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Generalized Projection Methods for Nonlinear Mappings

Preedaporn Kanjanasamranwong, Sarapee Chairat and Siwaporn Saewan[1]

Department of Mathematics and Statistics, Faculty of Science
Thaksin University, Thailand
e-mail : ypreedaporn@hotmail.com (P. Kanjanasamranwong)
sarapee@tsu.ac.th (S. Chairat)
siwaporn@scholar.tsu.ac.th (S. Saewan)

Abstract : We present a new hybrid iterative process for finding an element in the solution of variational inequality problem and the fixed point set of relatively nonexpansive multi-valued mapping in Banach spaces. This theorem improve and extend some recent results.

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1 Introduction

Let C be a nonempty closed and convex subset of a Banach space E with dual E^*. A mapping $A : C \to E^*$ is said to be:

(1) *monotone* if

$$\langle x - y, Ax - Ay\rangle \geq 0$$

for all $x, y \in C$;

(2) *α-inverse-strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2$$

[1]Corresponding author.

for all $x, y \in C$.

If $A : C \to E^*$ is α-inverse-strongly monotone, then it is Lipschitz continuous with constant $\frac{1}{\alpha}$, that is,

$$\|Ax - Ay\| = \frac{1}{\alpha}\|x - y\|$$

for all $x, y \in C$. Clearly, the class of monotone mappings include the class of α-inverse-strongly monotone mappings.

The class of inverse-strongly monotone have been studied by many authors to approximate a common fixed point (see [1, 2] for more details).

The *variational inequality problem* for an operator A is to find $\hat{z} \in C$ such that

$$\langle y - \hat{z}, A\hat{z}\rangle \geq 0, \ \forall y \in C. \tag{1.1}$$

The set of solution of (1.1) is denoted by $VI(A, C)$.

Let E be a Banach space with the dual space E^*. The *normalized duality mapping* from E to 2^{E^*} defined by

$$J(x) = \{f^* \in E^* : \langle x, f^*\rangle = \|x\|^2, \|f^*\| = \|x\|\}.$$

If E is a Hilbert space, then $J = I$, where I is the identity mapping.

Consider the functional $\phi : E \times E \to \mathbb{R}$ defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \tag{1.2}$$

for all $x, y \in E$, where J is the normalized duality mapping. It is obvious from the definition of function ϕ that

$$(\|y\| - \|x\|)^2 \leq \phi(y, x) \leq (\|y\| + \|x\|)^2 \tag{1.3}$$

for all $x, y \in E$ and

$$\phi(x, y) = \phi(z, y) + \phi(x, z) + 2\langle z - x, Jy - Jz\rangle \tag{1.4}$$

for all $x, y, z \in E$.

If E is a Hilbert space, then $\phi(y, x) = \|y - x\|^2$.

Remark 1.1. *If E is a reflexive, strictly convex and smooth Banach space, then, for any $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that, if $\phi(x, y) = 0$, then $x = y$. From (1.2), we have $\|x\| = \|y\|$. This implies that $\langle x, Jy\rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, one has $Jx = Jy$. Therefore, we have $x = y$ (see [3, 4] for more details).*

The *generalized projection* $\Pi_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem:

$$\phi(\bar{x}, x) = \inf_{y \in C} \phi(y, x). \tag{1.5}$$

The existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(y,x)$ and the strict monotonicity of the mapping J (see, for example, [3–7]). If E is a Hilbert space, then Π_C becomes the metric projection of E onto C. If E is a smooth, strictly convex and reflexive Banach space, then Π_C is a closed relatively quasi-nonexpansive mapping from E onto C with $F(\Pi_C) = C$ ([8]).

Let C be a nonempty closed and convex subset of a real Banach space E. A mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. A point $x \in C$ is a *fixed point* of T provided $Tx = x$. Denoted by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in C : Tx = x\}$. A point p in C is called an *asymptotic fixed point* of T [6] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The asymptotic fixed point set of T is denoted by $\widehat{F}(T)$.

