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Quasinormality and Fuglede-Putnam Theorem for w-Hyponormal Operators

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Abstract : We investigate several properties of Aluthge transform $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ of an operator $T = U|T|$. We prove (i) if T is a w-hyponormal operator and $\widetilde{T}$ is quasi-normal (resp., normal), then T is quasi-normal (resp., normal), (ii) if T is a contraction with ker $T =$ ker T^2 and $\widetilde{T}$ is a partial isometry, then T is a quasi-normal partial isometry, and (iii) we show that if either (a) T is a w-hyponormal operator such that $\ker(T) \subset \ker(T^*)$ and S^* is w-hyponormal operator such that $\ker(S^*) \subset \ker(S)$ or (b) T is an invertible w-hyponormal operator and S^* is w-hyponormal operator or (c) T is a w-hyponormal such that $\ker(T) \subset \ker(T^*)$ and S^* is a class $\mathcal{Y}$, then the pair (T, S) satisfy Fuglede-Putnam property.

Keywords : w-hyponormal operators; Fuglede-Putnam theorem; quasinormal operators; partial isometry.
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1 Introduction

For complex infinite dimensional Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$, $\mathscr{L}(\mathscr{H})$, $\mathscr{L}(\mathscr{K})$ and $\mathscr{L}(\mathscr{H},\mathscr{K})$ denote the set of bounded linear operators on $\mathscr{H}$, the set of bounded linear operators on $\mathscr{K}$ and the set of bounded linear operators from $\mathscr{H}$ to $\mathscr{K}$, respectively. Every operator T can be decomposed into $T = U|T|$ with a partial isometry U, where $|T|$ is the square root of T^*T. If U is determined uniquely by the kernel condition $\ker(U) = \ker(|T|)$, then this decomposition is called the *polar decomposition*, which is one of the most important results in operator theory ([1] and [2]). In this paper, $T = U|T|$ denotes the polar decomposition satisfying the kernel condition $\ker(U) = \ker(|T|)$.

Recall that an operator $T \in \mathscr{L}(\mathscr{H})$ is *positive*, $T \geq 0$, if $\langle Tx, x\rangle \geq 0$ for all $x \in \mathscr{H}$. An operator $T \in \mathscr{L}(\mathscr{H})$ is said to be *hyponormal* if $T^*T \geq TT^*$. Hyponormal operators have been studied by many authors and it is known that hyponormal operators have many interesting properties similar to those of normal operators ([3] and [4]). An operator T is said to be *p-hyponormal* if$(T^*T)^p \geq (TT^*)^p$ for $p \in (0, 1]$ and an operator T is said to be log-*hyponormal* if T is invertible and $\log|T| \geq \log|T^*|$. p-hyponormal and log-hyponormal operators are defined as extension of hyponormal operator. An operator $T \in \mathscr{L}(\mathscr{H})$ is said to be *paranormal* if it satisfies the following norm inequality $\left\|T^2\right\| \|x\| \geq \|Tx\|^2$ for all $x \in \mathscr{H}$. Ando [5] proved that every log-hyponormal operators is paranormal. Recall [6], an operator $T \in \mathscr{L}(\mathscr{H})$ is called w-hyponormal if $|\widetilde{T}| \geq |T| \geq |\widetilde{T}^*|$, where $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is the Aluthge transformation. The classes of log- and w-hyponormal operators were introduced, and their properties were studied in [6]. In particular, it was shown in [6] that the class of w-hyponormal operators contains both p-and log-hyponormal operators.

2 Quasinormality

Let $T = U|T|$ be the polar decomposition of $T \in \mathscr{L}(\mathscr{H})$. T is said to be quasinormal if $|T|U = U|T|$, or equivalently, $TT^*T = T^*TT$. Patel [7] proved that if T is p-hyponormal and its Aluthge transform $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is normal, then T is normal and $T = \widetilde{T}$. Aluthge and Wang [6] proved that if T is w-hyponormal, $\ker(T) \subset \ker(T^*)$ and its Aluthge transform $\widetilde{T}$ is normal, then T is normal and $T = \widetilde{T}$. The following is a generalization of these results.

Theorem 2.1. *Let T be a w-hyponormal operator with the polar decomposition $T = U|T|$. If $\widetilde{T}$ is quasinormal, then T is also quasinormal. Hence T coincides with its Aluthge transform $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$.*

Proof. Since T is a w-hyponormal operator,

$$|\widetilde{T}| \geq |T| \geq |\widetilde{T}^*|. \tag{2.1}$$

Then Douglass theorem [8] implies

$$\overline{\Re(\widetilde{T})} = \overline{\Re(\widetilde{T}^*)} \subset \overline{\Re(|T|)} = \overline{\Re|\widetilde{T}|}$$

where $\overline{\mathscr{M}}$ denotes the norm closure of $\mathscr{M}$. Let $\widetilde{T} = W|\widetilde{T}|$ be the polar decomposition of $\widetilde{T}$. Then $E := W^*W = U^*U \geq WW^* =: F$. Put

$$|\widetilde{T}^*| = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}, W = \begin{pmatrix} W_1 & W_2 \\ 0 & 0 \end{pmatrix}$$

on $\mathscr{H} = \overline{\Re(\widetilde{T})} \oplus \ker(\widetilde{T}^*)$.
Then X is injective and has a dense range. Since $\widetilde{T}$ is quasinormal, W commutes with $|\widetilde{T}|$ and

$$\begin{aligned}|\widetilde{T}| &= W^*W|\widetilde{T}| = W^*|\widetilde{T}|W \\ &\geq W^*|T|W \geq W^*|\widetilde{T}^*|W = |\widetilde{T}|.\end{aligned}$$

Hence

$$|\widetilde{T}| = W^*|\widetilde{T}|W = W^*|T|W,$$

and

$$|\widetilde{T}^*| = W|\widetilde{T}|W^* = WW^*|\widetilde{T}|WW^* \tag{2.2}$$

$$= WW^*|T|WW^* = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.3}$$

Since $WW^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, (2.1), (2.2) and (2.3) imply that $|\widetilde{T}|$ and $|T|$ are of the forms

$$|\widetilde{T}| = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \geq |T| = \begin{pmatrix} X & 0 \\ 0 & Z \end{pmatrix}, \tag{2.4}$$

where $\overline{\Re(Y)} = \overline{\Re(Z)} = \overline{\Re(|T|)} \ominus \overline{\Re(\widetilde{T})} = \ker(\widetilde{T}^*) \ominus \ker(T)$.
Since W commutes with $|\widetilde{T}|$,

$$\begin{pmatrix} W_1 & W_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}\begin{pmatrix} W_1 & W_2 \\ 0 & 0 \end{pmatrix}.$$

