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Weakened Mannheim Curves in Pseudo-Galilean 3-Space

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Abstract : In this study, Frenet-Mannheim curves and Weakened Mannheim curves are investigated in pseudo-Galilean 3-space. Some characterizations for this curves are obtained.

Keywords : Mannheim cuves; Frenet-Mannheim curves; Weakened-Mannheim curves; pseudo-Galilean 3-space.
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1 Introduction

In the study of the fundamental theory and the characterizations of space curves, the corresponding relations between the curves are the very interesting and important problem. The well-known Bertrand curve is characterized as a kind of such corresponding relation between the two curves. For the Bertrand curve α, it shares the normal lines with another curve β, called *Bertrand mate* or *Bertrand partner curve of* α [1].

In 1967, Lai investigated the properties of two types of similar curves (the Frenet-Bertrand curves and the Weakened Bertrand curves) under weakened conditions [2].

In recent works, Liu and Wang [1, 3] studied the Mannheim curves in both Euclidean and Minkowski 3-space and they obtained the necessary and sufficient conditions between the curvature and the torsion for a curve to be the Mannheim

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partner curves. Meanwhile, the detailed discussion concerned with the Mannheim curves can be found in literature (see Wang and Liu [3], Liu and Wang [1], Orbay and Kasap [4]) and references therein [4]. Karacan and Tuncer investigated the properties of two types of similar curves (the Frenet-Mannheim curves and the Weakened Mannheim curves) under weakened conditions, in [5, 6]. Öztekin investigated Weakened Bertrand curves in [7] under weakened conditions.

In this paper, our main purpose is to extend some results which were given in [2] to Frenet-Mannheim curves and Weakened Mannheim curves in pseudo-Galilean 3-space and we assume that, the angle between tangent vectors T_β and T_α is constant such that $\langle T_\alpha, T_\beta \rangle = \cosh\theta \neq 0$.

2 Preliminaries

The geometry of the pseudo-Galilean space is similar (but not the same) to the Galilean space. The pseudo-Galilean space G_3^1 is a three-dimensional projective space in which the absolute consists of a real plane w (the absolute plane), a real line $f \subset w$ (the absolute line) and a hyperbolic involution on f. Projective transformations which preserve the absolute form of a group H_8 and are in nonhomogeneous coordinates can be written in the form

$$\begin{aligned} \overline{x} &= a + bx \\ \overline{y} &= c + dx + r\cosh\theta \cdot y + r\sinh\theta \cdot z \\ \overline{z} &= e + fx + r\sinh\theta \cdot y + r\cos\theta \cdot z \end{aligned} \tag{2.1}$$

where a, b, c, d, e, f, r and θ are real numbers. Particularly, for $b = r = 1$, the group (2.1) becomes the group $B_6 \subset H_8$ of isometries (proper motions) of the pseudo-Galilean space G_3^1. The motion group remains invariant the absolute figure and defines the other invariants of this geometry. It has the following form

$$\begin{aligned} \overline{x} &= a + x \\ \overline{y} &= c + dx + \cosh\theta \cdot y + \sinh\theta \cdot z \\ \overline{z} &= e + fx + \sinh\theta \cdot y + \cos\theta \cdot z. \end{aligned} \tag{2.2}$$

According to the motion group in the pseudo-Galilean space, there are nonisotropic vectors $X(x, y, z)$ (for which holds $x \neq 0$) and four types of isotropic vectors: spacelike $\left(x = 0, y^2 - z^2 > 0\right)$, timelike $\left(x = 0, y^2 - z^2 < 0\right)$ and two types of lightlike vectors $(x = 0, y = \mp z)$. The scalar product of two vectors $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ in G_3^1 is defined by

$$\langle A, B\rangle_{G_3^1} = \begin{cases} a_1 b_1, & \text{if} \quad a_1 \neq 0 \ \vee b_1 \neq 0 \\ a_2 b_2 - a_3 b_3, & \text{if} \quad a_1 = 0 \ \wedge b_1 = 0. \end{cases} \tag{2.3}$$

The *pseudo-Galilean cross product* is defined for $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$ by

$$a \wedge_{G_3^1} b = \begin{vmatrix} 0 & -e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

