac{d((x,1),(v,1)) + d((y,1),(u,1))}{2} \Big\}.
\end{aligned}$$

Hence S is both a proximally G-edge preserving and a G-proximal generalized contraction. By Theorem 2.1 we can conclude that S has a best proximity point in A and $(0,1)$ is a proximity point of S.

Next, we will use the following property instead of continuity of S in Theorem 2.1 for proving the existence of a best proximity point.

Property (A) [24]. Let $\{x_n\}$ be any sequence in X, if $x_n \to x$, for some $x \in X$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, then there is a subsequence $\{x_{n_k}\}$ with $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$.

Theorem 2.3. *Suppose that all assumptions of Theorem 2.1 hold, except the continuity of S. In addition, suppose that X has the Property (A) and W_0 is* *closed. Then there exists an element $w \in W$ such that $d(w, Sw) = d(W, V)$. Further, the sequence $\{x_n\}$, defined by*

$$d(x_{n+1}, Sx_n) = d(W, V), \text{ for all } n \in \mathbb{N},$$

converges to the element w.

Proof. Following the proof of Theorem 2.1, there exists a sequence $\{x_n\}$ in W_0 satisfying

$$d(x_n, Sx_{n-1}) = d(W, V) \text{ with } (x_{n-1}, x_n) \in E(G), \text{ for all } n \in \mathbb{N}, \tag{2.8}$$

and $x_n \to u \in W$. Since W_0 is closed, we get $u \in W_0$. Again, by using (i) of Theorem 2.1, we have $S(W_0) \subseteq V_0$, so $Su \in V_0$. Then there exists $w \in W$ such that

$$d(w, Su) = d(W, V). \tag{2.9}$$

Since X has Property (A) and $(x_{n-1}, x_n) \in E(G)$, and $x_n \to u$ as $n \to \infty$, there exists a subsequence $\{x_{n_r}\}$ of $\{x_n\}$ such that $(x_{n_r}, u) \in E(G)$, for all $r \in \mathbb{N}$. Indeed, by using (2.8), (2.9), and S is a G-proximal generalized contraction, we get

$$d(x_{n_r+1}, w) \leq kM(x_{n_r}, u), \tag{2.10}$$

where

$$M(x_{n_r}, u) = \max\left\{d(x_{n_r}, u), d(x_{n_r}, x_{n_r+1}), d(u, w), \tfrac{d(x_{n_r}, w) + d(x_{n_r+1}, u)}{2}\right\}.$$

By taking the limit in the above inequality, we get

$$\lim_{r \to \infty} M(x_{n_r}, u) = d(u, w).$$

Suppose that $d(u, w) > 0$. From (2.10), we have

$$\lim_{r \to \infty} d(x_{n_r+1}, w) \leq kd(u, w).$$

Since $x_n \to u$ and $k \in [0, 1)$, we get

$$0 = \lim_{r \to \infty} [d(x_{n_r+1}, w) - d(u, w)] \leq (k - 1)d(u, w) < 0,$$

which is a contradiction. Hence $u = w$. Therefore there exists $w \in W$ such that $d(w, Sw) = d(W, V)$. The proof is completed. □

The following corollaries are obtained directly from Theorems 2.1 and 2.3.

Corollary 2.4. *Let (X, d) be a complete metric space, $G = (V(G), E(G))$ a directed graph such that $V(G) = X$. Let W and V be nonempty closed subsets of X with W_0 is nonempty and let $S : W \to V$ be proximally G-edge-preserving and G-proximal contraction such that $S(W_0) \subseteq V_0$. Assume that there exist $x_0, x_1 \in W_0$ such that*

$$d(x_1, Sx_0) = d(W, V) \text{ and } (x_0, x_1) \in E(G).$$

Suppose that either

(*i*) *S is continuous or*

(*ii*) *X has the Property (A) and W_0 is closed.*

Then there exists an element $w \in W$ such that $d(w, Sw) = d(W, V)$.
Further, the sequence $\{x_n\}$, defined by

$$d(x_n, Sx_{n-1}) = d(W, V), \text{ for all } n \in \mathbb{N},$$

converges to the element w.

