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Minimal enclosures by rectangles for few regions of given areas

B. Sroysang and W. Wichiramala

Abstract : We find minimal enclosures by rectangles for two and three regions of given areas. We show that each minimizer has connected regions and has shape depending on ratio of areas.

Keywords : enclosing by rectangles, minimizing enclosures
2000 Mathematics Subject Classification : 49Q10

1 Introduction

The planar soap bubble problem deals with the least-perimeter way to enclose and separate m regions of m given areas. Rigorously, speaking, an enclosure for m regions of areas $A_1, \ldots, A_m$ is a closed subset E of $\mathbb{R}^2$ for which (1) its complement is composed of $m+1$ regions $R_1, \ldots, R_m$, and R_0, (2) each region is a union of open (connected) components, and (3) for each $i = 1, \ldots, m$, the region R_i has area A_i and $R_0 = (\mathbb{R}^2 \setminus E) \setminus \cup_{i=1}^{m} R_i$. The region R_0 is called the exterior region. We define the length of an enclosure to be its one-dimensional measure. The existence of the solutions to this problem is guaranteed by Morgan [3]. Together with Bleicher's work [1], the regularity of solutions is well studied. A minimizing enclosure is composed of finitely many circular arcs or straight segments meeting in threes at 120 degree angles. Although it is natural to believe that each region, including the exterior region, is connected, there is no proof for the case $m \geq 4$. For a single area ($m = 1$), the unique solution is a circle. This has just been proved in 1880. For two areas, Foisy et al showed that the standard double bubble is the unique solution (see Figure 1) [2]. In 2002, Wichiramala proved that the standard triple bubble is the unique minimizing enclosure for three areas (see Figure 2) [7, 6]. The problem is still open for $m \geq 4$.

Minimal enclosing of two areas are studied on many surfaces, e.g. flat torus, cylinder, and cone. In 1998, Morgan, French and Greenleaf proved that the minimizing enclosure for two areas by vertical and horizontal segments is one of those in Figure 3 , depending on the ratio of the two areas. For a single area, the obvious solution is a square.

In this work, we study minimal enclosures of two and three given areas by

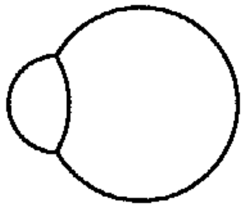

Figure 1: The standard double bubble.

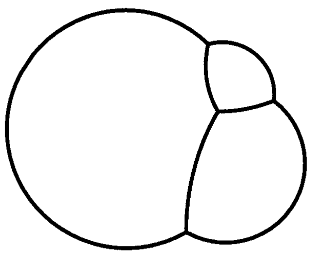

Figure 2: The standard triple bubble.

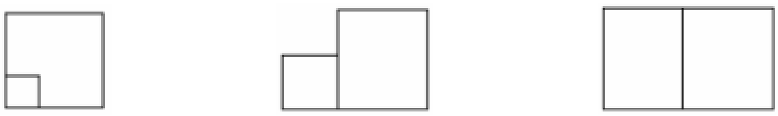

Figure 3: Minimal enclosures for two areas by vertical and horizontal segments.

rectangles. In other words, we only consider enclosures for two or three regions whose components are rectangles. Figure 4 shows an enclosure E for two regions, light gray and dark gray. Each region is a union of several components. Both regions and the exterior region are disconnected and E is not path connected. For the case of two areas, progress has been made by Morgan et. al. [4], but the problem is not yet fully solved. The reason is that the minimal enclosures are not necessary composed of only rectangles. The technique we used here provides a simpler proof of the main result in [4] (see [5].) However, there are too many complications in studying the case of three given areas enclosed by vertical and horizontal segments.

2 Elementary

In our figures, capital letters inside rectangles denote their areas and side lengths are indicated by lower-case letters or expressions. Lengths of dotted segments will not be considered. For clarification, we break up our result into many lemmas.

Lemma 2.1. *For $A > 0$, we have*
(1)Figure 5(a) and (c) have minimum length if and only if $x = \sqrt{A}$.
(2)Figure 5(b) has minimum length if and only if $x = \sqrt{2A}$.

