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Problems and Properties for p-Valent Functions Involving a New Generalized Differential Operator

Ala' Ali Amourah[†], **Tariq Al-Hawary**[‡] and **Maslina Darus**[†,1]

[†]School of Mathematical Sciences, Faculty of Science and Technology
Universiti Kebangsaan Malaysia, Bangi 43600
Selangor D. Ehsan, Malaysia
e-mail : alaammour@yahoo.com (A.A. Amourah)
maslina@ukm.edu.my (M. Darus)
[‡]Department of Applied Science, Ajloun College, Al-Balqa Applied University, Ajloun 26816, Jordan
e-mail : tariq-amh@yahoo.com (T. Al-Hawary)

Abstract : In this paper, a new differential operator $A_p^n f(z)$ defined in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ is introduced. We then, using this operator and introduce a new subclass of analytic functions $G(\mu, \lambda, \alpha, \beta, b, p)$. Moreover, we discuss coefficient estimates, growth and distortion theorems and inclusion properties for the function f belonging to the class $G_n(\mu, \lambda, \alpha, \beta, b, p)$.

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1 Introduction and Preliminaries

Let $\mathcal{A}(p)$ denote the class of function f of the form

[1]Corresponding author.

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic and normalized in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. For a function f in $\mathcal{A}(p)$, we defind the following differential operator

$$A_p^0 f(z) = f(z); \tag{1.2}$$

$$A_p^1 f(z) = \left(\frac{\alpha - p(\beta+\lambda)}{\alpha}\right) f(z) + \left(\frac{\beta+\lambda}{\alpha}\right) z f'(z); \tag{1.3}$$

and for $n = 1, 2, 3, \cdots$

$$A_p^n f(z) = A_\lambda(A_p^{n-1} f(z)). \tag{1.4}$$

If f is given by (1.1), then from (1.4) we get

$$A_p^n f(z) = z^p + \sum_{k=p+1}^{\infty} (\frac{\alpha + (\beta+\lambda)(k-p)}{\alpha})^n a_k z^k \tag{1.5}$$

where $f \in \mathcal{A}(p)$, $p \in \mathbb{N}$ and $n \in \mathbb{N}_0$. This generalizes many operators as follows:

(i) When $p = 1$, we get $A^n f(z) = z + \sum_{k=2}^{\infty} (\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha})^n a_k z^k$, introduced by Darus and Faisal [1].

(ii) When $\alpha = 1, \beta = 0$ and $p = 1$, we get $A^n f(z) = z + \sum_{k=2}^{\infty} (1 + \lambda(k-1))^n a_k z^k$ gven by Al-Oboudi [2].

(iii) When $\alpha = 1, \beta = 0, \lambda = 1$ and $p = 1$, we get $A^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k$ the Sălăgean's differential operator [3].

(iv) When $\alpha = 2, \beta = 0, \lambda = 1$ and $p = 1$, we get $A^n f(z) = z + \sum_{k=2}^{\infty} \left(\frac{k+1}{2}\right)^n a_k z^k$ given by Uralegaddi and Somanatha [4].

(v) When $\beta = 1, \lambda = 0$, $p = 1$ and replacing α by $\alpha+1$, we get $A^n f(z) = z + \sum_{k=2}^{\infty} \left(\frac{k+\alpha}{\alpha}\right)^n a_k z^k$, the differential operator of Cho and Srivastava [5].

(vi) When $\beta = 0$, $p = 1$ and replacing α by $\alpha + 1$, we get $A^n f(z) = z + \sum_{k=2}^{\infty} (\frac{\alpha+\lambda(k-1)+1}{\alpha+1})^n a_k z^k$, a well known differential operator of Aouf et al.[6].

Let $G_n(\mu,\lambda,\alpha,\beta,b,p)$ denote the subclass of $\mathcal{A}(p)$ consisting of functions f which satisfy

$$\operatorname{Re}\left\{1+\frac{1}{b}\left[p(1-\mu)\frac{A_p^n f(z)}{z}+\mu(A_p^n f(z))'-p\right]\right\}>0, \tag{1.6}$$

where $A_p^n f(z)$ is given by (1.5).

This implies that it satisfies the following inequality

$$\left|\frac{p(1-\mu)\frac{A_p^n f(z)}{z}+\mu(A_p^n f(z))'-p}{p(1-\mu)\frac{A_p^n f(z)}{z}+\mu(A_p^n f(z))'-1+2b}\right|<1 \tag{1.7}$$

where $z\in U;\mu\geq 0;p\in\mathbb{N};n\in\mathbb{N}_0;b\in\mathbb{C}-\{0\}$.