Corollary 2.5. *Let (X, d) be a complete metric space, $G = (V(G), E(G))$ a directed graph such that $V(G) = X$. Let W and V be nonempty closed subsets of X with W_0 is nonempty and let $S : W \to V$ be proximally G-edge-preserving and a G-proximal Kannan mapping such that $S(W_0) \subseteq V_0$. Assume that there exist $x_0, x_1 \in W_0$ such that*

$$d(x_1, Sx_0) = d(W, V) \text{ and } (x_0, x_1) \in E(G).$$

Suppose that either

(*i*) *S is continuous or*

(*ii*) *X has the Property (A) and W_0 is closed.*

Then there exists an element $w \in W$ such that $d(w, Sw) = d(W, V)$.
Further, the sequence $\{x_n\}$, defined by

$$d(x_n, Sx_{n-1}) = d(W, V), \text{ for all } n \in \mathbb{N},$$

converges to the element w.

Corollary 2.6. *Let (X, d) be a complete metric space, $G = (V(G), E(G))$ a directed graph such that $V(G) = X$. Let W and V be nonempty closed subsets of X with W_0 is nonempty and let $S : W \to V$ be proximally G-edge-preserving and a G-proximal C-contraction such that $S(W_0) \subseteq V_0$. Assume that there exist $x_0, x_1 \in W_0$ such that*

$$d(x_1, Sx_0) = d(W, V) \text{ and } (x_0, x_1) \in E(G).$$

Suppose that either

(*i*) *S is continuous or*

(*ii*) *X has the Property (A) and W_0 is closed.*

Then there exists an element $w \in W$ such that $d(w, Sw) = d(W, V)$.
Further, the sequence $\{x_n\}$, defined by

$$d(x_n, Sx_{n-1}) = d(W, V), \text{ for all } n \in \mathbb{N},$$

converges to the element w.

3 Applications to Coupled Best Proximity Point Theorems

In this section, we prove the existence of a coupled best proximity point by applications of Theorems 2.1 and 2.3 in Section 2. Now, we recall some definitions and notions regarding coupled best proximity points in a complete metric space endowed with a directed graph.

Let W and V be nonempty subsets of any set X, $F : W \times W \to V$ a non-self mapping. An element $(x, y) \in W \times W$ is called *a coupled best proximity point* of F if $d(x, F(x, y)) = d(W, V)$ and $d(y, F(y, x)) = d(W, V)$. Recently, the coupled best proximity point theorems were investigated by many authors (see [25–27] and the references therein). We assume throughout this section that W and V are nonempty subsets of a metric space (X, d). We define a new mapping $\eta : Y \times Y \to [0, \infty)$ by

$$\eta((x, y),(u, v)) = d(x, u) + d(y, v), \text{ for all } (x, y),(u, v) \in Y, \tag{3.1}$$

where $Y = X \times X$.

It is easy to show that (X, d) is a metric space if and only if (Y, η) is a metric space. Moreover, we can prove that (X, d) is a complete metric space if and only if (Y, η) is a complete metric space. We set $W^* = W \times W$, $V^* = V \times V$, $W_0^* = W_0 \times W_0$, $V_0^* = V_0 \times V_0$ and the following notions are used in this section:

$$\eta(W^*, V^*) := \inf\{\eta(x, y) \ : \ x = (x_1, y_1) \in W^* \text{ and } y = (x_2, y_2) \in V^*\},$$

$$W_0^* := \{x = (x_1, y_1) \in W^* \ : \ \eta(x, y) = \eta(W^*, V^*) \text{ for some } y = (x_2, y_2) \in V^*\},$$

$$V_0^* := \{y = (x_2, y_2) \in V^* \ : \ \eta(x, y) = \eta(W^*, V^*) \text{ for some } x = (x_1, y_1) \in W^*\}.$$

Remark 3.1. *We have the following facts:*

(1) $\eta(W^*, V^*) = 2d(W, V)$.

(2) If $x = (x_1, y_1) \in W^*$ *and* $y = (x_2, y_2) \in V^*$ *such that* $\eta(x, y) = \eta(W^*, V^*)$, *then* $d(x_1, x_2) = d(y_1, y_2) = d(W, V)$.