Proof. The proof is omitted. □

Lemma 2.2. *For $0 < A \leq B$ and $w > 0$, we have*
(1) if $A < B$, then Figure 6(c) has more length than Figure 6(a) for some b.
(2) if $2A < B$, then Figure 6(b) has more length than Figure 6(a) for some b.
(3) if $2A \geq B$, then Figure 6(a) has more length than Figure 6(b) for some b.

Proof. (1) Assume than $A < B$. Choose $b = c$. From Figure 6(c), we have $bw = cw > A + B$. Then $bwB - bwA > B^2 - A^2$. Hence $bwB - B^2 > bwA - A^2$. Thus $\frac{B}{bw-A} > \frac{A}{bw-B}$. Therefore Figure 6(a) has less length than Figure 6(c).
(2) Assume that $2A < B$. Choose $b = \frac{6B(A+B)}{w(2A+5B)}$. Therefore the length of Figure 6(a) is $w + \frac{3}{w}(\frac{-(2A+B)^2(A+B)}{(2A+5B)(4A+B)} + A + B) < w + \frac{3}{w}(A + B)$, which is the length of Figure 6(b).
(3) Assume that $2A \geq B$. From Figure 6(a), we have $0 < a < \frac{A+B}{w}$ and $\frac{a}{A} < w$. Then $\frac{A}{a} > \frac{A+B}{3a} > \frac{w}{3}$. Let $t = \frac{A+B}{w} - a$. Hence $3(\frac{A+B}{w}) = 3t + 3a = 2(\frac{\frac{w}{3}}{w-\frac{w}{3}})t + 2t + 3a < 2(\frac{\frac{A}{a}}{w-\frac{A}{a}})t + 2t + 3a = a + 2\frac{B}{w-\frac{A}{a}}$. Therefore Figure 6(b) has less length than Figure 6(a). □

Lemma 2.3. *For $0 < A_1 \leq A_2$, we have*
(1) if $A_1 = A_2$ and both rectangles are squares, then Figure 7(a,b,c) have the same length.
(2) otherwise, Figure 7(a) has more length than Figure 7(b) or (c).

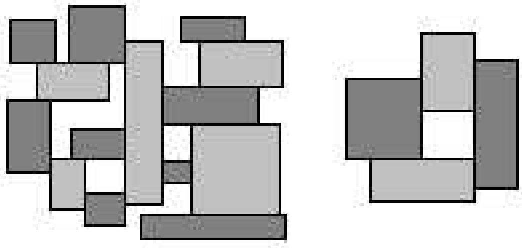

Figure 4: An enclosure for two regions, each of which consists of several components.

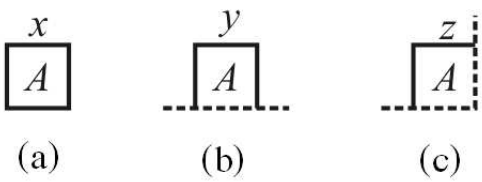

Figure 5: Rectangles on the plane, a half plane, and in a right-angle corner.

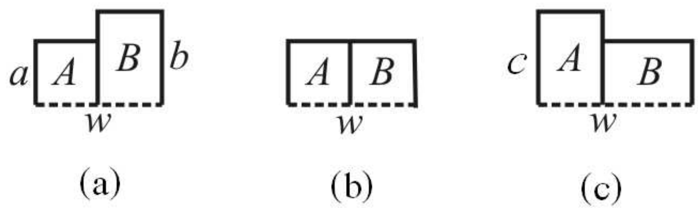

Figure 6: Double rectangles on a wall of length w.

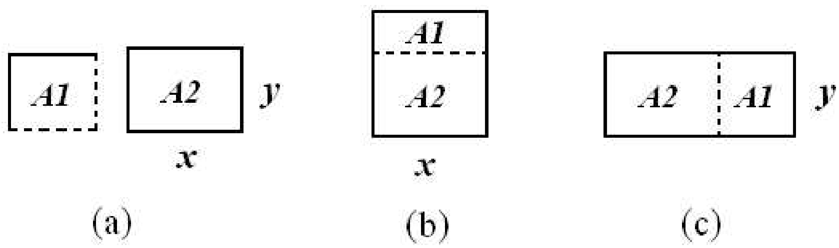

Figure 7: The smaller rectangle is attached to the bigger rectangle.