We note that

- $G_n(\mu,\lambda,\alpha,\beta,b,1)=G(\mu,\lambda,\alpha,\beta,b)$

  $=\operatorname{Re}\left\{f\in A(p):\operatorname{Re}\left\{1+\frac{1}{b}\left[(1-\mu)\frac{A^n f(z)}{z}+\mu(A^n f(z))'-1\right]\right\},z\in U\right\}>0;$
- $G_0(\mu,\lambda,\alpha,\beta,b,1)=G(\lambda,b)$

  $=\operatorname{Re}\left\{f\in\mathcal{A}(p):\operatorname{Re}\left\{1+\frac{1}{b}\left[(1-\mu)\frac{f(z)}{z}+\mu(f(z))'-1\right]\right\},z\in U\right\}>0;$
- $G_n(1,1,1,0,b,1)=R_n(b)$

  $=\operatorname{Re}\left\{f\in\mathcal{A}(p):\operatorname{Re}\left\{1+\frac{1}{b}\left[(A^n f(z))'-1\right]\right\},z\in U\right\}>0;$
- $G_0(0,1,1,0,b,1)=G(b)$

  $=\operatorname{Re}\left\{f\in\mathcal{A}(p):\operatorname{Re}\left\{1+\frac{1}{b}\left[\frac{f(z)}{z}-1\right]\right\},z\in U\right\}>0;$
- $G_n(0,1,1,0,b,1)=G_n(b)$

  $=\operatorname{Re}\left\{f\in\mathcal{A}(p):\operatorname{Re}\left\{1+\frac{1}{b}\left[\frac{A^n f(z)}{z}-1\right]\right\},z\in U\right\}>0;$
- $G_n(1,1,1,0,b,1)=R(b)$

  $=\operatorname{Re}\left\{f\in\mathcal{A}(p):\operatorname{Re}\left\{1+\frac{1}{b}\left[(f(z))'-1\right]\right\},z\in U\right\}>0;$
- $G_0(0,1,1,0,1-\alpha,1)$

  $=G_\alpha=\operatorname{Re}\left\{f\in\mathcal{A}(p):\operatorname{Re}\frac{f(z)}{z}>\alpha,0\leq\alpha<1,z\in U\right\}>0;$

- $G_0(0,1,1,0,1-\alpha,1)$

$$= R_\alpha = \text{Re}\left\{f \in \mathcal{A}(p) : \text{Re}(f(z))' > \alpha, 0 \le \alpha < 1, z \in U\right\} > 0.$$

The class $R(b)$ was studied by Halim [7], the class G_α by Chen [8, 9] and the class R_α by Ezrohi [10].

2 Coefficient Inequalities

In this section, we find the coefficient inequality for the class $G_n(\mu,\lambda,\alpha,\beta,b,p)$.

Theorem 2.1. *Let the function f defined by (1.1) satisfies the condition*

$$\sum_{k=p+1}^{\infty} [p+\mu(k-p)] \left[\frac{\alpha+(\beta+\lambda)(k-p)}{\alpha}\right]^n |a_k| \le |b| \tag{2.1}$$

where $\mu \ge 0, p \in \mathbb{N}, n \in \mathbb{N}_0, \alpha > 0, \beta+\lambda > 0$. Then $f \in G_n(\mu,\lambda,\alpha,\beta,b,p)$.

Proof. Suppose that the inequality (2.1) holds. Then we have for $z \in U$ and $|z| < 1$

$$\left|p(1-\mu)\tfrac{A_p^n f(z)}{z} + \mu(A_p^n f(z))' - p\right| - \left|p(1-\mu)\tfrac{A_p^n f(z)}{z} + \mu(A_p^n f(z))' - 1 + 2b\right|$$