A curve $\alpha(t) = (x(t), y(t), z(t))$ is *admissible* if it has no inflection points, no isotropic tangents or tangents or normals whose projections on the absolute plane would be light-like vectors. For an admissible curve $\alpha : I \subseteq R \to G_3^1$ the *curvature* $\kappa(t)$ and the *torsion* $\tau(t)$ are defined by

$$\kappa(t) = \frac{\sqrt{\left(y''(t)\right)^2 - \left(z''(t)\right)^2}}{\left(x'(t)\right)^2}\ , \quad \tau(t) = \frac{y''(t)z'''(t) - y'''(t)z''(t)}{\left|x'(t)\right|^5 \kappa^2(t)}\ . \tag{2.4}$$

expressed in components. Hence, for an admissible curve $\alpha : I \subseteq R \to G_3^1$ *parameterized by the arc length* s *with differential form* $ds = dx$, given by

$$\alpha(t) = (x, y(s), z(s))\,, \tag{2.5}$$

the formulas (2.5) have the following form

$$\kappa(s) = \sqrt{\left|\left(y''(s)\right)^2 - \left(z''(s)\right)^2\right|}\ , \quad \tau(s) = \frac{y''(s)z'''(s) - y'''(s)z''(s)}{\kappa^2(s)}\ . \tag{2.6}$$

The *associated trihedron* is given by

$$\begin{aligned} T &= \alpha'(s) = (1, y'(s), z'(s)) \\ N &= \frac{1}{\kappa(s)}\alpha''(s) = \frac{1}{\kappa(s)}\,(0, y''(s), z''(s)) \\ B &= \frac{1}{\kappa(s)}\,(0, \epsilon z''(s), \epsilon y''(s)) \end{aligned} \tag{2.7}$$

where $\epsilon = \mp 1$, chosen by criterion $\det(T, N, B) = 1$, that means

$$\left|\left(y''(s)\right)^2 - \left(z''(s)\right)^2\right| = \epsilon\left(\left(y''(s)\right)^2 - \left(z''(s)\right)^2\right).$$

We derive an important relation

$$\alpha'''(s) = \kappa'(s)N(s) + \kappa(s)\tau(s)B(s).$$

The curve α given by (2.5) is timelike (resp. spacelike) if $N(s)$ is a spacelike(resp. timelike) vector. The principal normal vector or simply normal is *spacelike* if $\epsilon = 1$ and *timelike* if $\epsilon = -1$. For derivatives of the tangent (vector) T, the normal N and the binormal B, respectively, the following *Serret-Frenet formulas* [8, 9] hold

$$\begin{aligned} T' &= \kappa N \\ N' &= \tau B \\ B' &= \tau N. \end{aligned} \tag{2.8}$$

Definition 2.1. Let G_3^1 be the 3-dimensional pseudo-Galilean space with the standard inner product $\langle , \rangle_{G_3^1}$. If there exists a corresponding relationship between the admissible curves α and β such that, at the corresponding points of the admissible curves, the principal normal lines of β coincides with the binormal lines of α, then β is called an *admissible Mannheim curve*, and α a *Mannheim partner curve of* β. The pair $\{\alpha, \beta\}$ is said to be a *Mannheim pair* [8].

Definition 2.2. An admissible Mannheim curve $\beta(s^\star)$, $s^\star \in I$ is a C^∞ regular curve with non-zero curvature for which there exists another (different) C^∞ regular curve $\alpha(s)$ where $\alpha(s)$ is of class C^∞ and $\alpha'(s) \neq 0$ (s being the arc length of $\alpha(s)$ only), also with non-zero curvature, in bijection with it in such a manner that the principal normal to $\beta(s^\star)$ and the binormal to $\alpha(s)$ at each pair of corresponding points coincide with the line joining the corresponding points. The curve $\alpha(s)$ is called a *Mannheim conjugate of* $\beta(s^\star)$.

Definition 2.3. An admissible Frenet-Mannheim curve $\beta(s^\star)$ (briefly called a *FM curve*) is a C^∞ Frenet curve for which there exists another C^∞ Frenet curve $\alpha(s)$, where $\alpha(s)$ is of class C^∞ and $\alpha'(s) \neq 0$, in bijection with it so that, by suitable choice of the Frenet frames the principal normal vector $N_\beta(s^\star)$ and binormal vector $B_\alpha(s)$ at corresponding points on $\beta(s^\star)$, $\alpha(s)$, both lie on the line joining the corresponding points. The curve $\alpha(s)$ is called a *FM conjugate of* $\beta(s^\star)$.

