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Generalized Fuzzy Ideals in Semigroups

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Abstract : The notion of $(\widetilde{\alpha}, \widetilde{\beta})$-fuzzy bi-ideals in semigroups is introduced, and related properties are investigated. Characterizations of $(\in, \in \vee\, q_0^\delta)$-generalized fuzzy bi-ideals are provided.

Keywords : $(\widetilde{\alpha}, \widetilde{\beta})$-fuzzy subsemigroup; $(\widetilde{\alpha}, \widetilde{\beta})$-fuzzy generalized bi-ideal.
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1 Introduction

The idea of quasicoincidence of a fuzzy point with a fuzzy set, which is mentioned in [1], played a vital role to generate some different types of fuzzy subgroups. It is worth pointing out that Bhakat and Das [2, 3] gave the concepts of (α, β)-fuzzy subgroups by using the "belongs to" relation $(\in)$ and "quasi-coincident with" relation (q) between a fuzzy point and a fuzzy subgroup, and introduced the concept of an $(\in, \in \vee\, q)$-fuzzy subgroup. In particular, $(\in, \in \vee\, q)$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. Bhakat et al. applied the $(\in, \in \vee\, q)$-fuzzy type to group and ring (see [2, 3, 4, 5]). Dudek et al. [6] characterized different types of (α, β)-fuzzy ideals of hemirings. In [7] Jun and Song initiated the study of (α, β)-fuzzy interior ideals of a semigroup. In [8] Kazanci and Yamak study $(\in, \in \vee\, q)$-fuzzy bi-ideals of a semigroup. Shabir et al. [9] introduced the concept of (α, β)-fuzzy ideal, (α, β)-fuzzy generalized bi-ideal,

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and characterized regular semigroups by the properties of these ideals. Jun et al. [10] considered more general form of quasi-coincident fuzzy point, and they [11] introduced the notions of $(\widetilde{\alpha}, \widetilde{\beta})$-fuzzy subsemigroups in semigroups, and investigate related properties. They provided characterizations of $(\in, \in\vee q_0^\delta)$-fuzzy subsemigroups, and considered a condition for an $(\in, \in\vee q_0^\delta)$-fuzzy subsemigroup to be an $(\in, \in)$-fuzzy subsemigroup. Given a fuzzy set with finite images, they established an $(\in, \in\vee q_0^\delta)$-fuzzy subsemigroup generated by the given fuzzy set. Yuan et al. [12] provided a generalization of fuzzy subgroups and $(\in, \in\vee q_0^\delta)$-fuzzy subgroups.

The aim of this paper is to generalize the notions and results in the paper [9]. We introduce the notions of $(\widetilde{\alpha}, \widetilde{\beta})$-fuzzy bi-ideals in semigroups, and investigate related properties. We discuss characterizations of $(\in, \in\vee q_0^\delta)$-fuzzy generalized bi-ideals.

2 Preliminaries

An element a of a semigroup S is called a regular element if there exists an element x of S such that $a = axa$. A semigroup S is said to be regular if every element of S is regular.

A nonempty subset B of a semigroup S is called

- a *subsemigroup of* S if $B^2 \subseteq B$,
- a *left (resp. right) ideal of* S if $SB \subseteq B$ (resp. $BS \subseteq B$),
- a *generalized bi-ideal of* S if $BSB \subseteq B$,
- a *bi-ideal of* S *if* it is both subsemigroup and a generalized bi-ideal of S.

For two fuzzy set λ and ν in S, we say $\lambda \le \nu$ if $\lambda(x) \le \nu(x)$ for all $x \in S$. We define $\lambda \wedge \nu$ and $\lambda \vee \nu$ as follows:

$$\lambda \wedge \nu : S \to [0,1],\ x \mapsto \min\{\lambda(x), \nu(x)\}$$

and

$$\lambda \vee \nu : S \to [0,1],\ x \mapsto \max\{\lambda(x), \nu(x)\}$$

respectively. The product $\lambda \circ \nu$ of λ and ν is defined to be fuzzy set in S as follows:

$$(\lambda \circ \nu)(x) := \begin{cases} \bigvee\limits_{x=yz} \min\{\lambda(y), \nu(z)\} & \text{if } \exists y, z \in S \text{ such that } x = yz, \\ 0 & \text{otherwise.} \end{cases}$$

