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Common Fixed Point Theorems of Integral Type for OWC Mappings under Relaxed Condition

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Abstract : In this paper, we prove a common fixed point theorem for a pair of occasionally weakly compatible (owc) self mappings satisfying a mixed contractive condition of integral type without using the triangle inequality. We prove also analogous results for two pairs of owc self mappings by assuming symmetry only on the set of points of coincidence. These results unify, extend and complement many results existing in the recent literature. Finally, we give an application of our results in dynamic programming.

Keywords : common fixed points; weakly compatible mappings; occasionally weakly compatible mappings; contractive condition of integral type; symmetric spaces.
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1 Introduction

Generalization of the Banach contraction mapping principle is one of pivotal results of analysis and has been an heavily investigated field of research. It is

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widely considered as the source of metric fixed point theory and the significance lies in its vast applicability in a number of branches of mathematics. In particular the establishment of fixed point theorems for a mapping satisfying a contractive condition without requirement of continuity at each point was firstly initiated by Kannan [1] in 1968. After that, there flows a flood of papers and several authors studied fixed point theorems for a pair of mappings. In this context, the notion of weakly commuting mappings was introduced by Sessa [2] that weakened the concept of commutativity of two mappings. Successively, Jungck [3, 4] enlarged the concept of weakly commuting mappings by adding the notions of compatible mappings and weakly compatible mappings. Then, Al-Thagafi and Shahzad [5] gave a definition of occasionally weakly compatible (owc) mappings which is a proper generalization of weakly compatible mappings. For other relaxed fixed point theorems in symmetric spaces and their applications, one may refer to [6, 7], [8]-[12] and [13, 14].
In 2002, Branciari [9] analyzed the existence of fixed points for a mapping f defined on a complete metric space (X, d) satisfying a contractive condition of integral type in the following manner:

Theorem 1.1. *Let (X, d) be a complete metric space, $\alpha \in (0, 1)$ and $f : X \to X$ be a mapping such that for each $x, y \in X$,*

$$\int_0^{d(fx,fy)} \varphi(s)ds \leq \alpha \int_0^{d(x,y)} \varphi(s)ds,$$

where $\varphi : [0, +\infty) \to [0, +\infty)$ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0, +\infty)$ and such that, for each $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(s)ds > 0$. Then f has a unique fixed point $a \in X$ such that for each $x \in X$, $\lim_{n\to\infty} f^n x = a$.

Recently, on similar lines, Vetro [15] proved the following theorem for two pairs of mappings.

Theorem 1.2. *Let (X, d) be a metric space and let A, B, S and T be self mappings of X with $S(X) \subseteq B(X)$ and $T(X) \subseteq A(X)$. We define, for each $x, y \in X$,*

$$m(x, y) = d(By, Ty)\frac{1 + d(Ax, Sx)}{1 + d(Ax, By)}$$

and

$$M(x, y) = \max\{d(Ax, By), d(Ax, Sx), d(By, Ty)\}.$$

We assume that for each $x, y \in X$,

$$\int_0^{d(Sx,Ty)} \varphi(s)ds \leq \alpha \int_0^{m(x,y)} \varphi(s)ds + \beta \int_0^{M(x,y)} \varphi(s)ds,$$

where $\alpha > 0$, $\beta > 0$, $\alpha + \beta < 1$ and $\varphi : [0, +\infty) \to [0, +\infty)$ is a Lebesgue integrable mapping on each compact subset of $[0, +\infty)$ and such that for all $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(s)ds > 0$. Suppose that one of $A(X), B(X), S(X)$ and $T(X)$ is a complete subset of X and the pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible. Then A, B, S and T have a unique common fixed point in X.

We recall the following concepts.

Definition 1.3. A pair of self mappings $\{f, g\}$ defined on a metric (symmetric) space (X, d) is said to be:

(i) (Sessa [2]) *weakly commuting* if $d(fgx, gfx) \leq d(fx, gx)$, for all $x \in X$;

(ii) (Jungck [3]) *compatible* if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$;

(iii) (Jungck [4]) *weakly compatible* if the mappings commute at their coincidence points, that is, if $fx = gx$ for some $x \in X$, then $fgx = gfx$;

(iv) (Pant [16]) *R-weakly commuting* if there exists some real $R > 0$ such that $d(fgx, gfx) \leq R\, d(fx, gx)$ for all $x \in X$.