So $W_1X = XW_1$ and $W_2Y = XW_2$, and hence $\overline{\Re(W_1)}$ and $\overline{\Re(W_2)}$ are reducing subspaces of X . Since $W^*W|\widetilde{T}| = |\widetilde{T}|$, we have $W_1^*W_1 = 1$ and

$$\begin{aligned}X^k &= W_1^*W_1X^k = W_1^*X^kW_1, \\ Y^k &= W_2^*W_2Y^k = W_2^*X^kW_2.\end{aligned}$$

Put $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$. Then $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} = W|\widetilde{T}|$ implies

$$\begin{pmatrix} X^{\frac{1}{2}} & 0 \\ 0 & Z^{\frac{1}{2}} \end{pmatrix}\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}\begin{pmatrix} X^{\frac{1}{2}} & 0 \\ 0 & Z^{\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} W_1 & W_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}.$$

Hence

$$\begin{aligned}X^{\frac{1}{2}}U_{11}X^{\frac{1}{2}} &= W_1X = X^{\frac{1}{2}}W_1X^{\frac{1}{2}}, \\ X^{\frac{1}{2}}U_{12}Z^{\frac{1}{2}} &= W_2Y = XW_2\end{aligned}$$

and

$$X^{\frac{1}{2}}(U_{11}-W_1)X^{\frac{1}{2}}=0,$$
$$X^{\frac{1}{2}}(U_{12}Z^{\frac{1}{2}}-X^{\frac{1}{2}}W_2)=0$$

Since X is injective and has a dense range, $U_{11}=W_1$ is isometry and $U_{12}Z^{\frac{1}{2}}=X^{\frac{1}{2}}W_2$ Then

$$U^*U=\begin{pmatrix} U_{11}^*U_{l1}+U_{21}^*U_{21} & U_{1l}^*U_{l2}+U_{21}^*U_{22} \\ U_{12}^*U_{ll}+U_{22}^*U_{21} & U_{12}^*U_{12}+U_{22}^*U_{22}\end{pmatrix}$$

on $\mathscr{H}=\overline{\Re(\widetilde{T})}\oplus\ker(\widetilde{T}^*)$ is the orthogonal projection onto $\overline{\Re(|T|)}\supset\overline{\Re(\widetilde{T})}$ and

$$U^*U=\begin{pmatrix} 1 & 0 \\ 0 & U_{l2}^*U_{12}+U_{22}^*U_{22}\end{pmatrix}.$$

Since $U_{12}Z^{\frac{1}{2}}=X^{\frac{1}{2}}W_2$, we have

$$Z\geq Z^{\frac{1}{2}}U_{12}^*U_{12}Z^{\frac{1}{2}}=W_2^*XW_2=Y,$$

and

$$Z\geq Z^{\frac{1}{2}}U_{12}^*U_{12}Z^{\frac{1}{2}}=W_2^*XW_2=Y\geq Z$$

by (2.4). Hence

$$Z^{\frac{1}{2}}U_{12}^*U_{12}Z^{\frac{1}{2}}=Z=Y,$$

so $Z=Y$ and $|\widetilde{T}|=|T|$. Since

$$\begin{aligned}Z&=Z^{\frac{1}{2}}U_{12}^*U_{12}Z^{\frac{1}{2}}\\&\leq Z^{\frac{1}{2}}U_{12}^*U_{12}Z^{\frac{1}{2}}+Z^{\frac{1}{2}}U_{22}^*U_{22}Z^{\frac{1}{2}}\leq Z\end{aligned}$$

$Z^{\frac{1}{2}}U_{22}^*U_{22}Z^{\frac{1}{2}}=0$ and $U_{22}Z^{\frac{1}{2}}=0$. This implies $\Re(U_{22}^*)\subset\ker(Z)$. Since $\overline{\Re(U_{12}^*U_{12}+U_{22}^*U_{22})}\subset\overline{\Re(Z)}$ and $U_{22}^*U_{22}\leq U_{12}^*U_{12}+U_{22}^*U_{22}$, we have $\Re(U_{22}^*)\subset\overline{\Re(Z)}$. Hence

$$U_{22}=0, U=\begin{pmatrix} W_1 & U_{12} \\ 0 & 0\end{pmatrix}$$

and

$$\Re(U)\subset\overline{\Re(\widetilde{T})}\subset\overline{\Re(|T|)}=\Re(E).$$

Since W commutes with $|\widetilde{T}|=|T|$, W commutes with $|T|$ and

$$\begin{aligned}|T|^{\frac{1}{2}}(W-U)|T|^{\frac{1}{2}}&=W|T|^{\frac{1}{2}}|T|^{\frac{1}{2}}-|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}\\&=W|\widetilde{T}|-\widetilde{T}=0.\end{aligned}$$

Hence $E(W-U)E=0$ and

$$U=UE=EUE=EWE=WE=W.$$

Thus $U=W$ commutes with $|T|$ and T is quasinormal. □

Corollary 2.2. *Let $T = U|T|$ be a w-hyponormal operator T. If $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is normal, then T is also normal.*

Proof. Since $\widetilde{T}$ is normal, T is quasinormal by Theorem 2.1. Hence $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ $= U|T|$ and $\widetilde{T}^* = |T|U^*$. Hence $|T|^2 = |\widetilde{T}|^2 = |\widetilde{T}^*|^2 = |T^*|^2$. This implies $|T| = |T^*|$ and T is normal. □

3 Partial Isometry

In this section, we deals with a partial isometry, i.e., $VV^*V = V$. Let V be a quasinormal partial isometry. Then VV^* is the orthogonal projection onto $V\mathcal{H}$ and V^*V is the orthogonal projection onto $V^*\mathcal{H}$. Let $V = U|V|$ be the polar decomposition of V. Since $V = U$ and $|V| = V^*V$, we have

$$\widetilde{V} = |V|^{\frac{1}{2}}U|V|^{\frac{1}{2}} = V^*VVV^*V = V.$$

Hence the Aluthge transform $\widetilde{V}$ of V is a partial isometry and coincides with V. In this section, we deal with converse situation in which either $\widetilde{T}$ is a partial isometry or $\widetilde{T} = T$. First we consider the situation in which $\widetilde{T}$ is a partial isometry. We start with the following lemma, which is well known.

Lemma 3.1 ([9])**.** *If $0 \leq A \leq 1$, and $\|Ax\| = \|x\|$. Then $Ax = x$.*

Lemma 3.2. *Let $T = U|T|$ be a contraction and $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ a partial isometry. Then $\widetilde{T} = \widetilde{T}(s,t) = |T|^sU|T|^t$ for all $s, t > 0$. In particular,* $\ker(\widetilde{T}) = \ker(\widetilde{T}(1,1)) = \ker(T^2)$.