For a non-self mapping $F : W \times W \to V$, we define the non-self mapping $S_F : W^* \to V^*$ by

$$S_F(x, y) = (F(x, y), F(y, x)) \text{ for all } (x, y) \in Y. \tag{3.2}$$

We note that an element $(x, y) \in W \times W$ is a coupled best proximity point of F if and only if (x, y) is a best proximity point of S_F.

Let (X, d) be a metric space and $G = (V(G), E(G))$ a directed graph which has no parallel edges such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges is a subset of $X \times X$. So, we define $G_Y = (V(G_Y), E(G_Y))$ such that $V(G_Y) = Y$ and $E(G_Y) = \{((x, y),(u, v)) \in Y \times Y :$

$(x,u) \in E(G)$ and $(y,v) \in E(G^{-1})\}$ where $Y = X \times X$. Hence G_Y is also a directed graph which has no parallel edges.

In 2014, Chifu and Petrusel [28] presented the concept of edge preserving as the following.

Definition 3.2. [28] We say that $F : X \times X \to X$ is *edge preserving* if $(x,u) \in E(G)$, $(y,v) \in E(G^{-1})$ implies $(F(x,y), F(u,v)) \in E(G)$ and $(F(y,x), F(v,u)) \in E(G^{-1})$.

Now, we give definition of a proximal mixed G-edge preserving for non-self mapping from the product space W^* into V as follows:

Definition 3.3. We say that $F : W^* \to V$ is a *proximally mixed G-edge preserving* if for each $x, y, u, v \in W$,

$$\left.\begin{array}{l}(x,u) \in E(G) \text{ and } (y,v) \in E(G^{-1})\\ d(u_1, F(x,y)) = d(W,V)\\ d(u_2, F(u,v)) = d(W,V)\end{array}\right\} \Longrightarrow (u_1,u_2) \in E(G),$$

and

$$\left.\begin{array}{l}(x,u) \in E(G) \text{ and } (y,v) \in E(G^{-1})\\ d(v_1, F(y,x)) = d(W,V)\\ d(v_2, F(v,u)) = d(W,V)\end{array}\right\} \Longrightarrow (v_1,v_2) \in E(G^{-1}).$$

Indeed, by taking $A = B = X$ in the above definition, the proximal mixed G-edge preserving reduces to edge preserving of Definition 3.2.

Theorem 3.4. *Let (X,d) be a complete metric space, $G = (V(G), E(G))$ a directed graph such that $V(G) = X$, and let W and V be two nonempty closed subsets of X such that W_0 is a nonempty subset of W. Let $F : W^* \to V$ is a mapping satisfying the following properties:*

(i) *F is proximally mixed G-edge preserving, continuous and $F(W_0^*) \subseteq V_0$;*

(ii) *there exist $(x_0,y_0),(x_1,y_1) \in W_0^*$ such that $d(x_1, F(x_0,y_0)) = d(W,V)$, $d(y_1, F(y_0,x_0)) = d(W,V)$, and $(x_0,x_1) \in E(G), (y_0,y_1) \in E(G^{-1})$;*

(iii) *there exists $k \in [0,1)$, for each $x, y, u, v, w_1, w_2, z_1, z_2 \in W_0$*

$$\left.\begin{array}{l}(x,u) \in E(G) \text{ and } (y,v) \in E(G^{-1})\\ d(w_1, F(x,y)) + d(z_1, F(y,x)) = 2d(W,V)\\ d(w_2, F(u,v)) + d(z_2, F(v,u)) = 2d(W,V)\end{array}\right\}$$

implies $d(w_1,w_2) + d(z_1,z_2) \le k \max\{d(x,u) + d(y,v), d(x,w_1) + d(y,z_1), d(u,w_2) + d(v,z_2), \frac{d(x,w_2)+d(y,z_2)+d(u,w_1)+d(v,z_1)}{2}\}$.

Then F has a coupled best proximity point in W^*, i.e., there exists an element $(x^*,y^*) \in W^*$ such that $d(x^*, F(x^*,y^*)) = d(W,V)$ and $d(y^*, F(y^*,x^*)) = d(W,V)$.