Definition 1.4. Let f, g be two self mappings of X. A point $x \in X$ is called *a coincidence point* of f and g iff $fx = gx$. We call $w = fx = gx$ a point of coincidence of f and g.

Definition 1.5. [5] Two self mappings f and g of a set X are *occasionally weakly compatible (owc)* iff there is a point x which is a coincidence point of f and g at which f and g commute.

Definition 1.6. A *symmetric* on X is a mapping $d : X \times X \to [0, +\infty)$ such that $d(x, y) \geq 0$ and $d(x, y) = 0$ iff $x = y$ and $d(x, y) = d(y, x)$. A set X endowed with a symmetric d is called *symmetric space.*

Lemma 1.7. [13] *Let X be a set and f, g be owc self mappings of X. If f and g have a unique point of coincidence, $w = fx = gy$, then w is a unique common fixed point of f and g.*

Now, we are ready to prove our results which are of three folds:

(i) We relax the containment of mappings.

(ii) We use occasionally weak compatibility that is more general than compatibility.

(iii) We consider the space (X, d) under relaxed condition, that is more general than metric (symmetric) space.

2 Fixed Point Theorems for a Pair of OWC Mappings

In this section, we prove a fixed point theorem for a pair of owc mappings satisfying a mixed contractive condition of integral type on the space (X, d), without imposing the triangular inequality or the symmetry on d.

Theorem 2.1. *Let X be a non-empty set and $d : X \times X \to [0, +\infty)$ be a function satisfying the condition $d(x, y) = 0$ iff $x = y$, for all $x, y \in X$. Suppose that f and g are owc mappings of X. We define for each $x, y \in X$,*

$$\begin{aligned} M(x,y) &= p\, d(fy, gy)\frac{1 + d(fx, gx)}{1 + d(fy, gx)} + \phi(\max\{d(gx, gy), d(gx, fy), \\ &\quad d(gy, fx), d(gy, fy)\}), \end{aligned} \tag{2.1}$$

where $0 < p < 1$ and $\phi : [0, +\infty) \to [0, +\infty)$ is a function satisfying the condition $\phi(t) < t$ for each $t > 0$ and $\phi(t) = 0$ if $t = 0$. We assume also that for each $x, y \in X$,

$$\int_0^{d(fx,fy)} \varphi(s)ds \le \int_0^{M(x,y)} \varphi(s)ds, \tag{2.2}$$

where $\varphi : [0, +\infty) \to [0, +\infty)$ is a Lebesgue integrable mapping on each compact subset of $[0, +\infty)$ and such that, for all $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(s)ds > 0$. Then f and g have a unique common fixed point.

Proof. Since f and g are owc, there exists a point $u \in X$ such that $fu = gu$ and $fgu = gfu$. We claim that fu is a unique common fixed point of f and g. We first assert that fu is a fixed point of f, if not, then by (2.1), we have

$$\begin{aligned} M(u, fu) &= p\, d(ffu, gfu)\frac{1 + d(fu, gu)}{1 + d(ffu, gu)} + \phi(\max\{d(gu, gfu), \\ &\quad d(gu, ffu), d(gfu, fu), d(gfu, ffu)\}) \\ &= \phi(\max\{d(fu, ffu), d(ffu, fu)\}). \end{aligned} \tag{2.3}$$

By use of (2.3), (2.2) becomes

$$\int_0^{d(fu,ffu)} \varphi(s)ds \le \int_0^{\phi(\max\{d(fu,ffu),d(ffu,fu)\})} \varphi(s)ds. \tag{2.4}$$

Let $\alpha = \max\{d(fu, ffu), d(ffu, fu)\} > 0$, then from (2.4), we have

$$\int_0^{d(fu,ffu)} \varphi(s)ds \le \int_0^{\phi(\alpha)} \varphi(s)ds < \int_0^{\alpha} \varphi(s)ds.$$

Similarly by (2.1), we get

$$\begin{aligned} M(fu, u) &= p\, d(fu, gu)\frac{1 + d(ffu, gfu)}{1 + d(fu, gfu)} + \phi(\max\{d(gfu, gu), \\ &\quad d(gfu, fu), d(gu, ffu), d(gu, fu)\}) \\ &= \phi(\max\{d(ffu, fu), d(fu, ffu)\}). \end{aligned} \tag{2.5}$$