$$\begin{aligned}
&= \left|\sum_{k=p+1}^{\infty} [p+\mu(k-p)] \left[\frac{\alpha+(\beta+\lambda)(k-p)}{\alpha}\right]^n\right| a_k \left|z^{k-1}\right| \\
&\quad - \left|2b + \sum_{k=p+1}^{\infty} [p+\mu(k-p)] \left[\frac{\alpha+(\beta+\lambda)(k-p)}{\alpha}\right]^n\right| a_k \left|z^{k-1}\right| \\
&\le \sum_{k=p+1}^{\infty} [p+\mu(k-p)] \left[\frac{\alpha+(\beta+\lambda)(k-p)}{\alpha}\right]^n |a_k| \left|z^{k-1}\right| - 2|b| \\
&\quad - \sum_{k=p+1}^{\infty} [p+\mu(k-p)] \left[\frac{\alpha+(\beta+\lambda)(k-p)}{\alpha}\right]^n |a_k| \left|z^{k-1}\right| \\
&\le \sum_{k=p+1}^{\infty} [p+\mu(k-p)] \left[\frac{\alpha+(\beta+\lambda)(k-p)}{\alpha}\right]^n |a_k| - |b| \le 0
\end{aligned}$$

where $A_p^n f(z)$ is given by (1.5).

This implies

$$\sum_{k=p+1}^{\infty}[p+\mu(k-p)]\left[\frac{\alpha+(\beta+\lambda)(k-p)}{\alpha}\right]^n|a_k|\le|b|,$$

which shows that $f\in G_n(\mu,\lambda,\alpha,\beta,b,p)$. □

Corollary 2.2. *Let the function f defined by (1.1) be in the class $G_n(\mu,\lambda,\alpha,\beta,b,p)$. Then we have*

$$|a_k|\le\frac{|b|}{[p+\mu(k-p)]\left[\frac{\alpha+(\beta+\lambda)(k-p)}{\alpha}\right]^n},k\ge p+1.$$

Corollary 2.3. *Let the hypotheses of (2.1) be satisfied. Then for $\beta=\lambda=0$ and $p=\mu=1$ we have*

$$|a_k|\le\frac{|b|}{k},k\ge p+1.$$

3 Growth and Distortion Theorems

A growth and distortion property for function f to be in the class $G_n(\mu,\lambda,\alpha,\beta,b,p)$ is given as follows:

Theorem 3.1. *If the function f defined by (1.1) is in the class $G_n(\mu,\lambda,\alpha,\beta,b,p)$, then for $|z|<1$, we have*

$$|r^p|-\frac{|b|\,|r|^k}{[p+\mu(k-p)]\left[\frac{\alpha+(\beta+\lambda)(k-p)}{\alpha}\right]^n}\le|f(z)|\le|r^p|+\frac{|b|\,|r|^k}{[p+\mu(k-p)]\left[\frac{\alpha+(\beta+\lambda)(k-p)}{\alpha}\right]^n}.$$

Proof. Let $f\in G_n(\mu,\lambda,\alpha,\beta,b,p)$. Then by (2.1), we have

$$\sum_{k=p+1}^{\infty}[p+\mu(k-p)]\left[\frac{\alpha+(\beta+\lambda)(k-p)}{\alpha}\right]^n|a_k|$$
$$\le|b|=\sum_{k=p+1}^{\infty}|a_k|\le\frac{|b|}{[p+\mu(k-p)]\left[\frac{\alpha+(\beta+\lambda)(k-p)}{\alpha}\right]^n}.$$

From equation (1.1) we have

$$|f(z)|=\left|z^p+\sum_{k=p+1}^{\infty}a_kz^k\right|\le|z^p|+\sum_{k=p+1}^{\infty}|a_k|\left|z^k\right|$$

which implies

$$|f(z)| \leq |z^p| + \sum_{k=p+1}^{\infty} |a_k| \left|z^k\right| \leq r^p + \frac{|b|}{[p+\mu(k-p)]\left[\frac{\alpha+(\beta+\lambda)(k-p)}{\alpha}\right]^n} r^k.$$

Similarly we can prove that

$$|f(z)| \geq r^p - \frac{|b|}{[p+\mu(k-p)]\left[\frac{\alpha+(\beta+\lambda)(k-p)}{\alpha}\right]^n} r^k. \qquad \square$$

Theorem 3.2. *Let the hypotheses of (2.1) be satisfied, then for* $|z| < 1$,

$$|r^p| - \frac{\alpha^n |b| |r|^2}{[p+\mu][\alpha+\beta+\lambda]^n} \leq |f(z)| \leq |r^p| + \frac{\alpha^n |b| |r|^2}{[p+\mu][\alpha+\beta+\lambda]^n}.$$

Proof. From(2.1) we have $f \in G_n(\mu, \lambda, \alpha, \beta, b, p)$ and hence

$$\sum_{k=p+1}^{\infty} [p+\mu(k-p)] \left[\frac{\alpha+(\beta+\lambda)(k-p)}{\alpha}\right]^n |a_k| \leq |b|.$$