Proof. Let $Y = X \times X$. By using the mapping $\eta : Y \times Y \to [0, \infty)$ according the equation (3.1), we get (Y, η) is a complete metric space. Set $G_Y = (V(G_Y), E(G_Y))$ such that $V(G_Y) = Y$ and

$$E(G_Y) = \{((x, y), (u, v)) \in Y \times Y : (x, u) \in E(G) \text{ and } (y, v) \in E(G^{-1})\}.$$

Hence G_Y is a directed graph which has no parallel edges. Let $S_F : W^* \to V^*$ be a non-self mapping defined by (3.2). Since $F(W_0^*) \subseteq V_0$, we get $S_F(W_0^*) \subseteq V_0^*$. Now, we will show that S_F satisfies all conditions of Theorem 2.1. We start by the proving that S_F is a proximally G-edge preserving as follows: Let $(x, y), (u, v) \in Y$ such that

$$\begin{aligned}
&((x, y), (u, v)) \in E(G_Y),\\
&\eta((u_1, v_1), S_F(x, y)) = \eta(W^*, V^*),\\
&\eta((u_2, v_2), S_F(u, v)) = \eta(W^*, V^*).
\end{aligned}$$

By using the definition of S_F and $E(G_Y)$, we have

$$\begin{aligned}
&(x, u) \in E(G) \text{ and } (y, v) \in E(G^{-1}),\\
&d(u_1, F(x, y)) + d(v_1, F(y, x)) = 2d(W, V),\\
&d(u_2, F(u, v)) + d(v_2, F(v, u)) = 2d(W, V).
\end{aligned}$$

Using the Remark 3.1(2), we obtain

$$\begin{aligned}
&(x, u) \in E(G) \text{ and } (y, v) \in E(G^{-1}),\\
&d(u_1, F(x, y)) = 2d(W, V),\\
&d(u_2, F(u, v)) = 2d(W, V),
\end{aligned}$$

and

$$\begin{aligned}
&(x, u) \in E(G) \text{ and } (y, v) \in E(G^{-1}),\\
&d(v_1, F(y, x)) = 2d(W, V),\\
&d(v_2, F(v, u)) = 2d(W, V).
\end{aligned}$$

Since F is a proximally mixed G-edge preserving, we have $(u_1, u_2) \in E(G)$ and $(v_1, v_2) \in E(G^{-1})$. Again, by the definition of $E(G_Y)$ it follows that $((u_1, v_1), (u_2, v_2)) \in E(G_Y)$. Hence S_F is a proximally G-edge preserving. From the continuity of F, it is easy to show that S_F is also continuous. Next, from (ii) there exist $(x_0, y_0), (x_1, y_1) \in W_0^*$ such that $d(x_1, F(x_0, y_0)) = d(y_1, F(y_0, x_0)) = d(W, V)$ and $(x_0, x_1) \in E(G), (y_0, y_1) \in E(G^{-1})$. It means that $((x_0, y_0), (x_1, y_1)) \in E(G_Y)$ and

$$d(x_1, F(x_0, y_0)) + d(y_1, F(y_0, x_0)) = 2d(W, V).$$

It implies that

$$\eta((x_1, y_1), (F(x_0, y_0), F(y_0, x_0))) = \eta(W, V),$$

that is,

$$\eta((x_1,y_1),S_F(x_0,y_0)=\eta(W,V) \text{ and } ((x_0,y_0),(x_1,y_1))\in E(G_Y).$$

Thus S_F satisfies the condition (ii) of Theorem 2.1. Finally, we will show that S_F is a G-proximal generalized contraction: Let $(x_0,y_0),(x_1,y_1)\in W_0^*$ such that

$$\begin{aligned}&((x_0,y_0),(x_1,y_1))\in E(G_Y),\\ &\eta((u_1,v_1),S_F(x_0,y_0))=\eta(W^*,V^*),\\ &\eta((u_2,v_2),S_F(x_1,y_1))=\eta(W^*,V^*),\end{aligned}$$

where $(u_1,v_1),(u_2,v_2)\in W_0^*$. Hence

$$\begin{aligned}&(x_0,x_1)\in E(G) \text{ and } (y_0,y_1)\in E(G^{-1}),\\ &d(u_1,F(x_0,y_0))+d(v_1,F(y_0,x_0))=2d(W,V),\\ &d(u_2,F(x_1,y_1))+d(v_2,F(y_1,x_1))=2d(W,V).\end{aligned}$$