By use of (2.5), (2.2) becomes

$$\int_0^{d(ffu,fu)} \varphi(s)ds \le \int_0^{\phi(\alpha)} \varphi(s)ds < \int_0^{\alpha} \varphi(s)ds. \tag{2.6}$$

Then, we have

$$\int_0^{\max\{d(ffu,fu),d(fu,ffu)\}} \varphi(s)ds \leq \int_0^{\phi(\alpha)} \varphi(s)ds < \int_0^{\alpha} \varphi(s)ds,$$

a contradiction. Hence $ffu = fu$ and $ffu = fgu = gfu = fu$. Thus, fu is a common fixed point of f and g. Finally, we prove uniqueness of the fixed point. Suppose that $u, v \in X$ are such that $fu = gu = u$, $fv = gv = v$ and $u \neq v$. Then by (2.1), we obtain

$$\begin{aligned} M(u,v) &= p\,d(fv,gv)\frac{1+d(fu,gu)}{1+d(fv,gu)} + \phi(\max\{d(gu,gv),d(gu,fv), \\ &\quad d(gv,fu),d(gv,fv)\}) \\ &= \phi(\max d(u,v),d(v,u)). \end{aligned} \tag{2.7}$$

By use of (2.7), (2.2) becomes

$$\int_0^{d(u,v)} \varphi(s)ds = \int_0^{d(fu,fv)} \varphi(s)ds \leq \int_0^{\phi(\max\{d(u,v),d(v,u)\})} \varphi(s)ds. \tag{2.8}$$

Now, let $\beta = \max\{d(u,v),d(v,u)\} > 0$, then from (2.8), we have

$$\int_0^{d(u,v)} \varphi(s)ds \leq \int_0^{\phi(\beta)} \varphi(s)ds < \int_0^{\beta} \varphi(s)ds.$$

Also, it is easy to show that

$$\int_0^{d(v,u)} \varphi(s)ds \leq \int_0^{\phi(\beta)} \varphi(s)ds < \int_0^{\beta} \varphi(s)ds.$$

Hence, we have

$$\int_0^{\max\{d(u,v),d(v,u)\}} \varphi(s)ds \leq \int_0^{\phi(\beta)} \varphi(s)ds < \int_0^{\beta} \varphi(s)ds,$$

that is a contradiction. Therefore $u = v$, i.e., the common fixed point is unique. □

Theorem 2.2. *Theorem 2.1, will remain true if the contractive condition* (2.1) *is replaced by anyone of the following conditions:*

(i) $M(x,y) = p\,d(fy,gy)\frac{1+d(fx,gx)}{1+d(fy,gx)} + \phi(d(gx,gy))$;

(ii) $M(x,y) = p\,d(fy,gy)\frac{1+d(fx,gx)}{1+d(fy,gx)} + k\,\max\{d(gx,gy),d(gx,fy), d(gy,fx),d(gy,fy)\}$, *where* $0 < k < 1$;

(iii) $M(x,y) = a\,d(gx,gy) + b\,\max\{d(fx,gx),d(fy,gy)\} + c\,\max\{d(gx,gy),d(gx,fx),d(gy,fy)\}$, *where* a, b *and* c *are nonnegative numbers such that* $a + b + c < 1$.

Remark 2.3. *Every contractive condition of integral type automatically induces the corresponding contractive condition not including integral, by setting* $\varphi(s) = 1$.

The following example shows that condition (2.2) in Theorem 2.1 is indeed a proper extension of the same condition with $\varphi(s) = 1$.

Example 2.1. Consider $X = \{1/n \mid n \in \mathbb{N}\} \cup \{0\}$ equipped with the usual metric $d(x, y) = \mid x - y \mid$ for all $x, y \in X$. Clearly (X, d) is a complete metric space. Define $f, g : X \to X$ as

$$fx = \begin{cases} \frac{1}{n+4} & \text{if } x = \frac{1}{n},\ n \text{ odd}, \\ 0 & \text{otherwise}, \end{cases} \qquad gx = \begin{cases} \frac{1}{n+2} & \text{if } x = \frac{1}{n},\ n \text{ odd}, \\ 0 & \text{otherwise}. \end{cases}$$