Since

$$[p+\mu]\left[\frac{\alpha+\beta+\lambda}{\alpha}\right]^n \sum_{k=p+1}^{\infty} |a_k| \leq \sum_{k=p+1}^{\infty} [p+\mu(k-p)]\left[\frac{\alpha+(\beta+\lambda)(k-p)}{\alpha}\right]^n |a_k| \leq |b|$$

we have

$$[p+\mu]\left[\frac{\alpha+\beta+\lambda}{\alpha}\right]^n \sum_{k=p+1}^{\infty} |a_k| \leq |b|.$$

From (1.1) we have

$$|f(z)| \leq |z^p| + \sum_{k=p+1}^{\infty} |a_k| \left|z^k\right| \leq |z^p| + \sum_{k=p+1}^{\infty} |a_k| \left|z^2\right|.$$

This proves that

$$|f(z)| \leq |r^p| + \frac{\alpha^n |b| |r|^2}{[p+\mu][\alpha+\beta+\lambda]^n}.$$

Similarly

$$|f(z)| \geq |z^p| - \sum_{k=p+1}^{\infty} |a_k| \left|z^k\right| \geq |z^p| - \sum_{k=p+1}^{\infty} |a_k| \left|z^2\right|$$

shows that

$$|f(z)| \geq |r^p| - \frac{\alpha^n |b| |r|^2}{[p+\mu][\alpha+\beta+\lambda]^n}. \qquad \square$$

Corollary 3.3. *Let the hypotheses of (2.1) be satisfied. If* $\alpha = \lambda = \mu = p = 1, \beta = 0$ *then for* $|z| < 1$, *we have*

$$|r^p| - \frac{|b| |r|^2}{2^{n+1}} \leq |f(z)| \leq |r^p| + \frac{|b| |r|^2}{2^{n+1}}.$$

Theorem 3.4. *If the function* f *defined by (1.1) is in the class* $G_n(\mu, \lambda, \alpha, \beta, b, p)$, *then for* $|z| < 1$, *we have*

$$p - \frac{k |b| |r|^{k-1}}{[p+\mu(k-p)]\left[\frac{\alpha+(\beta+\lambda)(k-p)}{\alpha}\right]^n} \leq \left|f^{'}(z)\right| \leq p + \frac{k |b| |r|^{k-1}}{[p+\mu(k-p)]\left[\frac{\alpha+(\beta+\lambda)(k-p)}{\alpha}\right]^n}.$$

Proof. Let $f \in G_n(\mu, \lambda, \alpha, \beta, b, p)$, by using (2.1) we have

$$\sum_{k=p+1}^{\infty} |a_k| \leq \frac{|b|}{[p+\mu(k-p)]\left[\frac{\alpha+(\beta+\lambda)(k-p)}{\alpha}\right]^n}.$$

Also

$$\left|f^{'}(z)\right| = \left|pz^{p-1} + \sum_{k=p+1}^{\infty} k a_k z^{k-1}\right| \leq p + k \sum_{k=p+1}^{\infty} |a_k| \left|z^{k-1}\right| \leq p + k \sum_{k=p+1}^{\infty} |a_k| \left|r^{k-1}\right|.$$

This shows that

$$\left|f^{'}(z)\right| \leq p + \frac{k |b| |r|^{k-1}}{[p+\mu(k-p)]\left[\frac{\alpha+(\beta+\lambda)(k-p)}{\alpha}\right]^n}.$$

Similarly we can prove that

$$\left|f^{'}(z)\right| \geq p - \frac{k |b| |r|^{k-1}}{[p+\mu(k-p)]\left[\frac{\alpha+(\beta+\lambda)(k-p)}{\alpha}\right]^n}.$$

$\square$

4 Inclusion Properties

The inclusion properties for the class $G_n(\mu,\lambda,\alpha,\beta,b,p)$ are given by the following theorem.