Proof. (1) It is obvious.
(2) We may assume that $x \geq y$. First we will consider the case that $x > y$ or that $x = y$ and $A_1 < A_2$. Then $x\sqrt{A_1} > A_1 = A_1 + A_2 - xy$. Figure 7(a) has length less than or equal to $2\sqrt{A_1} + 2x + 2y < 2(\frac{A_1+A_2}{x} - y) + 2x + 2y$ which is the length of Figure 7(b). The other case where $x = y$ and $A_1 = A_2$ (and rectangle A_1 is not square) is similar. □

3 Enclosures for two areas

In this section, we show that a minimizing enclosure for two areas is described by Figure 8. First we will show in the next theorem that each region of a minimizing enclosure must be a single rectangle.

Theorem 3.1. *A minimizing enclosure for two areas has connected regions.*

Proof. Suppose that a minimizing enclosure E has some disconnected region. By translation, we can assume that E is (path) connected. For region R_i, $i = 1, 2$, we choose a component with maximum area C_i. These two maximum components may not share any part of their boundaries. We now consider Figure 9. First, for each rectangular component, we remove its lower segment(s) and right segment(s) which are not parts of the upper side or the lower side of another component. For each region R_i, we combine its components, except C_i, to a single rectangle as illustrated in Figure 10. This process reduces total length and still works when there are infinitely many components since A_i is finite. If it is possible, we translate C_1 vertically or horizontally to touch C_2 The total length of removed segments is enough to fill up the boundary of the paired C_1 and C_2. Next we use Lemma 2.3, for each region, to add the combined rectangle into the maximum component. Finally, we create an enclosure with connected regions and less length, a contradiction. Therefore every region must be connected. □

The previous proof can be adapted to solve the problem of finding minimal enclosures by vertical and horizontal segments. In [5], we provide a simpler proof to the one in [4]. The key point is that we allow areas to increase and then show later that areas should be decrease back to the original ones. This approach is called the weak approach as described in [7] and [6].

Theorem 3.2. *For $0 < A \leq B$, among enclosures of areas A and B with connected regions,*
(1) if $2A < B$, then Figure 11(a) is minimizing.
(2) if $2A \geq B$, then Figure 11(b) is minimizing.

Proof. The result comes from elementary calculations. □

By the previous theorem, we know the shapes of minimizing enclosures for two areas. Note that the corresponding shapes depend on ratio of the areas.

Figure 8: Minimal enclosures for two area by rectangles.

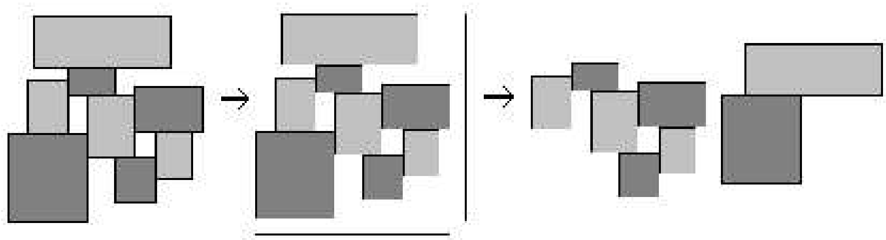

Figure 9: The process of pairing the two maximum components.

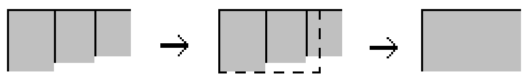

Figure 10: The process of combining components of a region to a single piece.

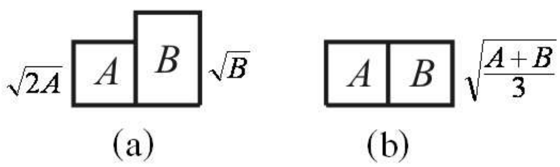

Figure 11: Minimal enclosures of connected regions.