Thus the reader, following the same lines of Example 3.6 in [9], can verify that condition (2.2) is satisfied assuming $0 < p < 1/2$, $\phi(t) = t/2$ and

$$\varphi(s) = \begin{cases} \max\{(1 - \log s)s^{(1/s)-2}, 0\} & \text{if } s > 0, \\ 0 & \text{if } s = 0. \end{cases}$$

However, for $x = 1/n$ with n odd and $y = 0$, condition (2.2) with $\varphi(s) = 1$ leads to the contradiction

$$\frac{1}{n+4} \leq p \cdot 0 + \frac{1}{2} \max\left\{\frac{1}{n+2},\ \frac{1}{n+2},\ \frac{1}{n+4},\ 0\right\} = \frac{1}{2n+4}.$$

The Class of symmetric spaces is more general than the class of metric spaces but we have also relaxed the symmetric condition on d. Therefore, Theorem 2.1, Corollary 2.1, Theorem 2.2 and Theorem 2.3 of [8] can be seen as special cases of Theorem 2.1 of this paper by setting $p = 0$.
The following example and remark are important in order to fully understand that relaxing the symmetric condition on d is really an useful tool to cover a wide range of problems.

Example 2.2. Let $X = [0, 1]$ and define $d : X \times X \to [0, +\infty)$ by

$$d(x, y) = \begin{cases} e^{x-y} - 1 & \text{if } x \geq y, \\ e^{y-x} & \text{if } y > x. \end{cases}$$

Define also $f, g : X \to X$ by $fx = \frac{1+x}{2}$ and $gx = \frac{1+3x}{4}$, for all $x \in X$. Assuming $0 < p < 1/2$, $\varphi(s) = 1$ and $\phi(t) = t/2$, the reader can show easily that all the hypotheses of Theorem 2.1 are satisfied and so $x = 1$ is the unique common fixed point of f and g.

Remark 2.4. *In the above Example, clearly, d is symmetric only on the set of points of coincidence of f and g. Therefore, in this case a very general fixed point theorem as is Theorem* 1 *of* [13] *cannot be applied.*

3 Fixed Point Theorems for Two Pairs of OWG Self Mappings

In this section, we prove several fixed point theorems for four self mappings on (X, d) satisfying a contractive condition of integral type assuming symmetry of d on the points of coincidence of the pairs $\{f, S\}$ and $\{g, T\}$.

Theorem 3.1. *Let X be a non-empty set and $d : X \times X \to [0, +\infty)$ be a function satisfying the condition $d(x, y) = 0$ iff $x = y$, for all $x, y \in X$. Suppose that f, g, S and T are self mappings of X, the pairs $\{f, S\}$ and $\{g, T\}$ are owc and $d(z, w) = d(w, z)$ whenever w and z are, respectively, points of coincidence of $\{f, S\}$ and $\{g, T\}$. We assume also that, for each $x, y \in X$ with $fx \neq gy$,*

$$\int_0^{d(fx,gy)} \varphi(s)ds < \int_0^{M(x,y)} \varphi(s)ds, \tag{3.1}$$

where

$$\begin{aligned} M(x, y) \quad = \quad & p\, d(gy, Ty)\frac{1 + d(fx, Sx)}{1 + d(fx, gy)} + \max\{d(Sx, Ty), d(Sx, fx), \\ & d(Ty, gy), d(Ty, fx)\}, \end{aligned} \tag{3.2}$$

with $0 < p < 1$ and $\varphi : [0, +\infty) \to [0, +\infty)$ is a Lebesgue integrable mapping on each compact subset of $[0, +\infty)$ and such that, for all $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(s)ds > 0$. Then f, g, S and T have a unique common fixed point.