By (iii), we have

$$\begin{aligned}d(u_1,u_2)+d(v_1,v_2)\le k\max\{&d(x_0,x_1)+d(y_0,y_1),d(x_0,u_1)+d(y_0,v_1),\\ &d(x_1,u_2)+d(y_1,v_2),\tfrac{d(x_0,u_2)+d(y_0,v_2)+d(x_1,u_1)+d(y_1,v_1)}{2}\}.\end{aligned}$$

It means that

$$\begin{aligned}\eta((u_1,v_1),(u_2,v_2))\le k\max\{&\eta((x_0,y_0),(x_1,y_1)),\eta((x_0,y_0),(u_1,v_1)),\\ &\eta((x_1,y_1),(u_2,v_2)),\tfrac{\eta((x_0,y_0),(u_2,v_2))+\eta((x_1,y_1),(u_1,v_1))}{2}\}.\end{aligned}$$

Therefore all conditions of Theorem 2.1 are satisfied. Hence S_F has a best proximity point in W^*, that is, there exists $w^*=(x^*,y^*)\in W^*$ such that $\eta(w^*,S_F(w^*))=\eta(W^*,V^*)$. Obviously, $w^*=(x^*,y^*)$ is a coupled best proximity point of F. The proof is completed. □

Next, we will use the following property instead of continuity of F in Theorem 3.4 for proving the existence of a coupled best proximity point.

Property (B). Let $\{x_n\}$ and $\{y_n\}$ be sequences in X such that $\{x_n\}$ and $\{y_n\}$ have the following properties:

- if $x_n\to x$, for some $x\in X$ and $(x_n,x_{n+1})\in E(G)$, for all $n\in\mathbb{N}$, then $(x_n,x)\in E(G)$ for all $n\in\mathbb{N}$ and
- if $y_n\to y$, for some $y\in X$ and $(y_n,y_{n+1})\in E(G^{-1})$, for all $n\in\mathbb{N}$, then $(y_n,x)\in E(G^{-1})$ for all $n\in\mathbb{N}$.

Theorem 3.5. *Suppose that all assumptions of Theorem 3.4 hold, except the continuity of F. In addition, suppose that X has the Property (B) and W_0 is closed. Then F has a coupled best proximity point in W^*, i.e., there exists an element $(x^*,y^*)\in W^*$ such that $d(x^*,F(x^*,y^*))=d(W,V)$ and $d(y^*,F(y^*,x^*))=d(W,V)$.*

Proof. Let Y, G_Y, and both mappings η and S_F be as in the proof of Theorem 3.4. Then it is sufficient to prove that Y has the Property (A). Let (x_n, y_n) be a sequence in Y with $(x_n, y_n) \to (x, y)$, for some $(x, y) \in Y$ as $n \to \infty$ and $((x_n, y_n), (x_{n+1}, y_{n+1})) \in E(G_Y)$, for all $n \in \mathbb{N}$. Then we get $x_n \to x$, $y_n \to y$ as $n \to \infty$, and $(x_n, x_{n+1}) \in E(G)$, $(y_n, y_{n+1}) \in E(G^{-1})$ for all $n \in \mathbb{N}$. Since X has the Property (B), we have $(x_n, x) \in E(G)$ and $(y_n, y) \in E(G^{-1})$. It implies that $((x_n, y_n), (x, y)) \in E(G_Y)$, for all $n \in \mathbb{N}$. Therefore Y has Property (A). By using Theorem 2.3, we obtain that S_F has a best proximity point in W^*, that is, there exists $w^* = (x^*, y^*) \in W^*$ such that $\eta(w^*, S_F(w^*)) = \eta(W^*, V^*)$ or $w^* = (x^*, y^*)$ is coupled best proximity point of F. □

Acknowledgements : The authors would like to thank the referees for many comments and suggestions to improve the exposition of this paper and the Thailand Research Fund under the project RTA5780007 and Chiang Mai University, Chiang Mai, Thailand for the financial support.

References

[1] S. Banach, Sur les operations dans les ensembles abstraits et leur application aux equations itegrales, Fundam. Math. 3 (1922) 133-181.

[2] R. Kannan, Some results on fixed points, Bull. Calcutta Math. Soc. 10 (1968) 71-76.

[3] S.K. Chatterjea, Fixed point theorems, C.R. Acad. Bulgare Sci. 25 (1972) 727-730.