Definition 2.4. An admissible weakened Mannheim curve $\beta(s^\star)$, $s^\star \in I^\star$ (briefly called a *WM curve*) is a C^∞ regular curve for which there exists another C^∞ regular curve $\alpha(s)$, $s \in I$, where s is the arclength of $\alpha(s)$, and a homeomorphism $\sigma : I \to I^\star$such that

(i) There exist two (disjoint) closed subsets Z, N of I with void interiors such that $\sigma \in C^\infty$ on $I \backslash N$, $\left(\frac{ds^\star}{ds}\right) = 0$ on Z, $\sigma^{-1} \in C^\infty$ on $\sigma\left(I \backslash Z\right)$ and $\left(\frac{ds}{ds^\star}\right) = 0$ on $\sigma(N)$;

(ii) The line joining corresponding points $s, s^\star$ of $\alpha(s)$ and $\beta(s^\star)$ is orthogonal to $\alpha(s)$ and $\beta(s^\star)$ at the points $s, s^\star$ respectively, and is along the principal normal to $\beta(s^\star)$ or $\alpha(s)$ at the points $s, s^\star$ whenever it is well defined.

The curve $\alpha(s)$ is called a *WM conjugate of* $\beta(s^\star)$.

Thus for a WM curve we not only drop the requirement of $\alpha(s)$ being a Frenet curve, but also allow $\left(\frac{ds^\star}{ds}\right)$ to be zero on a subset with void interior $\left(\frac{ds^\star}{ds}\right) = 0$ on an interval would destroy the injectivity of the mapping σ. Since $\left(\frac{ds^\star}{ds}\right) = 0$ implies that $\left(\frac{ds}{ds^\star}\right)$ does not exist, the apparently artificial requirements in (i) are in fact quite natural.

It is clear that an admissible Mannheim curve is necessarily a FM curve, and a FM curve is necessarily a WM curve. It will be proved in Theorem 4.3 that under certain conditions a WM curve is also a FM curve.

3 Frenet-Mannheim Curves

In this section we study the structure and characterization of FM curves. We begin with a lemma, the method used in which is classical.

Lemma 3.1. *Let $\beta(s^\star)$, $s^\star \in I^\star$ be a FM curve and $\alpha(s)$ a FM conjugate of $\beta(s^\star)$. Let*

$$\beta(s^\star) = \alpha(s) + \lambda(s)B_\alpha(s) \tag{3.1}$$

Then the distance $|\lambda|$ between corresponding points of $\alpha(s)$, $\beta(s^\star)$ is constant, and there is a constant angle θ such that $\langle T_\alpha, T_\beta\rangle = \cos\theta$ and

(*i*) $\sinh\theta = \lambda\tau_\alpha\cosh\theta$;

(*ii*) $\sinh\theta = \lambda\tau_\beta\cosh\theta$;

(*iii*) $\cosh^2\theta = 1$;

(*iv*) $\sinh^2\theta = \lambda^2\tau_\alpha\tau_\beta$.

Proof. From (3.1) it follows that

$$\lambda(s) = \langle \beta(s^\star) - \alpha(s), B_\alpha(s)\rangle$$

is of class C^∞. Differentiation of (3.1) with respect to s gives

$$T_\beta \frac{ds^\star}{ds} = T_\alpha + \lambda' B_\alpha + \lambda\tau_\alpha N_\alpha. \tag{3.2}$$

Since by hypothesis we have $B_\alpha = \epsilon N_\beta$ with $\epsilon = \pm 1$, scalar multiplication of (3.2) by B_α gives

$$\lambda' = 0,$$

then we have λ is a constant function. Therefore

$$T_\beta \frac{ds^\star}{ds} = T_\alpha + \lambda\tau_\alpha N_\alpha. \tag{3.3}$$

But by definition of FM curve we have $\frac{ds^\star}{ds} \neq 0$, so that T_β is C^∞ function of s. Hence

$$\langle T_\alpha, T_\beta\rangle'_{G_3^1} = \kappa_\alpha \langle N_\alpha, T_\beta\rangle_{G_3^1} + \frac{ds^\star}{ds}\kappa_\beta \langle T_\alpha, N_\beta\rangle_{G_3^1} = 0.$$