Proof. Since the pairs $\{f, S\}$ and $\{g, T\}$ are owc, there exist points $x, y \in X$ such that $fx = Sx$ and $gy = Ty$. We claim that $fx = gy$. If not, then by (3.2), we have

$$\begin{aligned} M(x, y) \quad = \quad & p\, d(gy, Ty)\frac{1 + d(fx, Sx)}{1 + d(fx, gy)} + \max\{d(Sx, Ty), d(Sx, fx), \\ & d(Ty, gy), d(Ty, fx)\} = \max\{d(fx, gy), d(gy, fx)\}. \end{aligned}$$

As, $fx = Sx = w$ and $gy = Ty = z$ are points of coincidence of $\{f, S\}$ and $\{g, T\}$, respectively, and by use of $d(z, w) = d(w, z)$, from (3.1), we obtain

$$\int_0^{d(fx,gy)} \varphi(s)ds < \int_0^{\max\{d(fx,gy),d(gy,fx)\}} \varphi(s)ds = \int_0^{d(fx,gy)} \varphi(s)ds.$$

This leads to a contradiction. Hence $fx = gy$ and so $fx = Sx = gy = Ty$. Moreover, if there is another point u such that $fu = Su$, then by (3.2), we get $fu = Su = gy = Ty$, and so $fx = fu$. Thus, $w = fx = Sx$ is the unique point of coincidence of f and S. By Lemma 1.7, w is the unique common fixed point of f and S. Similarly, there is a unique point $z \in X$ such that $z = gz = Tz$. Suppose that $w \neq z$. By use of (3.1) and (3.2), we get

$$\begin{aligned}\int_0^{d(w,z)} \varphi(s)ds &= \int_0^{d(w,gz)} \varphi(s)ds \\ &< \int_0^{\max\{d(w,z),d(z,w)\}} \varphi(s)ds = \int_0^{d(w,z)} \varphi(s)ds.\end{aligned}$$

This is a contradiction. Therefore, $w = z$ is the unique common fixed point of f, g, S and T. □

Remark 3.2. *Theorem 3.1 complements and extends Theorem 2 of* [15].

Example 3.1. Let X and d as in Example 2.2 and define $f, g, S, T : X \to X$ by $fx = 1$, $gx = x$, $Sx = \frac{1+x}{2}$ and $Tx = x^2$ for all $x \in X$. Assuming $0 < p < 1$ and $\varphi(s) = 1$, the reader can show easily that all the hypotheses of Theorem 3.1 are satisfied and so $x = 1$ is the unique common fixed point of f, g, S and T.
However, d is symmetric only on the set of points of coincidence of f, g, S and T and so Theorem 1 of [13] cannot be applied.

Theorem 3.3. *Let X be a non-empty set and $d : X \times X \to [0, +\infty)$ be a function satisfying the condition $d(x, y) = 0$ iff $x = y$, for all $x, y \in X$. Suppose that f, g, S and T are self mappings of X, the pairs $\{f, S\}$ and $\{g, T\}$ are owc and $d(z, w) = d(w, z)$ whenever w and z are, respectively, points of coincidence of $\{f, S\}$ and $\{g, T\}$. We assume also that, for each $x, y \in X$ with $fx \neq gy$,*

$$\int_0^{(d(fx,gy))^k} \varphi(s)ds < \int_0^{M(x,y)} \varphi(s)ds, \tag{3.3}$$

where

$$\begin{aligned}M(x,y) &= p(d(gy,Ty))^k \frac{1+(d(fx,Sx))^k}{1+(d(fx,gy))^k} + a(d(fx,Ty))^k + \\ &\quad (1-a)\max\{(d(fx,Sx))^k, (d(gy,Ty))^k, (d(fx,Sx))^{k/2} \\ &\quad (d(fx,Ty))^{k/2}, (d(Ty,fx))^{k/2}(d(Sx,gy))^{k/2}\}\end{aligned} \tag{3.4}$$

with $0 < a, p < 1$, $k \geq 1$ and $\varphi : [0, +\infty) \to [0, +\infty)$ is a Lebesgue integrable mapping on each compact subset of $[0, +\infty)$ and such that, for all $\varepsilon > 0$, $\int_0^\varepsilon \varphi(s)ds > 0$. Then f, g, S and T have a unique common fixed point.