Theorem 4.1. *Let the hypotheses of (2.1) be satisfied. Then*

$$\begin{aligned} G_n(\mu_2,\lambda,\alpha,\beta,b,p) &\subseteq G_n(\mu_1,\lambda,\alpha,\beta,b,p),\\ G_n(\mu,\lambda_2,\alpha,\beta,b,p) &\subseteq G_n(\mu,\lambda_1,\alpha,\beta,b,p),\\ G_n(\mu,\lambda,\alpha_1,\beta,b,p) &\subseteq G_n(\mu,\lambda,\alpha_2,\beta,b,p),\\ G_n(\mu,\lambda,\alpha,\beta_2,b,p) &\subseteq G_n(\mu,\lambda,\alpha,\beta_1,b,p) \end{aligned}$$

where $\alpha_2 \geq \alpha_1, \beta_2 \geq \beta_1, \mu_2 \geq \mu_1$ *and* $\lambda_2 \geq \lambda_1$.

Proof. Let $f \in G_n(\mu,\lambda,\alpha,\beta,b,p)$. Then by using (2.1) we have

$$\sum_{k=p+1}^{\infty} [p+\mu(k-p)] \left[\frac{\alpha+(\beta+\lambda)(k-p)}{\alpha}\right]^n |a_k| \leq |b|\,.$$

If $\mu_2 \geq \mu_1$, implying that $p+\mu_2(k-p) \geq p+\mu_1(k-p)$ in such that

$$\begin{aligned} &\sum_{k=p+1}^{\infty} [p+\mu_2(k-p)] \left[\frac{\alpha+(\beta+\lambda)(k-p)}{\alpha}\right]^n \\ &\geq \sum_{k=p+1}^{\infty} [p+\mu_1(k-p)] \left[\frac{\alpha+(\beta+\lambda)(k-p)}{\alpha}\right]^n \end{aligned}$$

this shows that

$$\begin{aligned} &\sum_{k=p+1}^{\infty} [p+\mu_1(k-p)] \left[\frac{\alpha+(\beta+\lambda)(k-p)}{\alpha}\right]^n |a_k| \\ &\leq |b| \leq \sum_{k=p+1}^{\infty} [p+\mu_2(k-p)] \left[\frac{\alpha+(\beta+\lambda)(k-p)}{\alpha}\right]^n |a_k| \leq |b| \end{aligned}$$

or

$$\sum_{k=p+1}^{\infty} [p+\mu_1(k-p)] \left[\frac{\alpha+(\beta+\lambda)(k-p)}{\alpha}\right]^n |a_k| \leq |b|\,.$$

Hence $f \in G_n(\mu,\lambda,\alpha,\beta,b,p)$, which shows that $G_n(\mu_2,\lambda,\alpha,\beta,b,p) \subseteq G_n(\mu_1,\lambda,\alpha,\beta,b,p)$. Similarly, let $f \in G_n(\mu,\lambda,\alpha,\beta,b,p)$, then by using

(2.1) we have $\alpha_2 \geq \alpha_1$. This implies that $(1+\frac{(\beta+\lambda)(k-p)}{\alpha_1})^n \geq (1+\frac{(\beta+\lambda)(k-p)}{\alpha_2})^n$,

$$\sum_{k=p+1}^{\infty} [p+\mu(k-p)] \left[\frac{\alpha_1+(\beta+\lambda)(k-p)}{\alpha_1}\right]^n \geq \sum_{k=p+1}^{\infty} [p+\mu_1(k-p)] \left[\frac{\alpha_2+(\beta+\lambda)(k-p)}{\alpha_2}\right]^n$$

$$\sum_{k=p+1}^{\infty} [p+\mu(k-p)] \left[\frac{\alpha_1+(\beta+\lambda)(k-p)}{\alpha_1}\right]^n |a_k| \leq |b|$$

and hence

$$\sum_{k=p+1}^{\infty} [p+\mu(k-p)] \left[\frac{\alpha_2+(\beta+\lambda)(k-p)}{\alpha_2}\right]^n |a_k| \leq |b|\,.$$

This proves that $f \in G_n(\mu,\lambda,\alpha,\beta,b,p)$ and finally implies that $G_n(\mu,\lambda,\alpha_1,\beta,b,p) \subseteq G_n(\mu,\lambda,\alpha_2,\beta,b,p)$. Employing a similar procedure we can prove that $G_n(\mu,\lambda_2,\alpha,\beta,b,p) \subseteq G_n(\mu,\lambda_1,\alpha,\beta,b,p)$ and $G_n(\mu,\lambda,\alpha,\beta_2,b,p) \subseteq G_n(\mu,\lambda,\alpha,\beta_1,b,p)$.

For more details about coefficient bounds we refer to Joshi [11], Aouf [12], Silverman [13], Raina [14], and Owa and Aouf [15], respectively. □

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