Recall that a mapping $T : C \to C$ is *closed* if, for each $\{x_n\}$ in C, $x_n \to x$ and $Tx_n \to y$ imply that $Tx = y$.

A mapping $T : C \to$ is called *relatively nonexpansive* ([9–11]) if

(R1) $F(T)$ is nonempty;

(R2) $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$;

(R3) $\widehat{F}(T) = F(T)$.

Let $CB(C)$ and $N(C)$ denoted the family of nonempty closed bounded subsets of C and nonempty subsets, respectively. The *Hausdorff metric* on $CB(C)$ is defined by

$$H(A_1, A_2) = \max\{\sup_{x\in A_2} d(x, A_1), \sup_{y\in A_1} d(y, A_2)\}$$

for all $A_1, A_2 \in CB(C)$, where $d(x, A_1) = \inf\{\|x - y\|; y \in A_1\}, x \in C$. A multi-valued mapping $T : C \to CB(C)$ is said to be *nonexpansive* if

$$H(T(x), T(y)) \leq \|x - y\|$$

for all $x, y \in C$. A multi-valued mapping $T : C \to CB(C)$ is said to be *quasi-nonexpansive* if $F(T)$ is nonempty and

$$H(T(x), T(p)) \leq \|x - p\|$$

for all $x \in C$ and all $p \in F(T)$. An element $p \in C$ is called a fixed point of $T : C \to N(C)$ if $p \in T(p)$. The set of fixed point of T is denoted by $F(T)$.

Let C be a nonempty closed convex subset of a smooth Banach space E. A point $p \in C$ is called an *asymptotic fixed point* of a multi-valued mapping $T : C \to N(C)$ if there exists a sequence $\{x_n\}$ in C which converges weakly to p and $\lim\limits_{n\to\infty} d(x_n, T(x_n)) = 0$.

A multi-valued mapping $T : C \to N(C)$ is said to be *relatively nonexpansive* if

($\acute{R}1$) $F(T)$ is nonempty;
($\acute{R}2$) $\phi(p,z) \leq \phi(p,x)$ for all $x \in C$, $z \in T(x)$ and, $p \in F(T)$;
($\acute{R}3$) $\widehat{F}(T) = F(T)$.

Sastry and Babu [12] proved that the Mann and Ishikawa iteration schemes for a multi-valued mapping T with a fixed point p converge to a fixed point q of T under certain conditions. Panyanak [13] extended the result of Sastry and Babu to uniformly convex Banach spaces. In 2009, Shahzad and Zegeye [14] proved strong convergence theorems for the Ishikawa iteration scheme involving quasi-nonexpansive multi-valued mappings in uniformly convex Banach spaces.

In 2014, Homaeipour and Razani [15] introduced an iterative sequence for two relatively nonexpansive multi-valued mappings in Banach spaces. Further. they proved that $\{x_n\}$ converges strongly to $\Pi_{F(T)\cap EP(f)}(x_0)$ under appropriate condition.

From the recent works, in this paper, we obtain new hybrid iterative scheme to find a common element of the fixed point set of relatively nonexpansive multi-valued mapping and the solution set of variational inequality problem in Banach spaces.

2 Preliminaries

Let E be a Banach space and let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E.

1. A Banach space E is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$.
2. A Banach space E is said to be *smooth* if the limit $\lim\limits_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for each $x, y \in U$.
3. The norm of E is said to be *Fréchet differentiable* if, for each $x \in U$, the limit is attained uniformly for $y \in U$.
4. A Banach space E is said to be *uniformly smooth* if the limit exists uniformly in $x, y \in U$.
5. The *modulus of convexity* of E is the function $\delta : [0,2] \to [0,1]$ defined by
$$\delta(\varepsilon) = \inf\Big\{1 - \Big\|\frac{x+y}{2}\Big\| : x, y \in E, \|x\| = \|y\| = 1, \|x-y\| \geq \varepsilon\Big\}.$$
6. E is said to be *uniformly convex* if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$.