A fuzzy set λ in a set S of the form

$$\lambda(y) := \begin{cases} t \in (0,1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases} \tag{2.1}$$

is said to be a *fuzzy point* with support x and value t and is denoted by x_t. It is clear that $x_t \circ y_r = (xy)_{\min\{t,r\}}$ for all fuzzy points x_t and y_r in a set S.

For a fuzzy point x_t and a fuzzy set λ in a set S, we say that

- $x_t \in \lambda$ (resp. $x_t \, q \, \lambda$) (see [1]) if $\lambda(x) \geq t$ (resp. $\lambda(x) + t > 1$). In this case, x_t is said to *belong to* (resp. *be quasi-coincident with*) a fuzzy set λ.
- $x_t \in \vee \, q \, \lambda$ (resp. $x_t \in \wedge \, q \, \lambda$) (see [1]) if $x_t \in \lambda$ or $x_t \, q \, \lambda$ (resp. $x_t \in \lambda$ and $x_t \, q \, \lambda$).

Let $\delta \in (0, 1]$. For a fuzzy point x_t and a fuzzy set λ in a set X, we say that

- x_t is a δ-*quasi-coincident* with λ, written $x_t \, q_0^\delta \, \lambda$, (see [10]) if $\lambda(x) + t > \delta$,
- $x_t \in \vee \, q_0^\delta \, \lambda$ (resp. $x_t \in \wedge \, q_0^\delta \lambda$) (see [10]) if $x_t \in \lambda$ or $x_t \, q_0^\delta \, \lambda$ (resp. $x_t \in \lambda$ and $x_t \, q_0^\delta \, \lambda$).

Obviously, $x_t \, q \, \lambda$ implies $x_t \, q_0^\delta \, \lambda$. If $\delta = 1$, then the δ-quasi-coincident with λ is the quasi-coincident with λ, that is, $x_t \, q_0^1 \, \lambda = x_t \, q \, \lambda$.

For $\alpha \in \{\in, q, \in \vee \, q, \in \wedge \, q, \in \vee \, q_0^\delta, \in \wedge \, q_0^\delta\}$, we say that $x_t \, \overline{\alpha} \, \lambda$ if $x_t \, \alpha \, \lambda$ does not hold.

3 Main Results

In what follows, let δ be an element of $(0, 1]$ and let S be a semigroup and $\widetilde{\alpha}$ and $\widetilde{\beta}$ denote any one of $\in$, q_0^δ, $\in \vee \, q_0^\delta$ and $\in \wedge \, q_0^\delta$ unless otherwise specified.

Definition 3.1 ([11])**.** A fuzzy set λ in S is called an $(\widetilde{\alpha}, \widetilde{\beta})$-*fuzzy subsemigroup of* S, where $\widetilde{\alpha} \neq \in \wedge \, q_0^\delta$, if

$$(\forall x, y \in S) \, (\forall t, r \in (0, \delta]) \left(x_t \, \widetilde{\alpha} \, \lambda, \; y_r \, \widetilde{\alpha} \, \lambda \; \Rightarrow \; x_t \circ y_r \, \widetilde{\beta} \, \lambda \right). \tag{3.1}$$

Let λ be a fuzzy set in S such that $\lambda(x) \leq \frac{\delta}{2}$ for all $x \in S$. Let $x \in S$ and $t \in (0, \delta]$ be such that $x_t \in \wedge \, q_0^\delta \, \lambda$. Then $\lambda(x) \geq t$ and $\lambda(x) + t > \delta$. It follows that $\delta < \lambda(x) + t \leq 2\lambda(x)$, so that $\lambda(x) \geq \frac{\delta}{2}$. This means that $\{x_t \mid x_t \in \wedge \, q_0^\delta \, \lambda\} = \emptyset$. Hence the case $\widetilde{\alpha} = \in \wedge \, q_0^\delta$ should be omitted.