Consequently $\langle T_\alpha, T_\beta\rangle$ is constant, and there exists a constant angle θ such that

$$T_\beta = T_\alpha\cosh\theta + N_\alpha\sinh\theta. \tag{3.4}$$

Taking the vector product of (3.3) and (3.4), we obtain

$$\sin\theta = \lambda\tau_\alpha\cosh\theta$$

which is (*i*). Now write

$$\alpha(s) = \beta(s^\star) - \epsilon\lambda(s)N_\beta(s).$$

Therefore

$$T_\alpha = \frac{ds^\star}{ds}\left[T_\beta - \lambda\epsilon\tau_\beta B_\beta\right]. \tag{3.5}$$

On the other hand, equation (3.4) gives

$$B_\beta = T_\beta \wedge_{G_3^1} N_\beta = \epsilon N_\alpha \cosh\theta.$$

Using (3.4) again, we get

$$T_\alpha = T_\beta \cosh\theta - \epsilon B_\beta \sinh\theta. \tag{3.6}$$

Taking the vector product of (3.5) and (3.6), we obtain

$$\sinh\theta = \lambda\tau_\beta \cosh\theta,$$

which is (*ii*). On the other hand, comparison of (3.3) and (3.4) gives

$$\frac{ds^\star}{ds}\cosh\theta = 1, \tag{3.7}$$

$$\frac{ds^\star}{ds}\sinh\theta = \lambda\tau_\alpha. \tag{3.8}$$

Similarly (3.5), (3.6) give

$$\frac{ds^\star}{ds} = \cosh\theta, \tag{3.9}$$

$$\frac{ds^\star}{ds}(\lambda\tau_\beta) = \sinh\theta. \tag{3.10}$$

The properties (*iii*) and (*iv*) then easily follow from (3.7) and (3.9), (3.6) and (3.8) and (3.10). □

Theorem 3.2. *Let $\beta(s^\star)$, $s^\star \in I^\star$ be a C^∞ Frenet curve with τ_β nowhere zero and satisfying the equation for constants λ with $\lambda \neq 0$. Then $\beta(s^\star)$ is a non-planar FM curve.*

$$\sinh\theta = \lambda\tau_\beta \cosh\theta. \tag{3.11}$$

Proof. Define the curve $\beta(s^\star)$ with position vector

$$\beta(s^\star) = \alpha(s) + \lambda(s)B_\alpha(s)$$

Then, denoting differentiation with respect to s by a dash, we have

$$\beta'(s^\star) = T_\alpha + \lambda\tau_\alpha N_\alpha.$$

Since $\tau_\alpha \neq 0$, it follows that $\beta(s^\star)$ is a C^∞ regular curve. Then

$$T_\beta \frac{ds^\star}{ds} = T_\alpha + \lambda\tau_\alpha N_\alpha.$$

Hence

$$\frac{ds^\star}{ds} = \sqrt{1+\lambda^2\tau_\alpha^2}.$$

And, using (3.11)

$$T_\beta = T_\alpha \cosh\theta + N_\alpha \sinh\theta,$$

notice that from (3.11) we have $\sinh\theta \neq 0$. Therefore

$$\frac{T_\beta}{ds^\star}\frac{ds^\star}{ds} = \kappa_\alpha N_\alpha \cosh\theta + \tau_\alpha B_\alpha \sinh\theta$$

Now define $N_\beta = \epsilon B_\alpha$,

$$\kappa_\beta = \frac{\epsilon}{\frac{ds^\star}{ds}}\tau_\alpha \sinh\theta.$$

These are C^∞ functions of s (and hence of $s^\star$), and

$$\frac{T_\beta}{ds^\star} = \kappa_\beta N_\beta.$$

Further define $B_\beta = T_\beta \wedge_{G_3^1} B_\alpha$ and $\tau_\beta = \left\langle \frac{B_\beta}{ds^\star}, N_\beta \right\rangle_{G_3^1}$. These are also C^∞ functions on $I^\star$. It is then easy to verify that with the frame $\{T_\beta, N_\beta, B_\beta\}$ and the functions κ_β, τ_β, the curve $\beta(s^\star)$ becomes a C^∞ Frenet curve. But B_α and N_β lie on the line joining corresponding points of $\alpha(s)$ and $\beta(s^\star)$. Thus $\beta(s^\star)$ is a FM curve and $\alpha(s)$ a FM conjugate of $\beta(s^\star)$. □

Lemma 3.3. *A necessary and sufficient condition for a C^∞ regular curve β to be a FM curve with a FM conjugate. Then β should be either a line or a non-planar circular helix.*