Remark 2.1. *Let E be a Banach space. Then the following are well known* (*see [3] for more details*)*:*
(1) *If E is an arbitrary Banach space, then J is monotone and bounded.*
(2) *If E is a strictly convex, then J is strictly monotone.*
(3) *If E is a smooth, then J is single valued and semi-continuous.*

(4) *If E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.*

(5) *If E is reflexive, smooth and strictly convex, then the normalized duality mapping J is single valued, one-to-one and onto.*

(6) *If E is reflexive, smooth and strictly convex, then J^{-1} is also single valued, one-to-one, onto and it is the duality mapping from E^* into E.*

(7) *If E is uniformly smooth, then E is smooth and reflexive.*

(8) *E is uniformly smooth if and only if E^* is uniformly convex.*

We also need the following lemmas for the proof of our main results.

Lemma 2.2 ([16]). *Let E be a strictly convex and smooth Banach space. Then, for all $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$.*

Lemma 2.3 ([6]). *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0$$

for all $y \in C$.

Lemma 2.4 ([6]). *Let E be a reflexive, strictly convex and smooth Banach space, C be a nonempty closed convex subset of E and $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x)$$

for all $y \in C$.

Lemma 2.5 ([15]). *Let E be a smooth and strictly convex Banach space and C be a nonempty closed convex subset of E. Suppose $T : C \to N(C)$ is a relatively nonexpansive multi-valued mapping. Then $F(T)$ is a closed convex subset of C.*

Lemma 2.6 ([17]). *Let E be a uniformly convex and smooth Banach space and $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty] \to [0, \infty]$ with $g(0) = 0$ such that*

$$g(\|y - z\|) \leq \phi(y, z)$$

for all $y, z \in B_r(0) = \{\|x\| \leq r\}$.

Lemma 2.7 ([18]). *Let E be a uniformly convex Banach space and $B_r(0)$ be a closed ball of E. Then there exists a strictly increasing, continuous and convex function $h : [0, \infty) \to [0, \infty)$ with $h(0) = 0$ such that*

$$\|\lambda x + \mu y + \gamma z\|^2 \leq \lambda\|x\|^2 + \mu\|y\|^2 + \gamma\|z\|^2 - \lambda\mu h(\|x - y\|)$$

for all $x, y, z \in B_r(0)$ and $\lambda, \mu, \gamma \in [0, 1]$ with $\lambda + \mu + \gamma = 1$.

Lemma 2.8 ([19]). *Let C be a nonempty closed convex subset of a uniformly smooth, strictly convex real Banach space E and $A : C \to E^*$ be a continuous monotone mapping. For any $r > 0$, define a mapping $F_r : E \to C$ as follows:*

$$F_r x = \{z \in C : \langle y - z, Az\rangle + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0,\ \forall y \in C\}$$

for all $x \in C$. Then the following hold:

(1) *F_r is a single-valued mapping;*
(2) *$F(F_r) = VI(A, C)$;*
(3) *$VI(A, C)$ is a closed and convex subset of C;*
(4) *$\phi(q, F_r x) + \phi(F_r x, x) \leq \phi(q, x)$ for all $q \in F(F_r)$.*

3 Main Results

In this section, we prove some new convergence theorems for finding a common solution of the set of common fixed points of relatively nonexpansive multi-valued mappings and the set of the variational inequality problems in a real uniformly smooth and uniformly convex Banach space.