Proof. Since $\widetilde{T}$ is an isometry on $\Re(\widetilde{T}^*)$, $\||T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}x\| = \|x\|$ for all $x \in \Re(\widetilde{T}^*)$. Since T is a contraction, $|T|^{\frac{1}{2}}$ is also contractions, hence we have

$$|T|^{\frac{1}{2}}x = x, |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}x = |T|^{\frac{1}{2}}Ux = Ux$$

by Lemma 3.1. Hence $|T|^tx = x$, $|T|^sUx = Ux$ and $|T|^sU|T|^tx = |T|^sUx = Ux$ for all s, $t > 0$. Hence we have $\widetilde{T} = \widetilde{T}(s,t) = U$ on $\Re(\widetilde{T}^*)$. To prove the rest, it suffices to show that $\ker(\widetilde{T}) = \ker(\widetilde{T}(s,t))$ because $\mathscr{H} = \Re(\widetilde{T}^*) \oplus \ker(\widetilde{T})$.
Since

$$\begin{aligned} |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}x = 0 &\Leftrightarrow U|T|^{\frac{1}{2}}x \in kerT = ker|T| \\ &\Leftrightarrow |T|^sU|T|^{\frac{1}{2}}x = 0, \end{aligned}$$

we have $\widetilde{T} = \widetilde{T}(s, \frac{1}{2})$. By using the same argument as above, we have $\widetilde{T}^* = \widetilde{T}(\frac{1}{2}, t)$ for all $t > 0$. Hence

$$\begin{aligned} \ker(\widetilde{T}) &= \Re(\widetilde{T}^*)^{\perp} = \Re(\widetilde{T}^*(\frac{1}{2}, t))^{\perp} \\ &= \ker(\widetilde{T}(\frac{1}{2}, t)) = \ker(\widetilde{T}(s,t)). \end{aligned}$$

Thus $\widetilde{T} = \widetilde{T}(s,t)$. It is clear that $\ker(\widetilde{T}(1,1)) = \ker(T^2)$. □

Theorem 3.3. *Let $T = U|T|$ be a contraction such that $\ker(T) = \ker(T^2)$. If $\widetilde{T}$ is a partial isometry, then $T = \widetilde{T} = U$ and T is a quasinormal partial isometry.*

Proof. By Lemma 3.2,

$$\ker(\widetilde{T}) = \ker(T^2) = \ker(T) = \ker(U),$$

so $\Re(\widetilde{T}^*) = \overline{\Re(T^*)} = \overline{\Re(|T|)}$. Since $\widetilde{T} = U$ on ran $\widetilde{T}^* = \overline{\Re(|T|)}$ and $\ker(\widetilde{T}) = \ker(U) = \mathscr{N}(T)$, $\widetilde{T} = U$ because $\mathscr{H} = \overline{\Re(|T|)} \oplus \ker(T)$. This shows

$$\Re(U) = \Re(\widetilde{T}) \subset \overline{\Re(|T|)} = \Re(U^*U).$$

Thus $U = UU^*U = U^*UU$. Let

$$|T| = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}, \quad U^*U = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{on } \mathscr{H} = \overline{\Re(|T|)} \oplus \ker(T).$$

Since T is a contraction, we have $U^*|T|U \le 1$ and $0 \le X \le 1$. Then

$$U^*U = \widetilde{T}^*\widetilde{T} = |T|^{\frac{1}{2}}U^*|T|U|T|^{\frac{1}{2}} \le |T| \le U^*U.$$

Hence $|T| = U^*U$ and $T = U|T| = UU^*U = U = \widetilde{T}$. Thus T is a quasinormal partial isometry. □

Corollary 3.4. *Let $T = U|T|$ be w-hyponormal operator. If $\widetilde{T}$ is a partial isometry, then $\widetilde{T} = T$ and T is a quasinormal partial isometry.*

Proof. Since $|\widetilde{T}|$ is a contraction and $|\widetilde{T}| \ge |T|$, it follows that T is a contraction and $\ker(T) = \ker(\widetilde{T}) = \ker(T^2)$ by Lemma 3.2. Now the result follows from Theorem 3.3. □

Theorem 3.5. *Let $T = U|T|$ and $T = \widetilde{T}$. Then the following assertions hold.*

(i) $(T^*T)^{\frac{1}{2}}(TT^*)^{\frac{1}{2}} = TT^*$, *hence T^*T commutes with TT^*.*

(ii) $\ker(T) \subset \ker(T^*)$.

Proof. (i) Since $T = \widetilde{T}$,

$$U|T|U^* = \widetilde{T}U^* = U\widetilde{T}^*.$$

Hence $|T|$ commute with $|T^*| = U|T|U^*$ and

$$\begin{aligned} TT^* &= U|T|U^*U|T|U^* \\ &= |T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}|T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}} = (T^*T)^{\frac{1}{2}}(TT^*)^{\frac{1}{2}}. \end{aligned}$$

(ii) Part(i) implies that $(T^*T)^{\frac{1}{2}}(TT^*)^{\frac{1}{2}} = TT^*$ and so (ii) is immediate. □

4 Fuglede-Putnam Type Theorem

A pair (T, S) is said to have the *Fuglede-Putnam property* if $T^*X = XS^*$ whenever $TX = XS$ for every $X \in \mathscr{L}(\mathscr{K}, \mathscr{H})$. The Fuglede-Putnam theorem is well-known in the operator theory. It asserts that for any normal operators T and S, the pair (T, S) has the Fuglede-Putnam property. There exist many generalization of this theorem which most of them go into relaxing the normality of T and S, see [9, 10, 11, 12, 13, 14, 15] and some references therein. The next lemma is concerned with the Fuglede-Putnam theorem and we need it in the future.

Lemma 4.1. ([15]) Let $T \in \mathscr{L}(\mathscr{H})$ and $S \in \mathscr{L}(\mathscr{K})$. Then the following assertions equivalent.

(i) The pair (T, S) has the Fuglede-Putnam property.

(ii) If $TX = SX$, then $\overline{\Re(X)}$ reduces T, $\ker(X)^\perp$ reduces S, and $T|_{\overline{\Re(X)}}$, $S|_{\ker(X)^\perp}$ are unitarily equivalent normal operators.

Lemma 4.2. ([16]) Let A, B and C be positive operators, $0 < p$ and $0 < r \leq 1$. If $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$ and $B \geq C$, then $(C^{\frac{r}{2}} A^p C^{\frac{r}{2}})^{\frac{r}{p+r}} \geq C^r$.