Proof. (⇒) : Let β have a FM conjugate α which is a line. Then $\kappa_\alpha = 0$. Using Lemma 3.1, (iii) and (i), (ii), we have

$$\cosh^2\theta = 1, \tag{3.12}$$

and then

$$\cosh^2\theta \sin\theta = \lambda\tau_\beta \cosh\theta, \tag{3.13}$$

$$\sinh\theta = \lambda\tau_\alpha \cosh\theta. \tag{3.14}$$

From (3.14) it follows that $\cosh\theta \neq 0$. Hence (3.13) is equivalent to

$$\lambda\tau_\beta = \cosh\theta \sinh\theta. \tag{3.15}$$

Case 1. $\sinh\theta = 0$. Then $\cosh\theta = \pm 1$, so that (3.12) implies that $\kappa_\beta = 0$, and β is a line. We note also that (3.15) implies that $\tau_\beta = 0$.

Case 2. $\sinh\theta \neq 0$. Then $\cosh\theta \neq \pm 1$, and (3.12), (3.15) imply that κ_β, τ_β are non-zero constants, and β is a non-planar circular helix.

$(\Leftarrow)$: If β is a non-planar circular helix

$$\beta = (as, b\cosh s, b\sinh s),$$

we may take

$$N_\beta = (0, \cosh s, \sinh s).$$

Now put $\lambda = b$, then the curve β with

$$\beta = \alpha + \lambda B_\alpha$$

will be a line along the $x-$axis, and can be made into a FM conjugate of β if N_β is defined to be equal to B_α. □

Theorem 3.4. *Let $\beta(s^\star)$ be a plane C^∞ Frenet curve with zero torsion and whose curvature is either bounded below or bounded above. Then β is a FM curve, and has FB conjugates which are plane curves.*

Proof. Let β be a curve satisfying the conditions of the hypothesis. Then there are non-zero numbers λ such that $\kappa_\beta < -\frac{1}{\lambda}$ on I or $\kappa_\beta > -\frac{1}{\lambda}$ on I. For any such λ, consider the plane curve α with position vector

$$\alpha = \beta - \lambda N_\beta.$$

Then

$$T_\alpha = T_\beta.$$

It is then a straightforward matter to verify that α is a FM conjugate of β. □

4 Weakened Mannheim Curves

Definition 4.1. Let D be a subset of a topological space X. A function on X into a set Y is said to be *D-piecewise constant* if it is constant on each component of D.

Lemma 4.2. *Let X be a proper interval on the real line and D an open subset of X. Then a necessary and sufficient condition for every continuous, D-piecewise constant real function on X to be constant is that $X \backslash D$ should have empty dense-in-itself kernel.*

We notice that if D is dense in X, any C^1 and D-piecewise constant real function on X must be constant, even if D has non-empty dense-in-itself kernel.

Theorem 4.3. *A WM curve for which N and Z have empty dense-in-itself kernels is a FM curve.*

Proof. Let $\beta(s^\star), s^\star \in I^\star$ be a WM curve and $\alpha(s)$, $s \in I$ a WM conjugate of $\beta(s^\star)$. It follows from the definition that $\alpha(s)$ and $\beta(s^\star)$ each has a C^∞ family of tangent vectors $T_\beta(s^\star)$, $T_\alpha(s)$. Let

$$\beta(s) = \beta(\sigma(s)) = \alpha(s) + \lambda(s)B_\alpha(s), \tag{4.1}$$

where $B_\alpha(s)$ is some unit vector function and $\lambda(s) \geq 0$ is some scalar function. Let $D = I \backslash N$, $D^\star = I^\star \backslash \sigma(Z)$. Then $s^\star(s) \in C^\infty$on $D^\star$.

Step 1. To prove $\lambda =$ constant.

Since $\lambda = \|\beta(s) - \alpha(s)\|$, it is continuous on I and is of class C^∞ on every interval of D on which it is nowhere zero. Let $P = \{s \in I : \lambda(s) \neq 0\}$ and X any component of P. Then P, and hence also X, is open in I. Let L be any component interval of $X \cap D$. Then on L, $\lambda(s)$ and $B_\alpha(s)$ are of class C^∞, and from (4.1) we have

$$\beta'(s) = \alpha'(s) + \lambda'(s)B_\alpha(s) + \lambda(s)B'_\alpha(s).$$

Now by definition of a WM curve we have $\langle \alpha'(s), B_\alpha(s)\rangle_{G_3^1} = 0 = \langle \beta'(s^\star), B_\alpha(s)\rangle_{G_3^1}$. Hence, using the identity $\langle B'_\alpha(s), B_\alpha(s)\rangle_{G_3^1} = 0$, we have

$$0 = \lambda'(s) \langle B_\alpha(s), B_\alpha(s)\rangle_{G_3^1}.$$

Therefore $\lambda =$ constant on L.