Definition 3.2. A fuzzy set λ in S is called an $(\widetilde{\alpha}, \widetilde{\beta})$-*fuzzy left (resp. right) ideal of* S if for any $x, y \in S$ and $t \in (0, \delta]$,

$$y_t \, \widetilde{\alpha} \, \lambda \; \Rightarrow \; (xy)_t \, \widetilde{\beta} \, \lambda \;\; \text{(resp. } (yx)_t \, \widetilde{\beta} \, \lambda). \tag{3.2}$$

A fuzzy set λ in S is called an $(\widetilde{\alpha}, \widetilde{\beta})$-*fuzzy ideal of* S if it is both an $(\widetilde{\alpha}, \widetilde{\beta})$-fuzzy left ideal and an $(\widetilde{\alpha}, \widetilde{\beta})$-fuzzy right ideal of S.

Definition 3.3. A fuzzy set λ in S is called an $(\widetilde{\alpha}, \widetilde{\beta})$-*fuzzy bi-ideal of* S if it satisfies the condition (3.1) and for any $x, y, z \in S$ and $t, r \in (0, \delta]$,

$$x_t \, \widetilde{\alpha} \, \lambda, \; z_r \, \widetilde{\alpha} \, \lambda \; \Rightarrow \; (xyz)_{\min\{t, r\}} \, \widetilde{\beta} \, \lambda. \tag{3.3}$$

If a fuzzy set λ in S satisfies the condition (3.3) only is called an $(\widetilde{\alpha}, \widetilde{\beta})$-*fuzzy generalized bi-ideal of* S.

Table 1: Cayley table of the operation $\cdot$

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Example 3.4 ([13]). Consider a semigroup $S = \{a, b, c, d\}$ with the multiplication $\cdot$ which is described by Table 1.
Let δ be a fuzzy set in S defined by

$$\delta : S \to [0,1],\ x \mapsto \begin{cases} \frac{\delta}{2} & \text{if } x = a, \\ \frac{\delta}{5} & \text{if } x = c, \\ \frac{\delta}{10} & \text{if } x \in \{b, d\}, \end{cases}$$

Then δ is an $(\in, \in\vee q_0^\delta)$-fuzzy generalized bi-ideal of S.

Let λ and ν be fuzzy sets in S defined by

$$\lambda : S \to [0,1],\ x \mapsto \begin{cases} \frac{\delta}{2} & \text{if } x = a, \\ \frac{3\delta}{5} & \text{if } x = b, \\ \frac{7\delta}{10} & \text{if } x = c, \\ 0 & \text{if } x = d, \end{cases}$$

and

$$\nu : S \to [0,1],\ x \mapsto \begin{cases} \frac{7\delta}{10} & \text{if } x = a, \\ \frac{\delta}{2} & \text{if } x = b, \\ \frac{3\delta}{5} & \text{if } x = c, \\ \frac{\delta}{5} & \text{if } x = d, \end{cases}$$

respectively. Then λ and ν are $(\in, \in\vee q_0^\delta)$-fuzzy ideals of S. It follows $\lambda \circ \nu$ and $\lambda \wedge \nu$ are $(\in, \in\vee q_0^\delta)$-fuzzy ideals of S. Moreover, we know that $\lambda \circ \nu \nleq \lambda \wedge \nu$ since

$$(\lambda \circ \nu)(b) = \bigvee_{b=xy} \min\{\lambda(x), \nu(y)\} = \tfrac{3\delta}{5} > \tfrac{\delta}{2} = (\lambda \wedge \nu)(b).$$

Lemma 3.5 ([13]). *A fuzzy set λ in S is an $(\in, \in\vee q_0^\delta)$-fuzzy generalized bi-ideal of S if and only if the following assertion is valid.*

$$(\forall x, y, z \in S) \left(\lambda(xyz) \geq \min\{\lambda(x), \lambda(z), \tfrac{\delta}{2}\}\right). \tag{3.4}$$

We provide a condition for an $(\in, \in\vee q_0^\delta)$-fuzzy generalized bi-ideal to be an $(\in, \in)$-fuzzy generalized bi-ideal.