Lemma 4.3. *Let T be a w-hyponormal operator and $\mathscr{M}$ an invariant subspace of T. Then the restriction $T|_{\mathscr{M}}$ is also w-hyponormal operator.*

Proof. Let $T = \begin{pmatrix} T_1 & S \\ 0 & T_2 \end{pmatrix}$ on $\mathscr{H} = \mathscr{M} \oplus \mathscr{M}^\perp$ and P the orthogonal projection onto $\mathscr{M}$. Let $T_0 = TP = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}$. Then $|T_0| = (P|T|^2P)^{\frac{1}{2}} \geq P|T|P$ by Hansens inequality, and $|T^*|^2 = TT^* \geq TPT^* = |T_0^*|^2$. Hence, T is w-hyponormal operator

$$\begin{aligned}
&\Leftrightarrow (|T^*|^{\frac{1}{2}}|T||T^*|^{\frac{1}{2}})^{\frac{1}{2}} \geq |T^*| \\
&\Rightarrow (|T_0^*|^{\frac{1}{2}}|T||T_0^*|^{\frac{1}{2}})^{\frac{1}{2}} \geq |T_0^*| \quad \text{(by Lemma 4.2)} \\
&\Rightarrow (|T_0^*|^{\frac{1}{2}}|T_0||T_0^*|^{\frac{1}{2}})^{\frac{1}{2}} \geq |T_0^*| \quad (\text{since } |T_0^*|^{\frac{1}{2}} = |T_0^*|^{\frac{1}{2}}P = P|T_0^*|^{\frac{1}{2}}).
\end{aligned}$$

Also

$$\begin{aligned}
|T_0| &\geq (|T_0|^{\frac{1}{2}}|T^*||T_0|^{\frac{1}{2}})^{\frac{1}{2}} \\
&\geq (|T_0|^{\frac{1}{2}}|T_0^*||T_0|^{\frac{1}{2}})^{\frac{1}{2}}.
\end{aligned}$$

Therefore, $T|_{\mathscr{M}}$ is w-hyponormal operator. □

Lemma 4.4. *Let $T \in \mathscr{L}(\mathscr{H})$ be a w-hyponormal operator with $\ker(T) \subset \ker(T^*)$. Then $T = T_1 \oplus T_2$ on $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$ where T_1 is normal, $\ker(T_2) = \{0\}$ and T_2 is pure w-hyponormal i.e., T_2 has no non-zero invariant subspace $\mathscr{M}$ such that $T_2|_{\mathscr{M}}$ is normal.*

Lemma 4.5. *Let $T = U|T| \in \mathscr{L}(\mathscr{H})$ be a w-hyponormal operator and* $\ker(T) \subset \ker(T^*)$*. Suppose $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ be of the form $N \oplus T'$ on $\mathscr{H} = \mathscr{M} \oplus \mathscr{M}^{\perp}$, where N is a normal operator on $\mathscr{M}$. Then $T = N \oplus T_1$ and $U = U_{11} \oplus U_{22}$ where T_1 is w-hyponormal operator with* $\ker(T_1) \subset \ker(T_1^*)$ *and $N = U_{11}|N|$ is the polar decomposition of N*

Proof. Since

$$|\widetilde{T}| \geq |T| \geq |\widetilde{T}^*|,$$

we have

$$|N| \oplus |T'| \geq |T| \geq |N| \oplus |T'^*|$$

by assumption. This implies that $|T|$ is of the form $|N| \oplus L$ for some positive operator L. Let $U = \begin{pmatrix} U_{11} & U_{l2} \\ U_{21} & U_{22} \end{pmatrix}$ be 2×2 matrix representation of U with respect to the decomposition $\mathscr{H} = \mathscr{M} \oplus \mathscr{M}^{\perp}$. Then the definition $\widetilde{T}$ means

$$\begin{pmatrix} N & 0 \\ 0 & T' \end{pmatrix} = \begin{pmatrix} |N|^{\frac{1}{2}} & 0 \\ 0 & L^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} U_{11} & U_{l2} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} |N|^{\frac{1}{2}} & 0 \\ 0 & L^{\frac{1}{2}} \end{pmatrix}$$

Hence, we have

$$N = |N|^{\frac{1}{2}}U_{11}|N|^{\frac{1}{2}}, \quad |N|^{\frac{1}{2}}U_{12}L^{\frac{1}{2}} = 0, \quad L^{\frac{1}{2}}U_{21}|N|^{\frac{1}{2}} = 0.$$

Since $\ker(T) \subset \ker(T^*)$,

$$\overline{\Re(U)} = \overline{\Re(T)} = \ker(T^*)^{\perp} \subset \ker(T)^{\perp} = \overline{\Re(|T|)}.$$

Let $Nx = 0$ for $x \in \mathscr{M}$. Then $x \in \ker(|T|) = \ker(U)$, and

$$Ux = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} U_{11}x \\ U_{21}x \end{pmatrix} = 0.$$

Hence

$$\ker(N) \subset \ker(U_{11}) \cap \ker(U_{21}).$$

Let $x \in \mathscr{M}$. Then

$$U \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} U_{11}x \\ U_{21}x \end{pmatrix} \in \overline{\Re(|T|)} = \overline{\Re(|N| \oplus L)}.$$

Hence

$$\Re(U_{11}) \subset \Re(|N|), \Re(U_{21}) \subset \overline{\Re(L)}.$$

Similarly

$$\Re(U_{12}) \subset \Re(|N|), \Re(U_{22}) \subset \overline{\Re(L)}.$$

Let $Lx = 0$ for $x \in \mathscr{M}^{\perp}$. Then $x \in \ker(|T|) = \ker(U)$ and

$$U \begin{pmatrix} 0 \\ x \end{pmatrix} = \begin{pmatrix} U_{12}x \\ U_{22}x \end{pmatrix} = 0.$$

Hence

$$\ker(L) \subset \ker(U_{12}) \cap \ker(U_{22}).$$

Let $N = V|N|$ be the polar decomposition of N. Then

$$(V|N|^{\frac{1}{2}} - |N|^{\frac{1}{2}}U_{11})|N|^{\frac{1}{2}} = 0.$$

Hence $V|N|^{\frac{1}{2}} - |N|^{\frac{1}{2}}U_{11} = 0$ on $\overline{\Re(|N|)}$. Since $\ker(N) \subset \ker(U_{11})$, this implies $0 = V|N|^{\frac{1}{2}} - |N|^{\frac{1}{2}}U_{11} = |N|^{\frac{1}{2}}(V - U_{11})$. Hence

$$\Re(V - U_{11}) \subset \ker(|N|) \cap \overline{\Re(|N|)} = \{0\}.$$

Hence $V = U_{11}$ and $N = U_{11}|N|$ is the polar decomposition of N. Since $|N|^{\frac{1}{2}}U_{12}L^{\frac{1}{2}} = 0$,

$$\Re(U_{11}L^{\frac{1}{2}}) \subset \ker(|N|) \cap \overline{\Re(|N|)} = \{0\}.$$