[4] B.E. Rhoades, Fixed point iterations using infinite matrices Trans. Amer. Math. Soc. 196 (1974) 161-176.

[5] T. Zamfirescu, Fixed point theorems in metric spaces Arch. Math. (Basel), 23 (1972) 292-298.

[6] F. Bojor, Fixed point theorems for Reich type contraction on metric spaces with a graph, Nonlinear Anal. 75 (2012) 3895-3901.

[7] L.B. Ciric, A generalization of Banachs contraction principle, Proc. Am. Math. Soc. 45 (1974) 267-273.

[8] K. Fan, Extensions of two fixed point theorems of F.E. Browder, Math. Z. 122 (1969) 234-240.

[9] W.A. Kirk, S. Reich, P. Veeramani, Proximinal retracts and best proximity pair theorems, Numer. Funct. Anal. Optim. 24 (7-8) (2003) 851-862.

[10] S. Reich, Approximate selections, best approximations, fixed points, and invariant sets, J. Math. Anal. Appl. 62 (1) (1978) 104-113.

[11] J.B. Prolla, Fixed point theorems for set-valued mappings and existence of best approximants, Numer. Funct. Anal. Optim. 5 (4) (1983) 449-455.

[12] V.M. Sehgal, S.P. Singh, A generalization to multifunctions of Fan's best approximation theorem, Proc. Am. Math. Soc. 102 (3) (1988) 534-537.

[13] V.M. Sehgal, S.P. Singh, A theorem on best approximations, Numer. Funct. Anal. Optim. 10 (1-2) (1989) 181-184.

[14] V. Vetrivel, P. Veeramani, P. Bhattacharyya, Some extensions of Fan's best approximation theorem, Numer. Funct. Anal. Optim. 13 (3-4) (1992) 397-402.

[15] J. Anuradha, P. Veeramani, Proximal pointwise contraction, Topol. Appl. 156 (18) (2009) 2942-2948 doi:10.1016/j.topol.2009.01.017.

[16] S.S. Basha, Best proximity points: optimal solutions, J. Optim. Theory Appl. 151 (1) (2011) 210-216.

[17] S.S. Basha, Best proximity point theorems, J. Approx. Theory 163 (11) (2011) 1772-1781.

[18] S.S. Basha, P. Veeramani, Best proximity pair theorems for multifunctions with open fibres, J. Approx. Theory 103 (1) (2000) 119-129.

[19] A.A. Eldred, P. Veeramani, Existence and convergence of best proximity points, J. Math. Anal. Appl. 323 (2) (2006) 1001-1006.

[20] A. Eldred, P. Veeramani, Existence and convergence of best proximity points, J. Math. Anal. Appl. 323 (2006) 1001-1006.

[21] V.S. Raj, A best proximity point theorem for weakly contractive non-self-mappings, Nonlinear Anal Theory Methods Appl. 74 (14) (2011) 4804-4808.

[22] A. Abkar, M. Gabeleh, Best proximity points of non-self mappings, Top 21 (2) (2013) 287-295.

[23] M. Gabeleh, Global optimal solutions of non-self mappings, U.P.B. Sci. Bull., Series A, 75 (3) (2013) 67-74.

[24] J. Jachymski, The contraction principle for mappings on a metric with a graph, Proc. Am. Math. Soc. 139 (2008) 1359-1373.

[25] P. Kumam, A.H. Ansari, K. Sitthithakerngkiet, Coupled best proximity points under the proximally coupled contraction in a complete ordered metric space, Comput. Math. Appl. 7 (3) (2016) 275289.

[26] W. Sintunavarat, P. Kumam, Coupled best proximity point theorem in metric spaces. Fixed Point Theory Appl. (2012) DOI: 10.1186/1687-1812-2012-93.

[27] A. Gupta, S.S. Rajput and P.S. Kaurav, Coupled best proximity point theorem in metric spaces, International Journal of Analysis and Applications 4 (2) (2014) 201-215.

[28] C. Chifu, G. Petrusel, New results on coupled fixed point theory in metric spaces endowed with a directed graph. Fixed Point Theory Appl. (2014) DOI: 10.1186/1687-1812-2014-151.

(Received 18 February 2017)
(Accepted 29 April 2017)