Lemma 3.6 ([11])**.** *Let λ be an $(\in, \in\vee q_0^\delta)$-fuzzy subsemigroup of S such that $\lambda(x) < \frac{\delta}{2}$ for all $x \in S$. Then λ is an $(\in,\in)$-fuzzy subsemigroup of S.*

Lemma 3.7. *Let λ be an $(\in, \in\vee q_0^\delta)$-fuzzy generalized bi-ideal of S. If $\mathrm{Im}(\lambda) \subseteq [0, \frac{\delta}{2})$, then λ is an $(\in,\in)$-fuzzy generalized bi-ideal of S.*

Proof. Let $x, y, z \in S$ and $t, r \in (0, \delta]$ be such that $x_t \in \lambda$ and $z_r \in \lambda$. Then $t \le \lambda(x) < \frac{\delta}{2}$ and $r \le \lambda(z) < \frac{\delta}{2}$. It follows from the hypothesis and Lemma 3.5 that

$$\lambda(xyz) \ge \min\{\lambda(x),\, \lambda(z),\, \tfrac{\delta}{2}\} = \min\{\lambda(x),\, \lambda(z)\} \ge \min\{t,\, r\}$$

so that $(xyz)_{\min\{t,\,r\}} \in \lambda$. Hence λ is an $(\in,\in)$-fuzzy generalized bi-ideal of S. □

Corollary 3.8. *Let λ be an $(\in, \in\vee q)$-fuzzy generalized bi-ideal of S such that $\lambda(x) < 0.5$ for all $x \in S$. Then λ is an $(\in,\in)$-fuzzy generalized bi-ideal of S.*

Using Lemmas 3.6 and 3.7 induce the following theorem.

Theorem 3.9. *Let λ be an $(\in, \in\vee q_0^\delta)$-fuzzy bi-ideal of S. If $\mathrm{Im}(\lambda) \subseteq [0, \frac{\delta}{2})$, then λ is an $(\in,\in)$-fuzzy bi-ideal of S.*

Lemma 3.10 ([11])**.** *A fuzzy set λ in S is an $(\in, \in)$-fuzzy subsemigroup of S if and only if the set*

$$Q_0^\delta(\lambda;t) := \{x \in S \mid x_t\, q_0^\delta\, \lambda\}$$

is a subsemigroup of S when it is nonempty for all $t \in (0, \delta]$.

Theorem 3.11. *A fuzzy set λ in S is an $(\in, \in)$-fuzzy generalized bi-ideal of S if and only if the set*

$$Q_0^\delta(\lambda;t) := \{x \in S \mid x_t\, q_0^\delta\, \lambda\}$$

is a generalized bi-ideal of S when it is nonempty for all $t \in (0, \delta]$.

Proof. Assume that λ is an $(\in, \in)$-fuzzy generalized bi-ideal of S. Let $y \in S$ and $x, z \in Q_0^\delta(\lambda;t)$. Then $x_t\, q_0^\delta\, \lambda$ and $z_t\, q_0^\delta\, \lambda$, i.e., $\lambda(x) + t > \delta$ and $\lambda(z) + t > \delta$. Hence

$$\lambda(xyz) \ge \min\{\lambda(x), \lambda(z)\} > \delta - t,$$

and so $(xyz)_t\, q_0^\delta\, \lambda$. Thus $xyz \in Q_0^\delta(\lambda;t)$, and therefore $Q_0^\delta(\lambda;t)$ is a generalized bi-ideal of S.