Theorem 3.1. *Let C be a nonempty closed and convex subset of a uniformly convex and uniformly smooth Banach space E. Let $T : C \to N(C)$ be a relatively nonexpansive multi-valued mapping and A be a continuous monotone mapping of C into E^*. Define a mapping $F_{r_n} : E \to C$ by*

$$F_{r_n} x = \{z \in C : \langle y - z, Az\rangle + \frac{1}{r_n}\langle y - z, Jz - Jx\rangle \geq 0,\ \forall y \in C\}.$$

Assume that $\Theta := F(T) \cap VI(A, C) \neq \emptyset$. For an initial point $x_1 \in C$, define the iterative sequence $\{x_n\}$ in C as follows:

$$\begin{cases} u_n = F_{r_n} x_n, \\ x_{n+1} = \Pi_C J^{-1}(\alpha_n Jx_n + \beta_n Ju_n + \gamma_n Jz_n) \end{cases} \tag{3.1}$$

for all $n \geq 1$, where $z_n \in Tx_n$. Assume that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are the sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Then $\{\Pi_\Theta x_n\}$ converges strongly to a point in Θ, where Π_Θ is the generalized projection from E onto Θ.

Proof. Let T be a relatively nonexpansive multi-value mapping. Since Θ is closed and convex, for any $p \in \Theta$, we have

$$\begin{aligned}
\phi(p, x_{n+1}) &= \phi(p, \Pi_C J^{-1}(\alpha_n Jx_n + \beta_n Ju_n + \gamma_n Jz_n)) \\
&\leq \phi(p, J^{-1}(\alpha_n Jx_n + \beta_n Ju_n + \gamma_n Jz_n)) \\
&= \|p\|^2 - 2\alpha_n\langle p, Jx_n\rangle - 2\beta_n\langle p, Ju_n\rangle - 2\gamma_n\langle p, Jz_n\rangle \\
&+ \|\alpha_n Jx_n + \beta_n Ju_n + \gamma_n Jz_n\|^2 \\
&\leq \|p\|^2 - 2\alpha_n\langle p, Jx_n\rangle - 2\beta_n\langle p, Ju_n\rangle - 2\gamma_n\langle p, Jz_n\rangle
\end{aligned}$$

$$
\begin{aligned}
&+ \alpha_n\|Jx_n\|^2 + \beta_n\|Ju_n\|^2 + \gamma_n\|Jz_n\|^2 \\
&= \alpha_n\phi(p, x_n) + \beta_n\phi(p, u_n) + \gamma_n\phi(p, z_n) \\
&= \alpha_n\phi(p, x_n) + \beta_n\phi(p, F_{r_n}x_n) + \gamma_n\phi(p, z_n) \\
&\le \alpha_n\phi(p, x_n) + \beta_n\phi(p, x_n) + \gamma_n\phi(p, x_n) \\
&= \phi(p, x_n). \qquad (3.2)
\end{aligned}
$$

Hence $\lim_{n\to\infty}\phi(p, x_n)$ exist. Thus $\{\phi(p, x_n)\}$ is bounded and, further, by (1.3), $\{x_n\}$ is bounded and so are $\{z_n\}$ and $\{u_n\}$. Let $y_n = \Pi_\Theta x_n$ for all $n \ge 1$. It follows from (3.2) that

$$\phi(y_n, x_{n+1}) \le \phi(y_n, x_n) \qquad (3.3)$$

and so, for all $m \ge 1$,

$$\phi(y_n, x_{n+m}) \le \phi(y_n, x_n). \qquad (3.4)$$

Thus it follows from Lemma 2.4 that

$$
\begin{aligned}
\phi(y_{n+1}, x_{n+1}) &= \phi(\Pi_\Theta x_n, x_{n+1}) \\
&\le \phi(y_n, x_{n+1}) - \phi(y_n, \Pi_\Theta x_{n+1}) \\
&= \phi(y_n, x_{n+1}) - \phi(y_n, y_{n+1}) \qquad (3.5) \\
&\le \phi(y_n, x_{n+1}) \\
&\le \phi(y_n, x_n).
\end{aligned}
$$