Proof. Since the pairs $\{f, S\}$ and $\{g, T\}$ are owc, there exist points $x, y \in X$ such that $fx = Sx$ and $gy = Ty$. We claim that $fx = gy$. If not, then by (3.4), we have

$$\begin{aligned}M(x,y) &= p\,0\frac{1+0}{1+(d(fx,gy))^k} + a(d(fx,gy))^k \\ &\quad +(1-a)\max\{0,0,0,(d(gy,fx))^{k/2}(d(fx,gy))^{k/2}\} \\ &= a(d(fx,gy))^k + (1-a)\{(d(gy,fx))^{k/2}(d(fx,gy))^{k/2}\}.\end{aligned}$$

As $fx = Sx = w$ and $gy = Ty = z$ are, respectively, points of coincidence of $\{f, S\}$ and $\{g, T\}$ and $d(z, w) = d(w, z)$, from (3.3), we obtain

$$\begin{aligned}\int_0^{d(fx,gy))^k} \varphi(s)ds &< \int_0^{a(d(fx,gy))^k+(1-a)(d(fx,gy))^k} \varphi(s)ds \\ &= \int_0^{(d(fx,gy))^k} \varphi(s)ds.\end{aligned}$$

It leads to contradiction and hence $d(fx, gy) = 0$, which yields $fx = gy$. Now, suppose that there exists another point u such that $fu = Su$. Then, by use of (3.4), we get $fu = Su = gy = Ty = fx = Sx$. Therefore, $w = fx = Sx$ is the unique point of coincidence of f and S. By Lemma 1.7, w is the unique common fixed point of f and S. Similarly the point of coincidence of the pair $\{g, T\}$ is unique and it is the unique common fixed point of the pair $\{g, T\}$. Following similar arguments to those in Theorem 3.1, it is easy to show that the point of coincidence of the pairs $\{f, S\}$ and $\{g, T\}$ is the same and it is the unique common fixed point of f, g, s and T. To avoid repetitions the details are omitted. □

Let $\psi : [0, +\infty) \to [0, +\infty)$ be such that $\psi(t) < t$, for each $t > 0$ and let $\Phi : [0, +\infty) \to [0, +\infty)$ be a continuous, monotonically increasing function such that $\Phi(2t) \leq 2\Phi(t)$ and $\Phi(0) = 0$ iff $t = 0$.

Theorem 3.4. *Let X be a non-empty set and $d : X \times X \to [0, +\infty)$ be a function satisfying the condition $d(x, y) = 0$ iff $x = y$, for all $x, y \in X$. Suppose that f, g, S and T are self mappings of X, the pairs $\{f, S\}$ and $\{g, T\}$ are owc and $d(z, w) = d(w, z)$, whenever w and z are, respectively, points of coincidence of $\{f, S\}$ and $\{g, T\}$. We assume also that, for each $x, y \in X$,*

$$\int_0^{d(fx,gy)} \varphi(s)ds \leq \int_0^{\psi(M_\Phi(x,y))} \varphi(s)ds, \tag{3.5}$$

where

$$\begin{aligned}M_\Phi(x,y) &= p\, d(gy,Ty)\frac{1+d(fx,Sx)}{1+d(fx,gy)} + \max\{\Phi(d(Sx,Ty)), \\ &\quad \Phi(d(Sx,fx)), \Phi(d(gy,Ty)), \frac{\Phi(d(fx,Ty)) + \Phi(d(Sx,gy))}{2}\}\end{aligned} \tag{3.6}$$

with $0 < p < 1$ and $\varphi : [0, +\infty) \to [0, +\infty)$ is a Lebesgue integrable mapping on each compact subset of $[0, +\infty)$ and such that, for all $\varepsilon > 0$, $\int_0^\varepsilon \varphi(s)ds > 0$. Then f, g, S and T have a unique common fixed point.

Proof. Since the pairs $\{f, S\}$ and $\{g, T\}$ are owc, there exist points $x, y \in X$ such that $fx = Sx$ and $gy = Ty$. We claim that $fx = gy$. If not, by $d(z, w) = d(w, z)$

and (3.5), we have

$$\begin{aligned}\int_0^{\Phi(d(fx,gy))}\varphi(s)ds &\leq \int_0^{\psi(M_\Phi(x,y))}\varphi(s)ds\\ &= \int_0^{\psi(\Phi(d(fx,gy)))}\varphi(s)ds < \int_0^{\Phi(d(fx,gy))}\varphi(s)ds.\end{aligned}$$

This leads to contradiction. Hence $\Phi(d(fx,gy)) = 0$ and so $d(fx,gy) = 0$, i.e., $fx = gy$. Now, by repeated use of the condition (3.5), it is easy to show that f, g, S and T have a unique common fixed point. Therefore the details are omitted. □

Define $\Gamma = \{\gamma : [0,+\infty)^5 \to [0,+\infty)\}$ such that

(γ1) γ is nondecreasing in its fourth and fifth variables;