Conversely, suppose that $Q_0^\delta(\lambda;t)$ is a generalized bi-ideal of S for all $t \in (0, \delta]$ with $Q_0^\delta(\lambda;t) \neq \emptyset$. If λ is not an $(\in, \in)$-fuzzy generalized bi-ideal of S, then

$$\lambda(abc) + t \le \delta < \min\{\lambda(a), \lambda(c)\} + t$$

for some $a, b, c \in S$ and $t \in (0, \delta]$. It follows that $a, c \in Q_0^\delta(\lambda;t)$, and so that $abc \in Q_0^\delta(\lambda;t)$ since $Q_0^\delta(\lambda;t)$ is a generalized bi-ideal of S. Thus $\lambda(abc) + t > \delta$, a contradiction. Therefore λ is an $(\in, \in)$-fuzzy generalized bi-ideal of S. □

Theorem 3.12. *A fuzzy set λ in S is an $(\in, \in)$-fuzzy bi-ideal of S if and only if the set*

$$Q_0^\delta(\lambda;t) := \{x \in S \mid x_t \, q_0^\delta \, \lambda\}$$

is a bi-ideal of S when it is nonempty for all $t \in (0, \delta]$.

Proof. It is by Lemma 3.10 and Theorem 3.11. □

Lemma 3.13 ([11])**.** *If λ is an $(\in, \in\vee q_0^\delta)$-fuzzy subsemigroup of S, then the set $Q_0^\delta(\lambda;t)$ is a subsemigroup of S when it is nonempty for all $t \in (\frac{\delta}{2}, 1]$.*

Theorem 3.14. *If λ is an $(\in, \in\vee q_0^\delta)$-fuzzy generalized bi-ideal of S, then the set $Q_0^\delta(\lambda;t)$ is a generalized bi-ideal of S when it is nonempty for all $t \in (\frac{\delta}{2}, 1]$.*

Proof. Assume that λ is an $(\in, \in\vee q_0^\delta)$-fuzzy generalized bi-ideal of S and let $t \in (\frac{\delta}{2}, 1]$ such that $Q_0^\delta(\lambda;t) \neq \emptyset$. Let $x, z \in Q_0^\delta(\lambda;t)$ and $y \in S$. Then $x_t \, q_0^\delta \, \lambda$ and $z_t \, q_0^\delta \, \lambda$, i.e., $\lambda(x) + t > \delta$ and $\lambda(z) + t > \delta$. Using (3.4), we have

$$\begin{aligned}
\lambda(xyz) &\geq \min\{\lambda(x), \lambda(z), \tfrac{\delta}{2}\} \\
&= \begin{cases} \min\{\lambda(x), \lambda(z)\} & \text{if } \lambda(x) < \frac{\delta}{2} \text{ or } \lambda(z) < \frac{\delta}{2}, \\ \frac{\delta}{2} & \text{if } \lambda(x) \geq \frac{\delta}{2} \text{ and } \lambda(z) \geq \frac{\delta}{2} \end{cases} \\
&> \delta - t.
\end{aligned}$$

Hence $xyz \in Q_0^\delta(\lambda;t)$, and $Q_0^\delta(\lambda;t)$ is a generalized bi-ideal of S for all $t \in (\frac{\delta}{2}, 1]$. □

Theorem 3.15. *If λ is an $(\in, \in\vee q_0^\delta)$-fuzzy bi-ideal of S, then the set $Q_0^\delta(\lambda;t)$ is a bi-ideal of S when it is nonempty for all $t \in (\frac{\delta}{2}, 1]$.*

Proof. It follows from Lemma 3.13 and Theorem 3.14. □

Corollary 3.16. *If λ is an $(\in, \in\vee q)$-fuzzy bi-ideal of S, then the set*

$$Q(\lambda;t) := \{x \in S \mid x_t \, q \, \lambda\}$$

is a bi-ideal of S when it is nonempty for all $t \in (0.5, 1]$.

Lemma 3.17 ([11])**.** *A fuzzy set λ in S is an $(\in, \in\vee q_0^\delta)$-fuzzy subsemigroup of S if and only if the set*

$$U(\lambda;t) := \{x \in S \mid \lambda(x) \geq t\}$$

is a subsemigroup of S for all $t \in (0, \frac{\delta}{2}]$.