Hence $U_{12}L^{\frac{1}{2}} = 0$ and $U_{12} = 0$. Similarly we have $U_{21} = 0$ by $L^{\frac{1}{2}}U_{21}|N|^{\frac{1}{2}} = 0$. Hence $U = U_{11} \oplus U_{22}$. So we obtain

$$T = U|T| = U_{11}|N| \oplus U_{22}L = N \oplus T_1,$$

where $T_1 = U_{22}L$. □

Lemma 4.6. *Let $T \in \mathscr{L}(\mathscr{H})$ be w-hyponormal operator and* $\ker(T) \subset \ker(T^*)$. *If L is self-adjoint and $TL = LT^*$, then $T^*L = LT$.*

Proof. Since $\ker(T) \subset \ker(T^*)$ and $TL = LT^*$, $\ker(T)$ reduces T and L. Hence

$$T = T_1 \oplus 0, \quad L = L_1 \oplus L_2 \quad \text{on } \mathscr{H} = \overline{\Re(T^*)} \oplus kerT,$$

$T_1L_1 = L_1T_1^*$ and $\{0\} = \ker(T_1) \subset \ker(T_1^*)$. Since $\overline{\Re(L_1)}$ is invariant under T_1 and reduces L_1,

$$T_1 = \begin{pmatrix} T_{11} & S \\ 0 & T_{22} \end{pmatrix}, \quad L_1 = L_{11} \oplus 0 \text{ on } \overline{\Re(T^*)} = \overline{\Re(L_1)} \oplus \ker(L_1).$$

T_{11} is an injective w-hyponormal operator by Lemma 4.4 and L_{11} is an injective self-adjoint operator (hence it has dense range) such that $T_{11}L_{11} = L_{11}T_{11}^*$. Let $T_{11} = V_{11}|T_{11}|$ be the polar decomposition of T_{11} and $\widetilde{T}_{11} = |T_{11}|^{\frac{1}{2}}V_{11}|T_{11}|^{\frac{1}{2}}$, $W = |T_{11}|^{\frac{1}{2}}L_{11}|T_{11}|^{\frac{1}{2}}$. Then

$$\begin{aligned}
\widetilde{T}_{11}W &= |T_{11}|^{\frac{1}{2}}V_{11}|T_{11}|L_{11}|T_{11}|^{\frac{1}{2}} \\
&= |T_{11}|^{\frac{1}{2}}T_{11}L_{11}|T_{11}|^{\frac{1}{2}} = |T_{11}|^{\frac{1}{2}}L_{11}T_{11}^*|T_{11}|^{\frac{1}{2}} \\
&= |T_{11}|^{\frac{1}{2}}L_{11}|T_{11}|^{\frac{1}{2}}|T_{11}|^{\frac{1}{2}}V_{11}^*|T_{11}|^{\frac{1}{2}} \\
&= W\widetilde{T}_{11}^*.
\end{aligned}$$

Since $\widetilde{T}_{11}$ is semi-hyponormal and $\Re(W)$ is dense (because $\ker(W) = \{0\}$), $\widetilde{T}$ is normal by [14, Theorem 2.6]. Hence T_{11} is normal and $T_{11} = \widetilde{T}_{11}$ by Corollary

2.2. Then $\overline{\Re(L_1)}$ reduces T_1 by Lemma 4.4 and $T_{11}^* L_{11} = L_{11}T_{11}$ by Lemma 4.1. Hence

$$\begin{aligned} T &= T_{11} \oplus T_{22} \oplus 0, \\ L &= L_{11} \oplus 0 \oplus L_2 \end{aligned}$$

and

$$T^* L = T_{11}^* L_{11} \oplus 0 \oplus 0 = L_{11}T_{11} \oplus 0 \oplus 0 = LT.$$

□

Corollary 4.7. *Let $T \in \mathscr{L}(\mathscr{H})$ be w-hyponormal operator and $\ker(T) \subset \ker(T^*)$. If $TX = XT^*$ for some $X \in \mathscr{L}(\mathscr{H})$ then $T^*X = XT$.*

Proof. Let $X = L + iJ$ be the Cartesian decomposition of X. Then we have $TL = LT^*$ and $TJ = JT^*$ by the assumption. By Lemma 4.6, we have $T^*L = LT$ and $T^*J = JT$. This implies that $T^*X = XT$. □

Corollary 4.8. *Let $T \in \mathscr{L}(\mathscr{H})$, $S^* \in \mathscr{L}(\mathscr{H})$ be w-hyponormal operators and $\ker(T) \subset \ker(T^*)$, $\ker(S^*) \subset \ker(S)$. If $SX = XT$ for some $X \in \mathscr{L}(\mathscr{K}, \mathscr{H})$, then $T^*X = XS^*$. Moreover, Then $\overline{\Re(X)}$ reduces T, $\ker(X)^\perp$ reduces S and $T|_{\overline{\Re(X)}}$, $S|_{\ker(X)^\perp}$ are unitarily equivalent normal operators.*

Proof. Put $A = \begin{pmatrix} S^* & 0 \\ 0 & T \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix}$ on $\mathscr{H} \oplus \mathscr{K}$. Then A is w-hyponormal operator with $\ker(A) \subset \ker(A^*)$, which satisfies $AY = YA^*$. Hence we have $A^*Y = YA$ by Corollary 4.7, and hence $T^*X = XS^*$.