Therefore $\{\phi(y_n, x_n)\}$ is a convergence sequence. For all $m, n \ge 1$ with $n > m$, it follows from Lemma 2.4 that

$$\phi(y_n, y_{n+m}) + \phi(y_{n+m}, x_{n+m}) \le \phi(y_n, x_{n+m})$$

and so, from (3.5),

$$
\begin{aligned}
\phi(y_n, y_{n+m}) &\le \phi(y_n, x_{n+m}) - \phi(y_{n+m}, x_{n+m}) \\
&\le \phi(y_n, x_n) - \phi(y_{n+m}, x_{n+m}).
\end{aligned}
$$

Let $r = sup_{n\in\mathbf{N}}\|y_n\|$. It follows from Lemma 2.6 that there exist a continuous, strictly increasing, and convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that

$$
\begin{aligned}
g(\|y_m - y_n\|) &\le \phi(y_m, y_n) \qquad (3.6) \\
&\le \phi(y_m, x_m) - \phi(y_n, x_n). \qquad (3.7)
\end{aligned}
$$

Thus, from the property of g, we can show that $\{y_n\}$ is a Cauchy sequence for all $m, n \ge 1$. Since E is complete and $\Theta := F(T) \cap VI(A, C)$ is closed and convex, there exist $q \in \Theta$ such that $\{y_n\}$ converges strongly to a point $q \in \Theta$, where $y_n = \Pi_\Theta x_n$. This completes the proof. □

In Theorem 3.1, if $\beta_n = 0$, then we have the following:

Corollary 3.2. *Let C be a nonempty closed and convex subset of a uniformly convex and uniformly smooth Banach space E. Let $T : C \to N(C)$ be a relatively nonexpansive multi-valued mapping. For an initial point $x_1 \in C$, define the iterative sequence $\{x_n\}$ in C as follows:*

$$\left\{ \; x_{n+1} = \Pi_C J^{-1}(\alpha_n Jx_n + (1-\alpha_n)Jz_n) \right. \tag{3.8}$$

for all $n \geq 1$, where $z_n \in Tx_n$. Assume that $\{\alpha_n\}$ is a sequence in $[0,1]$. Then $\{\Pi_\Theta x_n\}$ converges strongly to some point of T. Where $\Pi_{F(T)}$ is the generalized projection from C onto $F(T)$.

Let f be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. *The equilibrium problem* (for short, EP) is to find $x^* \in C$ such that

$$f(x^*, y) \geq 0, \quad \forall y \in C. \tag{3.9}$$

The set of solutions of (3.9) is denoted by $EP(f)$.

For solving the equilibrium problem for a bifunction $f : C \times C \to \mathbb{R}$, let us assume that f satisfies the following conditions:

(A1) $f(x,x) = 0$ for all $x \in C$;

(A2) f is monotone, i.e., $f(x,y) + f(y,x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\lim_{t \downarrow 0} f(tz + (1-t)x, y) \leq f(x,y);$$

(A4) for each $x \in C$, $y \mapsto f(x,y)$ is convex and lower semi-continuous.

Lemma 3.3 (Blum and Oettli [20]). *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4), and let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$f(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C.$$

Lemma 3.4 (Takahashi and Zembayashi [21]). *Let C be a closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space E and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4). For $r > 0$ and $x \in E$, define a mapping $T_r : E \to C$ as follows:*

$$T_r x = \{z \in C : f(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C\},$$

for all $x \in C$. Then the following hold:

(1) *T_r is single-valued;*

(2) *T_r is a firmly nonexpansive-type mapping, for all $x, y \in E$,*

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \le \langle T_r x - T_r y, Jx - Jy\rangle;$$

(3) $F(T_r) = EP(f)$;

(4) *$EP(f)$ is closed and convex.*

Lemma 3.5 (Takahashi and Zembayashi [21])**.** *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $r > 0$. Then, for $x \in E$ and $q \in F(T_r)$,*

$$\phi(q, T_r x) + \phi(T_r x, x) \le \phi(q, x).$$

In Theorem 3.1, if $\langle y - z, Az\rangle = f(z, y)$, then we have the following:

Corollary 3.6. *Let C be a nonempty closed and convex subset of a uniformly convex and uniformly smooth Banach space E. Let $T : C \to N(C)$ be a relatively nonexpansive multi-valued mapping and Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4). Define a mapping $F_{r_n} : E \to C$ by*

$$T_{r_n} x = \{z \in C : f(z, y) + \frac{1}{r_n}\langle y - z, Jz - Jx\rangle \ge 0,\ \forall y \in C\}.$$

Assume that $\Theta := F(T) \cap EP(f) \ne \emptyset$. For an initial point $x_1 \in C$, define the iterative sequence $\{x_n\}$ in C as follows:

$$\begin{cases} u_n = T_{r_n} x_n, \\ x_{n+1} = \Pi_C J^{-1}(\alpha_n Jx_n + \beta_n Ju_n + \gamma_n Jz_n) \end{cases} \tag{3.10}$$

for all $n \ge 1$, where $z_n \in Tx_n$. Assume that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are the sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Then $\{\Pi_\Theta x_n\}$ converges strongly to a point in Θ, where Π_Θ is the generalized projection from E onto Θ.

In the following theorem, we can show that the sequence $\{x_n\}$ defined in (3.1) also converges strongly to some point of Θ.

Theorem 3.7. *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E. Let $T : C \to N(C)$ be a relatively nonexpansive multi-valued mapping and A be a continuous monotone mapping of C into E^*. Define a mapping $F_{r_n} : E \to C$ by*

$$F_{r_n} x = \{z \in C : \langle y - z, Az\rangle + \frac{1}{r_n}\langle y - z, Jz - Jx\rangle \ge 0,\ \forall y \in C\}.$$

Assume that $\Theta := F(T) \cap VI(A, C) \ne \emptyset$. For an initial point $x_1 \in C$, define the iterative sequence $\{x_n\}$ in C as follows:

$$\begin{cases} u_n = F_{r_n} x_n, \\ x_{n+1} = \Pi_C J^{-1}(\alpha_n Jx_n + \beta_n Ju_n + \gamma_n Jz_n) \end{cases} \tag{3.11}$$

for all $n \geq 1$, where $z_n \in Tx_n$. Assume that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are the sequences in $[0, 1]$ satisfying the conditions:

(a) $\alpha_n + \beta_n + \gamma_n = 1$;

(b) $\liminf_{n\to\infty} \alpha_n\beta_n > 0$, $\liminf_{n\to\infty} \alpha_n\gamma_n > 0$;

(c) $\{r_n\} \subset [d, \infty)$ *for some* $d > 0$.

Then $\{x_n\}$ converges strongly to some point of Θ.

Proof. As in the proof of Theorem 3.1, we have $\{x_n\}$, $\{u_n\}$ and $\{z_n\}$ are bounded. So, there exists $r_1 = \sup_{n\geq 1}\{\|x_n\|, \|z_n\|, \|u_n\|\}$ such that $x_n, z_n \in B_r(0)$ for all $n \geq 1$. Since E is a uniformly smooth Banach space, E^* is a uniformly convex Banach space. Since Θ is nonempty, there exist $p \in \Theta$. By Lemma 2.7, there exists a continuous, strictly increasing and convex function $h : [0, \infty) \to [0, \infty)$ with $h(0) = 0$ such that