Theorem 3.18. *A fuzzy set λ in S is an $(\in, \in\vee q_0^\delta)$-fuzzy generalized bi-ideal of S if and only if the set*

$$U(\lambda;t) := \{x \in S \mid \lambda(x) \geq t\}$$

is a generalized bi-ideal of S for all $t \in (0, \frac{\delta}{2}]$.

Proof. Assume that λ is an $(\in, \in\vee q_0^\delta)$-fuzzy generalized bi-ideal of S. Let $t \in (0, \frac{\delta}{2}]$ and $x, z \in U(\lambda; t)$. Then $\lambda(x) \geq t$ and $\lambda(z) \geq t$. It follows from (3.4) that

$$\lambda(xyz) \geq \min\{\lambda(x), \lambda(z), \tfrac{\delta}{2}\} \geq \min\{t, \tfrac{\delta}{2}\} = t$$

for all $y \in S$ and so that $xyz \in U(\lambda; t)$. Therefore $U(\lambda; t)$ is a generalized bi-ideal of S.

Conversely, let λ be a fuzzy set in S such that $U(\lambda; t)$ is a generalized bi-ideal of S for all $t \in (0, \frac{\delta}{2}]$. Suppose that there are elements a, b and c of S such that

$$\lambda(abc) < \min\{\lambda(a), \lambda(c), \tfrac{\delta}{2}\},$$

and take $t \in (0, \delta]$ such that $\lambda(abc) < t \leq \min\{\lambda(a), \lambda(c), \frac{\delta}{2}\}$. Then $a, c \in U(\lambda; t)$ and $t \leq \frac{\delta}{2}$, which implies that $abc \in U(\lambda; t)$ since $U(\lambda; t)$ is a generalized bi-ideal of S. This induces $\lambda(abc) \geq t$, and this is a contradiction. Hence $\lambda(xyz) \geq \min\{\lambda(x), \lambda(z), \frac{\delta}{2}\}$ for all $x, y, z \in S$, and therefore λ is an $(\in, \in\vee q_0^\delta)$-fuzzy generalized bi-ideal of S by Lemma 3.5. □

Using Lemma 3.17 and Theorem 3.18, we have the following theorem.

Theorem 3.19. *A fuzzy set λ in S is an $(\in, \in\vee q_0^\delta)$-fuzzy bi-ideal of S if and only if the set*

$$U(\lambda; t) := \{x \in S \mid \lambda(x) \geq t\}$$

is a bi-ideal of S for all $t \in (0, \frac{\delta}{2}]$.

Corollary 3.20. *A fuzzy set λ in S is an $(\in, \in\vee q)$-fuzzy bi-ideal of S if and only if the set*

$$U(\lambda; t) := \{x \in S \mid \lambda(x) \geq t\}$$

is a bi-ideal of S for all $t \in (0, 0.5]$.

For a subset Q of S, a fuzzy set χ_Q^δ in S defined by

$$\chi_Q^\delta : S \to [0,1],\ x \mapsto \begin{cases} \delta & \text{if } x \in Q, \\ 0 & \text{otherwise,} \end{cases}$$

is called a *δ-characteristic fuzzy set* of Q in S (see [10]).

The following is a corollary of Theorem 3.18.

Corollary 3.21. *For any subset Q of S and the δ-characteristic fuzzy set χ_Q^δ of Q in S, the following are equivalent:*

(i) *Q is a generalized bi-ideal of S.*

(ii) *χ_Q^δ is an $(\in, \in\vee q_0^\delta)$-fuzzy generalized bi-ideal of S.*

Lemma 3.22 ([11]). *For any subset Q of S and the δ-characteristic fuzzy set χ_Q^δ of Q in S, the following are equivalent:*

(i) *Q is a subsemigroup of S.*

(ii) χ_Q^δ *is an* $(\in, \in\vee q_0^\delta)$*-fuzzy subsemigroup of* S*.*

Using Lemma 3.22 and Corollary 3.21, we have the following theorem.