4 Enclosures for three areas

In this section, we show that a minimizing enclosure for three areas is described by Figure 12. First we will show in the next theorem that each region of a minimizing enclosure must be a single rectangle.

Theorem 4.1. *A minimizing enclosures for three areas has connected regions.*

Proof. Argument for two areas also work here. □

Proposition 4.2. *Each enclosure in Figure 13 has more length than some enclosure in Figure 12.*

Proof. Figure 13(r) has more length than some enclosure in Figure 12. Each enclosure in Figure 13(m-q) has more length than Figure 13(r). Each enclosure in Figure 13(a,d,g,j) has more length than Figure 12(b) or (e). Each enclosure in Figure 13(b,h,f,l) has more length than Figure 12(c) or (f). Each enclosure in Figure 13(c,e,i,k) has more length than Figure 12 (a), (b), (c), (f) or Figure 13(r). □

Theorem 4.3. *For $0 < A \leq B \leq C$, a minimizing enclosure of areas A, B, C is as follows.*
(1) For $2A < B$, we have
(1.1) if $\sqrt{C} < \frac{\sqrt{A}+\sqrt{2A}}{2}$, then Figure 12(a) is minimizing where $a_1 = \sqrt{2A}$, $b_1 = \sqrt{B}$, $c_1 = \sqrt{2C}$.
(1.2) if $\sqrt{c} > \sqrt{A}+\sqrt{2B}$, then Figure 12(b) is minimizing where $a_2 = \sqrt{A}, b_2 = \sqrt{2B}, c_2 = \sqrt{C}$.
(1.3) if $\frac{\sqrt{A}+\sqrt{2B}}{2} \leq \sqrt{C} \leq \sqrt{A} + \sqrt{2B}$, then Figure 12(c) is minimizing where $a_3 = \frac{\sqrt{\frac{A}{3}((\sqrt{A}+\sqrt{2B})^2+2C)}}{\sqrt{A}+\sqrt{2B}}, b_3 = a_3\sqrt{\frac{2B}{A}}$.
(2) For $2A \geq B$, we have
(2.1) if $C < \frac{3}{4}(A+B)$ then Figure 12(d) is minimizing where $a_4 = \sqrt{\frac{2}{3}(A+B)}$, $c_4 = \sqrt{2C}$.
(2.2) if $C > 3(A+B)$ then Figure 12(e) is minimizing where $a_5 = \sqrt{\frac{A+B}{3}}$, $c_5 = \sqrt{C}$.
(2.3) if $\frac{3}{4}(A+B) \leq C \leq 3(A+B)$ then Figure 12(f) is minimizing where $a_6 = \frac{A+B}{\sqrt{A+B+\frac{2}{3}C}}$.

Proof. The result comes from elementary calculations. □

In conclusion, the shapes of minimizing enclosures depends on ratio of the areas as described in Figure 14. Figure 15 shows when each of the six patterns is minimizing.

We believe that our technique can be used to shed some light to the problem of finding minimizing enclosures by vertical and horizontal segments for three regions of given areas.

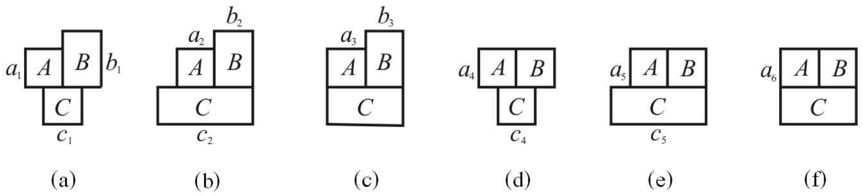

Figure 12: Six patterns of minimizing enclosures for three areas.