$$\begin{aligned}
\phi(p, x_{n+1}) &= \phi(p, \Pi_C J^{-1}(\alpha_n Jx_n + \beta_n Ju_n + \gamma_n Jz_n)) \\
&\leq \phi(p, J^{-1}(\alpha_n Jx_n + \beta_n Ju_n + \gamma_n Jz_n)) \\
&= \|p\|^2 - 2\alpha_n\langle p, Jx_n\rangle - 2\beta_n\langle p, Ju_n\rangle - 2\gamma_n\langle p, Jz_n\rangle \\
&\quad +\|\alpha_n Jx_n + \beta_n Ju_n + \gamma_n Jz_n\|^2 \\
&\leq \|p\|^2 - 2\alpha_n\langle p, Jx_n\rangle - 2\beta_n\langle p, Ju_n\rangle - 2\gamma_n\langle p, Jz_n\rangle \\
&\quad +\alpha_n\|Jx_n\|^2 + \beta_n\|Ju_n\|^2 + \gamma_n\|Jz_n\|^2 - \alpha_n\gamma_n h(\|Jx_n - Jz_n\|) \\
&= \alpha_n\phi(p, x_n) + \beta_n\phi(p, u_n) + \gamma_n\phi(p, z_n) - \alpha_n\gamma_n h(\|Jx_n - Jz_n\|) \\
&\leq \phi(p, x_n) - \alpha_n\gamma_n h(\|Jx_n - Jz_n\|)
\end{aligned} \tag{3.12}$$

and so

$$\alpha_n\gamma_n h(\|Jx_n - Jz_n\|) \leq \phi(p, x_n) - \phi(p, x_{n+1}).$$

Since $\{\phi(p, x_n)\}$ is convergent and $\liminf_{n\to\infty} \alpha_n\gamma_n > 0$, we have

$$\lim_{n\to\infty} h(\|Jx_n - Jz_n\|) = 0 \tag{3.13}$$

and so

$$\lim_{n\to\infty} \|Jx_n - Jz_n\| = 0. \tag{3.14}$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.15}$$

Therefore,

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{3.16}$$

Let $p \in \Theta$ and $r > 0$. Then there exists $p + rk \in \Theta$, whenever $\|k\| \leq 1$. Thus, by (1.4), for any $q \in \Theta$, we have

$$\phi(q, x_n) = \phi(x_{n+1}, x_n) + \phi(q, x_{n+1}) + 2\langle x_{n+1} - q, Jx_n - Jx_{n+1}\rangle, \tag{3.17}$$

which implies

$$\frac{1}{2}(\phi(q, x_n) - \phi(q, x_{n+1})) = \frac{1}{2}\phi(x_{n+1}, x_n) + \langle x_{n+1} - q, Jx_n - Jx_{n+1}\rangle. \tag{3.18}$$

Follow from (3.2), we have

$$0 \leq \frac{1}{2}\phi(x_{n+1}, x_n) + \langle x_{n+1} - q, Jx_n - Jx_{n+1}\rangle \tag{3.19}$$

and

$$-\langle x_{n+1} - q, Jx_n - Jx_{n+1}\rangle \leq \frac{1}{2}\phi(x_{n+1}, x_n). \tag{3.20}$$

Since

$$\begin{aligned}\langle x_{n+1} - p, Jx_n - Jx_{n+1}\rangle &= \langle x_{n+1} - (p + rk) + rk, Jx_n - Jx_{n+1}\rangle \\ &= \langle x_{n+1} - (p + rk), Jx_n - Jx_{n+1}\rangle + r\langle k, Jx_n - Jx_{n+1}\rangle\end{aligned} \tag{3.21}$$

and so

$$r\langle k, Jx_n - Jx_{n+1}\rangle = \langle x_{n+1} - p, Jx_n - Jx_{n+1}\rangle - \langle x_{n+1} - (p+rk) + rk, Jx_n - Jx_{n+1}\rangle.$$

Thus it follows from (3.19), we have

$$-\langle x_{n+1} - (p + rk), Jx_n - Jx_{n+1}\rangle \leq \frac{1}{2}\phi(x_{n+1}, x_n),$$

we obtain that

$$\begin{aligned}r\langle k, Jx_n - Jx_{n+1}\rangle &\leq \langle x_{n+1} - p, Jx_n - Jx_{n+1}\rangle + \frac{1}{2}\phi(x_{n+1}, x_n) \\ &= \frac{1}{2}(\phi(p, x_n) - \phi(p, x_{n+1}))\end{aligned}$$

and hence

$$\langle k, Jx_n - Jx_{n+1}\rangle \leq \frac{1}{2r}(\phi(p, x_n) - \phi(p, x_{n+1})).$$