Theorem 3.23. *For any subset* Q *of* S *and the* δ*-characteristic fuzzy set* χ_Q^δ *of* Q *in* S*, the following are equivalent:*

(i) Q *is a bi-ideal of* S*.*

(ii) χ_Q^δ *is an* $(\in, \in\vee q_0^\delta)$*-fuzzy bi-ideal of* S*.*

Lemma 3.24 ([11])**.** *For any fuzzy set* λ *in* S*, the following are equivalent.*

(i) λ *is an* $(\in, \in\vee q_0^\delta)$*-fuzzy subsemigroup of* S*.*

(ii) *The set* $UQ_0^\delta(\lambda;t) := \{x \in S \mid x_t \in\vee q_0^\delta \lambda\}$ *is a subsemigroup of* S *for all* $t \in (0,\delta]$ *with* $UQ_0^\delta(\lambda;t) \neq \emptyset$*.*

Theorem 3.25. *For any fuzzy set* λ *in* S*, the following are equivalent.*

(i) λ *is an* $(\in, \in\vee q_0^\delta)$*-fuzzy generalized bi-ideal of* S*.*

(ii) *The set* $UQ_0^\delta(\lambda;t) := \{x \in S \mid x_t \in\vee q_0^\delta \lambda\}$ *is a generalized bi-ideal of* S *for all* $t \in (0,\delta]$ *with* $UQ_0^\delta(\lambda;t) \neq \emptyset$*.*

Proof. Assume that λ is an $(\in, \in\vee q_0^\delta)$-fuzzy generalized bi-ideal of S. Let $t \in (0,\delta]$ be such that $UQ_0^\delta(\lambda;t) \neq \emptyset$. Note that $UQ_0^\delta(\lambda;t) = U(\lambda;t) \cup Q_0^\delta(\lambda;t)$. Let $y \in S$ and $x, z \in UQ_0^\delta(\lambda;t)$. Then $x_t \in\vee q_0^\delta \lambda$ and $z_t \in\vee q_0^\delta \lambda$, that is, $\lambda(x) \geq t$ or $\lambda(x) + t > \delta$, and $\lambda(z) \geq t$ or $\lambda(z) + t > \delta$. We consider four cases:

(1) $\lambda(x) \geq t$ and $\lambda(z) \geq t$,

(2) $\lambda(x) \geq t$ and $\lambda(z) + t > \delta$,

(3) $\lambda(x) + t > \delta$ and $\lambda(z) \geq t$,

(4) $\lambda(x) + t > \delta$ and $\lambda(z) + t > \delta$.

For the first case, (3.4) implies that

$$\lambda(xyz) \geq \min\{\lambda(x), \lambda(z), \tfrac{\delta}{2}\} \geq \min\{t, \tfrac{\delta}{2}\} = \begin{cases} \frac{\delta}{2} & \text{if } t > \frac{\delta}{2}, \\ t & \text{if } t \leq \frac{\delta}{2}. \end{cases}$$

Hence $(xyz)_t \in \lambda$ or $\lambda(xyz) + t = \frac{\delta}{2} + t > \frac{\delta}{2} + \frac{\delta}{2} = \delta$. It follows that $xyz \in U(\lambda;t) \cup Q_0^\delta(\lambda;t) = UQ_0^\delta(\lambda;t)$. For the second case, if $t > \frac{\delta}{2}$ then $\delta - t < \frac{\delta}{2}$. Hence

$$\lambda(xyz) \geq \min\{\lambda(z), \tfrac{\delta}{2}\} > \delta - t$$

whenever $\min\{\lambda(z), \frac{\delta}{2}\} \leq \lambda(x)$, and $\lambda(xyz) \geq \lambda(x) \geq t$ whenever $\min\{\lambda(z), \frac{\delta}{2}\} > \lambda(x)$. Thus $xyz \in U(\lambda;t) \cup Q_0^\delta(\lambda;t) = UQ_0^\delta(\lambda;t)$. If $t \leq \frac{\delta}{2}$, then $\delta - t \geq \frac{\delta}{2}$ and so