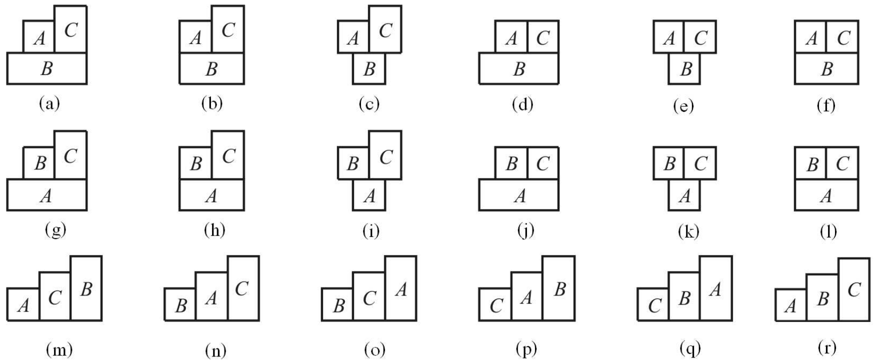

Figure 13: All possible enclosures that may be minimizing for three areas.

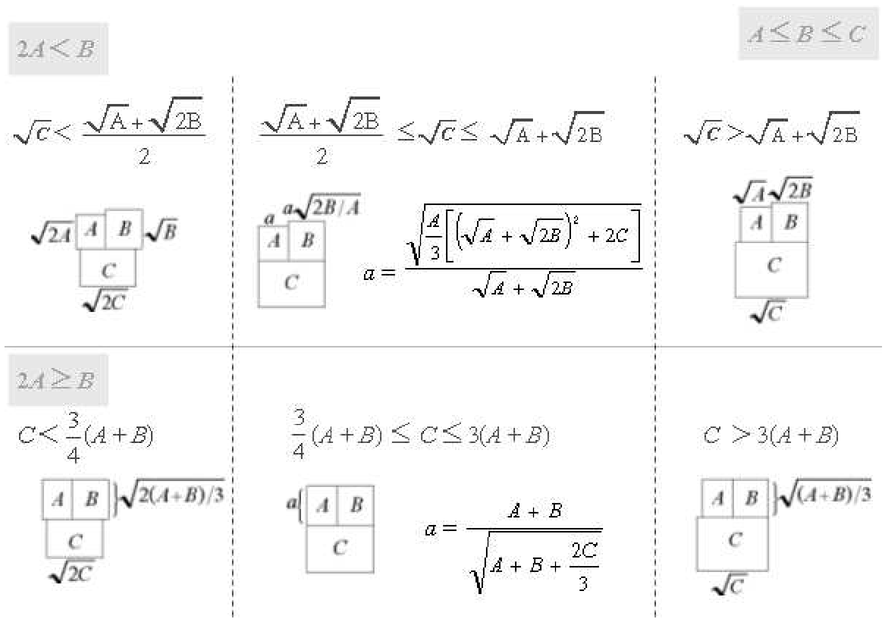

Figure 14: Table indicating shapes of minimizing enclosures for three areas.

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B. Sroysang and W. Wichiramala
Department of Mathematics
Faculty of Science
Chulalongkorn University
Bangkok 10330, Thailand.
e-mail: `wacharin.w@chula.ac.th`

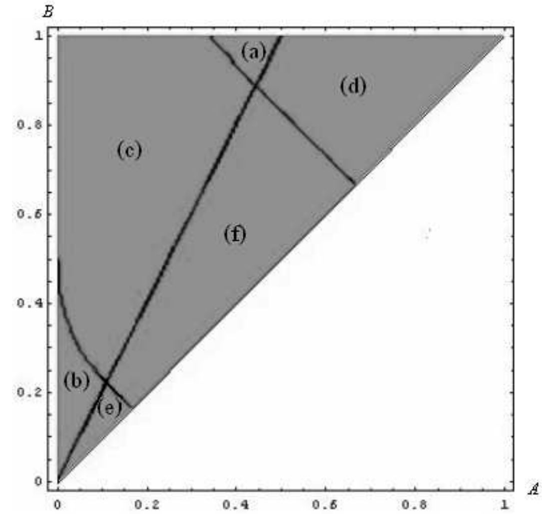

Figure 15: Ratio of the three areas $A \leq B \leq C = 1$ indicating shapes of minimizing enclosures.