On the other hand, since $p + rk \in \Theta$, it follows from Theorem 3.1 that

$$\phi(p + rk, x_{n+1}) \leq \phi(p + rk, x_n). \tag{3.22}$$

Since $\|k\| \leq 1$, we obtain

$$\|Jx_n - Jx_{n+1}\| \leq \frac{1}{2r}(\phi(p, x_n) - \phi(p, x_{n+1})) \tag{3.23}$$

and so, for all $m, n \geq 1$ with $n > m$, we have

$$\begin{aligned}\|Jx_m - Jx_n\| &\leq \textstyle\sum_{i=m}^{n-1} \|Jx_i - Jx_{i+1}\| \\ &\leq \textstyle\frac{1}{2r}\sum_{i=m}^{n-1}(\phi(p, x_i) - \phi(p, x_{i+1})) \\ &= \textstyle\frac{1}{2r}(\phi(p, x_m) - \phi(p, x_n)).\end{aligned} \tag{3.24}$$

Since $\{\phi(p, x_n)\}$ converges, $\{Jx_n\}$ is a Cauchy sequence. Since E is uniformly convex and uniformly smooth and E^* is complete, $\{Jx_n\}$ converge strongly to some point in E^*. Since E^* has a Fréchet differentiable norm, J^{-1} is norm-to-norm continuous on E^*. Hence $\{x_n\}$ converges strongly to some point x in C. Thus, from 3.16 and T is a relatively nonexpansive, we have $x \in F(T)$.

Also, from (3.12) and $\liminf_{n\to\infty} \alpha_n\beta_n > 0$, we have

$$\lim_{n\to\infty} h(\|Jx_n - Ju_n\|) = 0 \tag{3.25}$$

and so

$$\lim_{n\to\infty} \|Jx_n - Ju_n\| = 0. \tag{3.26}$$

Since J^{-1} is norm-to-norm continuous on E^*, it follows that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.27}$$

Thus, from (3.26), for all $r_n > 0$, we obtain

$$\lim_{n\to\infty} \tfrac{\|Jx_n - Ju_n\|}{r_n} = 0. \tag{3.28}$$

Thus, from $F_{r_n}x_n = u_n \in C$ and $u_n \to x$, we have

$$\langle v - u_n, Au_n\rangle + \tfrac{1}{r_n}\langle v - u_n, Ju_n - Jx_n\rangle \geq 0 \tag{3.29}$$

for all $v \in C$, that is,

$$\langle v - u_n, Au_n\rangle + \langle v - u_n, \frac{Ju_n - Jx_n}{r_n}\rangle \geq 0.$$

For all $t \in (0,1)$, define $v_t = tv + (1-t)x$. Then $v_t \in C$ and it follows from (3.29) that

$$\langle v_t - u_n, Au_n\rangle + \langle v_t - u_n, \tfrac{Ju_n - Jx_n}{r_n}\rangle \geq 0$$

for all $v \in C$ and

$$\langle v_t - u_n, Av_t\rangle \quad\geq\quad \langle v_t - u_n, Av_t\rangle - \langle v_t - u_n, Au_n\rangle - \langle v_t - u_n, \tfrac{Ju_n - Jx_n}{r_n}\rangle \geq 0. \tag{3.30}$$

From (3.30), we have $\frac{Ju_n - Jx_n}{r_n} \to 0$. Since A is monotone, we have

$$\langle v_t - u_n, Av_t\rangle \geq \langle v_t - u_n, Av_t - Au_n\rangle \geq 0$$

and

$$\lim_{n\to\infty} \langle v_t - u_n, Av_t\rangle = \langle v_t - x, Av_t\rangle \geq 0.$$

Taking $t \to 0$, it follows that

$$\langle v - x, Ax\rangle \geq 0$$

for all $v \in C$ and so $x \in VI(A,C)$. Therefore, $x \in F(T) \cap VI(A,C)$. This completes the proof. □

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