$$\lambda(xyz) \geq \min\{\lambda(x), \tfrac{\delta}{2}\} \geq t$$

whenever $\min\{\lambda(x), \frac{\delta}{2}\} \leq \lambda(z)$, and $\lambda(xyz) \geq \lambda(z) > \delta - t$ whenever $\min\{\lambda(x), \frac{\delta}{2}\} > \lambda(z)$. Hence $xyz \in U(\lambda; t) \cup Q_0^\delta(\lambda; t) = UQ_0^\delta(\lambda; t)$. We have similar result for the third case. For the final case, if $t > \frac{\delta}{2}$ then $\delta - t < \frac{\delta}{2}$. It follows that

$$\begin{aligned}\lambda(xyz) &\geq \min\{\lambda(x), \lambda(z), \tfrac{\delta}{2}\}\\ &= \begin{cases} \frac{\delta}{2} > \delta - t & \text{if } \min\{\lambda(x), \lambda(z)\} \geq \frac{\delta}{2},\\ \min\{\lambda(x), \lambda(z)\} > \delta - t & \text{if } \min\{\lambda(x), \lambda(z)\} < \frac{\delta}{2},\end{cases}\end{aligned}$$

and so that $xyz \in Q_0^\delta(\lambda; t) \subseteq UQ_0^\delta(\lambda; t)$. If $t \leq \frac{\delta}{2}$, then $\delta - t \geq \frac{\delta}{2}$ and thus

$$\begin{aligned}\lambda(xyz) &\geq \min\{\lambda(x), \lambda(z), \tfrac{\delta}{2}\}\\ &= \begin{cases} \frac{\delta}{2} \geq t & \text{if } \min\{\lambda(x), \lambda(z)\} \geq \frac{\delta}{2},\\ \min\{\lambda(x), \lambda(z)\} > \delta - t & \text{if } \min\{\lambda(x), \lambda(z)\} < \frac{\delta}{2}.\end{cases}\end{aligned}$$

Hence $xyz \in U(\lambda; t) \cup Q_0^\delta(\lambda; t) = UQ_0^\delta(\lambda; t)$. Consequently, $UQ_0^\delta(\lambda; t)$ is a generalized bi-ideal of S for all $t \in (0, \delta]$ with $UQ_0^\delta(\lambda; t) \neq \emptyset$.

Conversely, suppose that (ii) is valid and there exist $a, b, c \in S$ such that

$$\lambda(abc) < \min\{\lambda(a), \lambda(c), \tfrac{\delta}{2}\}.$$

Then $\lambda(abc) < r \leq \min\{\lambda(a), \lambda(c), \frac{\delta}{2}\}$ for some $r \in (0, \frac{\delta}{2}]$. It follows that $a, c \in U(\lambda; r) \subseteq UQ_0^\delta(\lambda; r)$ so that $abc \in UQ_0^\delta(\lambda; r)$. Hence $\lambda(abc) \geq r$ or $\lambda(abc) + r > \delta$, which is a contradiction. Therefore $\lambda(xyz) \geq \min\{\lambda(x), \lambda(z), \frac{\delta}{2}\}$ for all $x, y, z \in S$. Using Lemma 3.5, we know that λ is an $(\in, \in\vee q_0^\delta)$-fuzzy generalized bi-ideal of S. □

Combining Lemma 3.24 and Theorem 3.25 induce the following theorem.

Theorem 3.26. *For any fuzzy set λ in S, the following are equivalent.*

(i) *λ is an $(\in, \in\vee q_0^\delta)$-fuzzy bi-ideal of S.*

(ii) *The set $UQ_0^\delta(\lambda; t)$ is a bi-ideal of S for all $t \in (0, \delta]$ with $UQ_0^\delta(\lambda; t) \neq \emptyset$.*

Corollary 3.27. *For any fuzzy set λ in S, the following are equivalent.*

(i) *λ is an $(\in, \in\vee q)$-fuzzy bi-ideal of S.*

(ii) *The set $UQ(\lambda; t) := \{x \in S \mid x_t \in\vee q\, \lambda\}$ is a bi-ideal of S for all $t \in (0, 1]$ with $UQ(\lambda; t) \neq \emptyset$.*

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