(γ2) if $u, v \in [0,+\infty)$ are such that $u \leq \Psi(v,v,u,u+v,0)$ or $u \leq \Psi(v,u,v,u+v,0)$ or $v \leq \Psi(u,u,v,u+v,0)$ or $u \leq \Psi(v,u,v,u,u+v)$, then $u \leq hv$, where $0 < h < 1$ is a constant;

(γ3) if $u \in [0,+\infty)$ is such that $u \leq \Psi(u,0,0,u,u)$ or $u \leq \Psi(0,u,0,u,u)$ or $u \leq \Psi(0,0,u,u,u)$, then $u = 0$.

Theorem 3.5. *Let X be a non-empty set and $d : X \times X \to [0,+\infty)$ be a function satisfying the condition $d(x,y) = 0$ iff $x = y$, for all $x, y \in X$. Suppose that f, g, S and T are self mappings of X, the pairs $\{f,S\}$ and $\{g,T\}$ are owc and $d(z,w) = d(w,z)$ whenever w and z are, respectively, points of coincidence of $\{f,S\}$ and $\{g,T\}$. We assume also that, for all $x, y \in X$,*

$$\int_0^{d(fx,gy)}\varphi(s)ds \leq \int_0^{M(x,y)}\varphi(s)ds, \tag{3.7}$$

where

$$\begin{aligned}M(x,y) &= p\,d(gy,Ty)\frac{1+d(fx,Sx)}{1+d(fx,gy)} + \Psi(\max\{d(Sx,Ty),\\ &\quad d(fx,Sx), d(gy,Ty), d(gy,Sx), d(fx,Ty)\}),\end{aligned} \tag{3.8}$$

with $0 < p < 1$ and $\varphi : [0,+\infty) \to [0,+\infty)$ is a Lebesgue integrable mapping on each compact subset of $[0,+\infty)$ and such that, for all $\varepsilon > 0$, $\int_0^\varepsilon \varphi(s)ds > 0$. Then f, g, S and T have a unique common fixed point.

Proof. Since the pairs $\{f,S\}$ and $\{g,T\}$ are owc, there exist points $x, y \in X$ such that $fx = Sx$ and $gy = Ty$. We claim that $fx = gy$. If not, then by (3.8), we have

$$M(x,y) = \Psi(\max\{d(fx,gy), 0, 0, d(fx,gy), d(gy,fx)\}).$$

As, $fx = Sx = w$ and $gy = Ty = z$ are, respectively, points of coincidence of $\{f,S\}$ and $\{g,T\}$ and $d(z,w) = d(w,z)$, from (3.7), we have

$$\int_0^{d(fx,gy)}\varphi(s)ds \leq \int_0^{\Psi(\max\{d(fx,gy),0,0,d(fx,gy),d(fx,gy)\})}\varphi(s)ds.$$

Using ($\gamma 3$), we have $d(fx, gy) = 0$, i.e. $fx = gy$. Suppose that there exists another point $u \in X$ such that $fu = Su$. Then, using (3.7), we get $fu = Su = gy = Ty = fx = Sx$. Hence $w = fx = Sx$ is the unique point of coincidence of f and S. Since $\{f, S\}$ is owc, by using Lemma 1.7, we conclude that w is the unique common fixed point of f and S. Similarly, there exists a unique point $z \in X$ such that $z = gv = Tv$. Following the same lines as above, one can show easily that $z = w$ and so w is the unique common fixed point of f, g, S and T. □

Remark 3.6. *Theorems 3.1, 3.2, 3.3 and 3.4 of* [8] *are, respectively, special cases of Theorems 3.1, 3.3, 3.4 and 3.5 of this paper by setting* $p = 0$ *and* $\varphi(s) = 1$.

4 Application to Dynamic Programming

Throughout this section, we assume that X and Y are Banach Spaces, $S \subseteq X$ is a state space and $D \subseteq Y$ a decision space. We denote by $B(S)$ the set of all bounded real valued functions defined on S. Bellman and Lee [17], first studied the existence of solutions for some classes of functional equations arising in dynamic programming. They pointed out that the basic form of the functional equations in dynamic programming is the following:

$$f(x) = opt_y H(x, y, f(T(x, y))),$$

where *opt* represents sup or inf, x and y denote the state and decision vectors, respectively, T stands for the transformation of the process, and $f(x)$ represents the optimal return function with the initial state x.