Now since $T^*X = XS^*$, then $T^*TX = XS^*S$ and so $|T|X = X|S|$. Let $T = U|T|$, $S = V|S|$ be polar decomposition. Then $UX|S| = U|T|X = TX = XS = XV|S|$. Let $x \in \ker(|S|)$. Then $Vx = 0$ and $TXx = XSx = 0$. Hence $Xx \in \ker(T) = \ker(U)$ and $UXx = 0$. Hence $UX = XV$. Since $\ker(U) = \ker(T) \subset \ker(T^*) = \ker(U^*)$, $UU^* \leq U^*U$. Hence $U^*UU = U^*UUU^*U = UU^*U = U$. This implies U and V^* are quasinormal. Hence $U^*X = XV^*$, $\overline{\Re(X)}$ reduces U, $|T|$, $\ker(X)^\perp$ reduces V, $|S|$. Since S, T^* are class w-hyponormal operators with reducing kernels. Let $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, $\widetilde{S} = |S|^{\frac{1}{2}}V|S|^{\frac{1}{2}}$. Then $\widetilde{T}$, $\widetilde{S^*} = |S^*|^{\frac{1}{2}}V^*|S^*|^{\frac{1}{2}} = V\widetilde{S}^*V^*$ are semi-hyponormal. Also, since $|(\widetilde{S})^*| - |\widetilde{S}| = V^*(|\widetilde{S^*}| - |(\widetilde{S^*})^*|)V \geq 0$, $\widetilde{S}^*$ is semi-hyponormal, too. Then

$$\begin{aligned} \widetilde{T}X &= |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}X = |T|^{\frac{1}{2}}UX|S|^{\frac{1}{2}} \\ &= |T|^{\frac{1}{2}}XV|S|^{\frac{1}{2}} = X\widetilde{S}, \end{aligned}$$

hence $\widetilde{T}^*X = X\widetilde{S}^*$, $\overline{\Re(X)}$ reduces $\widetilde{T}$, $\ker(X)^\perp$ reduces $\widetilde{S}$ and $\widetilde{T}|_{\overline{\Re(X)}}$, $\widetilde{S}|_{\ker(X)^\perp}$ are unitarily equivalent normal operators. Hence $T|_{\overline{\Re(X)}}$, $S|_{\ker(X)^\perp}$ are normal operators by Corollary 2.2, and that they are unitarily equivalent follows from the fact that if $N = U|N|$ are $M = W|M|$ are normal operators, then for a unitary operator V, $N = V^*MV$ if and only if $U = V^*WV$ and $|N|^{\frac{1}{2}} = V^*|M|^{\frac{1}{2}}V$. □

Theorem 4.9. *Let $T = U|T| \in \mathscr{L}(\mathscr{H})$ be a w-hyponormal operator and N a normal operator. Let $TX = XN$. Then the following assertions hold.*

(i) If the range $\Re(X)$ is dense, then T is normal.

(ii) If $\ker(X^*) \subset \ker(T^*)$, then T is quasinormal.

Proof. Let $Z = |T|^{\frac{1}{2}}X$. Then

$$\begin{aligned}\widetilde{T}Z &= |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}|T|^{\frac{1}{2}}X = |T|^{\frac{1}{2}}TX\\ &= |T|^{\frac{1}{2}}XN = ZN.\end{aligned}$$

Since $\widetilde{T}$ is semi-hyponormal, we have

$$\widetilde{T}^*Z = ZN^*$$

by [14]. Hence

$$\begin{aligned}&(\widetilde{T}^*\widetilde{T} - \widetilde{T}\widetilde{T}^*)|T|^{\frac{1}{2}}X\\ &= \widetilde{T}^*\widetilde{T}Z - \widetilde{T}T^*Z\\ &= \widetilde{T}^*ZN - \widetilde{T}ZN^* = ZN^*N - ZNN^* = 0.\end{aligned}$$

(i) If $\Re(X)$ is dense, then

$$(\widetilde{T}^*\widetilde{T} - \widetilde{T}\widetilde{T}^*)|T|^{\frac{1}{2}} = 0.$$

Since

$$\ker(|T|^{\frac{1}{2}}) \subset \ker(\widetilde{T}) \cap \ker(\widetilde{T}^*),$$

this implies $\widetilde{T}$ is normal. Hence T is normal by Corollary 2.2.

(ii) Let $X^*|T|^{\frac{1}{2}}x = 0$. Then $|T|^{\frac{1}{2}}x \in \ker(X^*) \subset \ker(T^*) = \ker(U^*)$ and $\widetilde{T}^*x = |T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}}x = 0$. Hence $\ker(X^*|T|^{\frac{1}{2}}) \subset \widetilde{T}^*$ and $\overline{\Re(\widetilde{T})} \subset \overline{\Re(|T|^{\frac{1}{2}}X)}$. Hence

$$(\widetilde{T}^*\widetilde{T} - \widetilde{T}\widetilde{T}^*)\widetilde{T} = 0$$

by (i). This implies $\widetilde{T}$ is quasinormal, and T is quasinormal by Theorem 2.1. □

Following [17], an operator $T \in \mathscr{L}(\mathscr{H})$ is said to be a class $\mathcal{Y}_\alpha$ operator for $\alpha \geq 0$ if there exists a positive number k_α such that

$$|TT^* - T^*T|^\alpha \leq k_\alpha^2(T-\lambda)^*(T-\lambda) \text{ for all } \lambda \in \mathbb{C}.$$

It is known that $\mathcal{Y}_\alpha \subset \mathcal{Y}_\beta$ if $1 \leq \alpha \leq \beta$. Let $\mathcal{Y} = \bigcup_{\alpha\geq 1}\mathcal{Y}_\alpha$. We remark that a class $\mathcal{Y}_1$ operator T is M-hyponormal and M-hyponormal operators are class $\mathcal{Y}_2$ operators.

Lemma 4.10. ([17]) Let $T \in \mathscr{L}(\mathscr{H})$ be a class $\mathcal{Y}$ and $\mathscr{M} \subset \mathscr{H}$ invariant under T. If $T|_{\mathscr{M}}$ is normal, then $\mathscr{M}$ reduces T.

Lemma 4.11. ([17]) If $T \in \mathcal{Y}_\alpha$ for some $\alpha \geq 1$ and if, for a closed set $S \subseteq \mathbb{C}$, there exists a bounded function $f(z) : \mathbb{C} \setminus S \longrightarrow \mathscr{H}$ and a non-zero $x \in \mathscr{H}$ such that $(T - z)f(z) \equiv x$, then $g(z) = (I - E(\{0\}))f(z)$ is analytic on $\mathbb{C} \setminus S$ where $E(.)$ denotes the spectral measure of $|TT^* - T^*T|^{\frac{\alpha}{2}}$. Moreover, if $0 \notin \sigma_p(TT^* - T^*T)$, then $f(z)$ is analytic on $\mathbb{C} \setminus S$.