Now we study existence and uniqueness of common solution for some kinds of functional equations arising in dynamic programming:

$$P(x) = \sup_{y \in D} H(x, y, P(T(x, y))),\ x \in S \tag{4.1}$$

$$Q(x) = \sup_{y \in D} F(x, y, Q(T(x, y))),\ x \in S \tag{4.2}$$

where $T : S \times D \to S$, H and $F : S \times D \times \mathbb{R} \to \mathbb{R}$.
For all $h, k \in B(S)$, we endow $B(S)$ with the metric

$$d(h, k) = \sup |h(x) - k(x)|,\ x \in S.$$

Now, we give the main result of this section.

Theorem 4.1. *Assume that the following conditions hold:*

(i) H *and* F *are bounded;*

(ii) $|H(x, y, h(t)) - H(x, y, k(t))| \le p|fk(t) - gk(t)|\frac{|fh(t)-gh(t)|}{|fk(t)-gh(t)|}$ $+ \phi(\max\{|gh(t) - gk(t)|, |gh(t) - fk(t)|, |gk(t) - fh(t)|, |gk(t) - k(t)|\})$. *For all* $(x, y) \in S \times D$, h, $k \in B(S)$ *and* $t \in S$, *where* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *is a*

nondecreasing function satisfying the condition $\phi(t) < t$ for each $t > 0$, f and g are defined as follows:

$$fh(x) = \sup_{y \in D} H(x, y, h(T(x, y))),\ x \in S,\ h \in B(S)$$

and

$$gk(x) = \sup_{y \in D} F(x, y, k(T(x, y))),\ x \in S,\ k \in B(S);$$

(iii) *$fu(x) = gu(x) = k(x)$ for some $u(x) \in B(S)$ implies $fgu(x) = gfu(x)$. Then $k(x)$ is the unique solution of* (4.1) *and* (4.2).

Proof. From conditions (i), (ii) and (iii), it follows that f and g are self mappings of $B(S)$. Let $h_1, h_2 \in B(S)$. Then, for every $\eta > 0$, there exist y_1, $y_2 \in D$ such that

$$fh_1(x) < H(x, y_1, h_1(x_1)) + \eta, \tag{4.3}$$

$$fh_2(x) < H(x, y_2, h_2(x_2)) + \eta,$$

$$fh_1(x) \geq H(x, y_2, h_1(x_2)),$$

$$fh_2(x) \geq H(x, y_1, h_2(x_1)). \tag{4.4}$$

Now, subtracting (4.4) from (4.3) and using (ii), we have

$$\begin{aligned} fh_1(x) - fh_2(x) &< H(x, y_1, h_1(x_1)) - H(x, y_1, h_2(x_1)) + \eta \\ &\leq |H(x, y_1, h_1(x_1)) - H(x, y_1, h_2(x_1))| + \eta \\ &\leq p|fh_2(x_1) - gh_2(x_1)|\frac{|fh_1(x_1) - gh_1(x_1)|}{|fh_2(x_1) - gh_1(x_1)|} \\ &\quad +\phi(\max\{|gh_1(x_1) - gh_2(x_1)|, |gh_1(x_1) - fh_2(x_1)|, \\ &\quad |gh_2(x_1) - fh_1(x_1)|, |gh_2(x_1) - fh_2(x_1)\}) + \eta. \end{aligned}$$

Letting $\eta \to 0^+$ in the above inequality, we obtain

$$\begin{aligned} d(fh_1, fh_2) \leq\ & p\, d(fh_2, gh_2)\frac{1 + d(fh_1, gh_1)}{1 + d(fh_2, gh_1)} + \phi(\max\{d(gh_1, gh_2), \\ & d(gh_1, fh_2), d(gh_2, fh_2), d(gh_2, fh_1)\}), \end{aligned}$$

for all $h_1, h_2 \in B(S)$. Therefore, the mappings f and g satisfy the hypotheses of Theorem 2.1 with $\varphi(s) = 1$. Thus, f and g have a unique common fixed point that is the unique common solution of functional equations (4.1) and (4.2) in $B(S)$. □

Remark 4.2. *Theorem 4.1 is a generalization of Theorem 4.1 of* [8].

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