Theorem 4.12. *Let $T \in \mathscr{L}(\mathscr{H})$ be an invertible w-hyponormal operator and $S^* \in \mathscr{L}(\mathscr{K})$ be class $\mathcal{Y}$. If $TX = XS$ for some $X \in \mathscr{L}(\mathscr{K}, \mathscr{H})$, then $T^*X = XS^*$. Moreover, $\overline{\Re(X)}$ reduces T, $\ker(X)^\perp$ reduces S, and $T|_{\overline{\Re(X)}}$, $S|_{\ker(X)^\perp}$ are unitarily equivalent normal operators.*

Proof. Since S^* is class $\mathcal{Y}$, then there exist positive numbers α and k_α such that

$$|SS^* - S^*S|^\alpha \leq k_\alpha^2 (S - \lambda)(S - \lambda)^*, \text{ for all } \lambda \in \mathbb{C}.$$

Hence for $x \in |SS^* - S^*S|^{\frac{\alpha}{2}}\mathscr{K}$ there exists a bounded function $f : \mathbb{C} \longrightarrow \mathscr{K}$ such that

$$(S - \lambda)f(\lambda) = x, \text{ for all } \lambda \in \mathbb{C}$$

by [8]. Let $T = U|T|$ be the polar decomposition of T, then the Aluthge transform $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is semi-hyponormal by [6]. Then

$$\begin{aligned}(\widetilde{T} - \lambda)|T|^{\frac{1}{2}}Xf(\lambda) &= |T|^{\frac{1}{2}}(T - \lambda)Xf(\lambda)\\ &= |T|^{\frac{1}{2}}X(S - \lambda)f(\lambda), \text{ for all } \lambda \in \mathbb{C}.\end{aligned}$$

We claim that $|T|^{\frac{1}{2}}Xx = 0$. Because if $|T|^{\frac{1}{2}}Xx \neq 0$, there exists a bounded entire analytic function $g : \mathbb{C} \longrightarrow \mathscr{H}$ such that $(\widetilde{T}(s,t) - \lambda)g(\lambda) = |T|^s Xx$ by Lemma 4.11. Since

$$g(\lambda) = (\widetilde{T} - \lambda)^{-1}|T|^{\frac{1}{2}}Xx \longrightarrow 0 \text{ as } \lambda \longrightarrow \infty,$$

we have $g(\lambda) = 0$ by Liouville's theorem, and hence $|\widetilde{T}|^{\frac{1}{2}}Xx = 0$. This is a contradiction. Thus

$$|T|^{\frac{1}{2}}X|SS^* - S^*S|^{2n-1}\mathscr{K} = \{0\}.$$

Since $\ker(T) = \ker(|T|) = \{0\}$, we have

$$X(SS^* - S^*S) = 0.$$

Since $\overline{\Re(X)}$ is invariant under T and $\ker(X)^\perp$ is invariant under S^*. We consider the following decompositions

$$\mathscr{H} = \overline{\Re(X)} \oplus \overline{\Re(X)}^\perp, \ \mathscr{K} = \ker(X)^\perp \oplus \ker(X),$$

then we have

$$T = \begin{pmatrix} T_1 & A \\ 0 & T_2 \end{pmatrix}, \qquad S = \begin{pmatrix} S_1 & 0 \\ B & S_2 \end{pmatrix}$$

and

$$X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} : \ker(X)^{\perp} \oplus \ker(X) \longrightarrow \overline{\Re(X)} \oplus \overline{\Re(X)}^{\perp}.$$

Then

$$\begin{aligned} 0 &= X(SS^* - S^*S) \\ &= \begin{pmatrix} X_1(S_1S_1^* - S_1^*S_1 - B^*B) & X_1(S_1B^* - B^*S_2) \\ 0 & 0 \end{pmatrix} \end{aligned}$$

and

$$X_1(S_1S_1^* - S_1^*S_1 - B^*B) = 0.$$

Since X_1 is injective with dense range, we have

$$S_1S_1^* - S_1^*S_1 - B^*B = 0$$

and

$$S_1S_1^* = S_1^*S_1 + B^*B \geq S_1^*S_1.$$

This implies that B_1^* is hyponormal. Since $TX = XS$, we have

$$T_1X_1 = X_1S_1$$

where T_1 is w-hyponormal by Lemma 4.3. Hence T_1, S_1 are normal and

$$T_1^*X_1 = X_1S_1^*$$

by 4.1. Then $A = 0$ by Lemma 4.4 and $B = 0$ by Lemma 4.10. Hence

$$T^*X = \begin{pmatrix} T_1^*X_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} X_1S_1^* & 0 \\ 0 & 0 \end{pmatrix} = XS^*.$$

Hence $T|_{\overline{(\Re(X))}}$, $S|_{\ker(X)^{\perp}}$ are normal by Lemma 4.1. □

Theorem 4.13. *Let $T \in \mathscr{L}(\mathscr{H})$ and $S^* \in \mathscr{L}(\mathscr{K})$. If either (i) T is a w-hyponormal operator such that* $\ker(T) \subset \ker(T^*)$ *and S^* is a class $\mathcal{Y}$ operator or (ii) T is a class $\mathcal{Y}$ operator and S^* is a w-hyponormal operator such that* $\ker(S^*) \subset \ker(S)$*, if $TX = XS$ for some operator $X \in \mathscr{L}(\mathscr{K}, \mathscr{H})$, then $T^*X = XS^*$. Moreover, $\overline{\Re(X)}$ reduces T,* $\ker(X)^{\perp}$ *reduces S, and $T|_{\overline{(\Re(X))}}$, $S|_{\ker(X)^{\perp}}$ are unitarily equivalent normal operators.*

Proof. (i) Decompose T and S^* into their normal and pure parts as in Lemma 4.4 and [17]. Then we have

$$\begin{aligned} T &= N \oplus A \quad \text{on} \quad \mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2 \\ S^* &= M^* \oplus B^* \quad \text{on} \quad \mathscr{K} = \mathscr{K}_1 \oplus \mathscr{K}_2 \end{aligned}$$

and

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} : \mathscr{K} = \mathscr{K}_1 \oplus \mathscr{K}_2 \longrightarrow \mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2,$$

where N, M are normal, A is a w-hyponormal and B^* is class $\mathcal{Y}$. Then $TX = XS$ implies that

$$\begin{pmatrix} NX_{11} & NX_{12} \\ AX_{21} & AX_{22} \end{pmatrix} = \begin{pmatrix} X_{11}M & X_{12}B \\ X_{21}M & X_{22}B \end{pmatrix}.$$

Let $A = U_2|A|$ be the polar decomposition of A and $\widetilde{A} = |A|^{\frac{1}{2}}U_2|A|^{\frac{1}{2}}, W = |A|^{\frac{1}{2}}X_{22}$. Then

$$\begin{aligned} \widetilde{A}W &= |A|^{\frac{1}{2}}U_2|A|^{\frac{1}{2}}|A|^{\frac{1}{2}}X_{22} \\ &= |A|^{\frac{1}{2}}U_2|A|X_{22} \\ &= |A|^{\frac{1}{2}}X_{22}(B^*)^* = W(B^*)^*. \end{aligned}$$

Since A is a w-hyponormal operator, then $\widetilde{A}$ is semi-hyponormal operator, B^* is a class $\mathcal{Y}$. Hence it follows from [18, Theorem 7] that $\overline{\Re(W)}$ reduces $\widetilde{A}$, $\ker(W)^\perp$ reduces B^* and $\widetilde{A}|_{\overline{\Re(W)}}$, $B^*|_{\ker(W)^\perp}$ are unitarily equivalent normal operators. Since A and B^* are pure, we have $W = 0$ by Lemma 4.4 and Lemma 4.10. Then $X_{22} = 0$ as A, B^* are injective. Since $AX_{21} = X_{21}M$ and $NX_{12} = X_{12}B$ we have $X_{21}M = 0$ and $NX_{12} = 0$ by similar arguments. Then $TX = XS$ implies

$$\begin{pmatrix} NX_{11} & 0 \\ AX_{21} & 0 \end{pmatrix} = \begin{pmatrix} X_{11}M & X_{12}B \\ 0 & 0 \end{pmatrix}$$

and $X_{12} = X_{21} = 0$. Hence $X = \begin{pmatrix} X_{11} & 0 \\ 0 & 0 \end{pmatrix}$ and

$$\Re(X) = \Re(X_{11}) \oplus \{0\}, \ker(X)^\perp = \ker(X_{11})^\perp \oplus \{0\}.$$

Since $NX_{11} = X_{11}M$, we have $N^*X_{11} = X_{11}M^*$, $\overline{\Re(X_{11})}$ reduces N, $\ker(X_{11})^\perp$ reduces M, $N|_{\overline{\Re(X_{11})}}$, $M|_{\ker(X_{11})^\perp}$ are unitarily equivalent normal operators. Then $N|_{\overline{\Re(X)}} \cong N|_{\overline{\Re(X_{11})}}$, $M|_{\ker(X)^\perp} \cong M|_{\ker(X_{11})^\perp}$ imply that $T^*X = XS^*$, $\overline{\Re(X)}$ reduces T, $\ker(X)^\perp$ reduces S, $T|_{\overline{\Re(X)}}, S|_{\ker(X)^\perp}$ are unitarily equivalent normal operators.

(ii) Since $TX = XS$, we have $S^*X^* = X^*T^*$. Hence $SX^* = S^{**}X^* = X^*T^{**}$ by part (i) and $T^*X = XS^*$. The rest of the proof follows from Lemma 4.1. □

Corollary 4.14. *Let $T \in \mathscr{L}(\mathscr{H})$. Then T is normal if and only if either (i) T is a w-hyponormal operator such that* $\ker(T) \subset \ker(T^*)$ *and T^* is a class $\mathcal{Y}$ operator or (ii) T is a class $\mathcal{Y}$ operator and T^* is a w-hyponormal operator such that* $\ker(S^*) \subset \ker(S)$.

Corollary 4.15. *Let $T \in \mathscr{L}(\mathscr{H})$ and $S^* \in \mathscr{L}(\mathscr{K})$ be such that $TX = XS$. If either T is pure w-hyponormal such that* $\ker(T) \subset \ker(T^*)$ *and S^* is class $\mathcal{Y}$ or T is w-hyponormal such that* $\ker(T) \subset \ker(T^*)$ *and S^* is pure class $\mathcal{Y}$, then $X = 0$.*

Proof. The hypotheses imply that $TX = XS$ and $T^*X = XS^*$ simultaneously by Theorem 4.13. Therefore $T|_{\overline{\Re(X)}}$ and $S|_{\ker(X)^\perp}$ are unitarily equivalent normal operators, which contradicts the hypotheses that T or S^* is pure. Hence we must have $X = 0$. □

References

[1] A. Aluthge, On p-hyponormat operators for $0 < p < 1$, Integral Equation Operater Theory 13 (1990) 307–315.

[2] M. Fujii, S. Izumino, R. Nakamoto, classes of operators determined by the Heinz-Kato-Furuta inequality and the Hölder-McCarthy inequality, Nihonkai Math. J. 5 (1994) 61–67.

[3] T. Furuta, M. Ito, T. Yamazaki, A subclass of paranormal operators including class of *log*-hyponormal and several related classes. Sci. math. 1 (1998) 389–403.

[4] I. H. Jeon, J.I. Lee, A. Uchiyama, On p-quasihyponormal operators and quasisimilarity. Math. Ineq. App. 6 (2) (2003) 309–315.

[5] T. Ando, Operators with norm condition, Acta. Sci. Math. 33 (4) (1972) 359–365.

[6] A. Aluthge, D. Wang, w-Hyponormal operators. Integral Equation Operator Theory 36 (2000) 1–10.

[7] S. M. Patel, A note on p-hyponormal operators for $0 < p < 1$, Integral Equations and Operator Theory 21 (1995) 498–503.

[8] R.G. Douglas, On majorization, factorization, and range inclusion of operators on Hilbert space, Proc. Amer. Math. Soc. 17 (1966) 413–415.

[9] S.M. Patel, K. Tanahashi, A. Uchiyama, M. Yanagida, Quasinormality and Fuglede-Putnam theorem for class $A(s,t)$ operators, Nihonkai Math. J 17 (2006) 49–67.

[10] M. Radjabalipour, An extension of Putnam-Fuglede theorem for hyponormal operators, Math. Z. 194 (1987) 117120.

[11] M.H.M. Rashid, Class $wA(s,t)$ operators and quasisimilarity, Portugaliae Math. 69 (4) (2012) 305-320, DOI: 10.4171/PM/1919.

[12] M.H.M. Rashid, An Extension of Fuglede-Putnam Theorem for w-Hyponormal Operators, Afr. Diaspora J. Math. 14 (1) (2012) 106–118.

[13] M.H.M. Rashid, Fuglede-Putnam type theorems via the generalized Aluthge transform, Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas 108 (2) (2014) 1021–1034.

[14] A. Uchiyama, K. Tanahashi, Fuglede-Putnam theorem for p-hyponormal or log-hyponormal operators, Glassgow Math. Jour. 44 (2002) 397-410.

[15] K. Takahashi, On the converse of Putnam-Fuglede theorem, Acta Sci. Math.(Szeged) 43 (1981) 123–125.

[16] M. Yanagida, Powers of class $wA(s,t)$ operators with generalized Aluthge transformation, J. Inequal. Appl. 7 (2002) 143–168.

[17] A. Uchiyama, T. Yochino, On the class $\mathcal{Y}$ operators, Nihonkai. Math. J. 8 (1997) 174–179.

[18] S. Mecheri, K. Tanahashi, A. Uchiyama, Fuglede-Putnam theorem for p-hyponormal or class $\mathcal{Y}$ operators, Bull. Korean. Math. Soc. 43 (2006) 747–753.

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