Hence λ is constant on each interval of the set $X \cap D$. But by hypothesis $X \backslash D$ has empty dense-in-itself kernel. It follows from Lemma 3.3 that λ is constant (and non-zero) on X. Since λ is continuous on I, X must be closed in I. But X is also open in I. Therefore by connectedness we must have $X = I$, that is, λ is constant on I.

Step 2. To prove the existence of two frames

$$\{T_\alpha(s), N_\alpha(s), B_\alpha(s)\}, \{T_\beta(s^\star), N_\beta(s^\star), B_\beta(s^\star)\}$$

which are Frenet frames for $\alpha(s)$, $\beta(s^\star)$ on D, $D^\star$ respectively.

Since λ is a non-zero constant, it follows from (4.1) that $B_\alpha(s)$ is continuous on I and C^∞ on D, and is always orthogonal to $T_\alpha(s)$. Now define $B_\alpha(s) = T_\alpha(s) \wedge_{G_3^1} N_\alpha(s)$. Then $\{T_\alpha(s), N_\alpha(s), B_\alpha(s)\}$ forms a right-handed orthonormal frame for $\alpha(s)$ which is continuous on I and C^∞ on D.

Now from the definition of WM curve we see that there exists a scalar function $\kappa_\beta(s^\star)$ such that $T'_\beta(s^\star) = \kappa_\beta(s^\star)N_\beta(s^\star)$ on $I^\star$. Hence $\kappa_\beta(s^\star) = \left\langle T'_\beta(s^\star), N_\beta(s^\star)\right\rangle_{G_3^1}$ is continuous on $I^\star$ and C^∞ on $D^\star$. Thus the first Frenet formula holds on $D^\star$. It is then straightforward to show that there exists a C^∞ function $\tau_\alpha(s)$ on D such that the Frenet formulas hold. Thus $\{T_\alpha(s), N_\alpha(s), B_\alpha(s)\}$ is a Frenet frame for $\alpha(s)$ on D.

Similarly there exists a right-handed orthonormal frame $\{T_\beta(s^\star), N_\beta(s^\star), B_\beta(s^\star)\}$ for $\beta(s^\star)$ which is continuous on $I^\star$ and is a Frenet frame for $\beta(s^\star)$ on $D^\star$. Moreover, we can choose

$$B_\alpha(s) = N_\beta(\sigma(s))$$

Step 3. To prove that $N=\emptyset$, $Z=\emptyset$. We first notice that on D we have

$$\langle T_\beta, T_\alpha\rangle'_{G_3^1} = \left\langle \kappa_\beta N_\beta \frac{ds^\star}{ds}, T_\alpha \right\rangle_{G_3^1} + \langle T_\beta, \kappa_\alpha N_\alpha\rangle_{G_3^1} = 0,$$

so that $\langle T_\beta, T_\alpha\rangle$ is constant on each component of D and hence on I by Lemma 4.2. Consequently there exists a angle θ such that

$$T_\beta = T_\alpha \cosh\theta + N_\alpha \sinh\theta.$$

Further,

$$B_\alpha(s) = N_\beta(\sigma(s))$$

and so

$$B_\beta(s^\star) = -T_\alpha \sinh\theta + N_\alpha \cosh\theta.$$

Thus $\{T_\beta(s^\star), N_\beta(s^\star), B_\alpha(s)\}$ are also of class C^∞ on D. On the other hand $\{T_\beta(s^\star), N_\beta(s^\star), B_\beta(s^\star)\}$ are of class C^∞ with respect to $s^\star$ on $D^\star$. Writing (4.1) in the form

$$\alpha = \beta - \lambda N_\beta.$$

and differentiating with respect to s on $D \cap \sigma^{-1}(D^\star)$, we have

$$T_\alpha = \frac{ds^\star}{ds}\left[T_\beta + \lambda\tau_\beta B_\beta\right].$$

But

$$T_\alpha = T_\beta \cosh\theta - B_\beta \sinh\theta.$$

Hence

$$\frac{ds^\star}{ds} = \cosh\theta \text{ and } \lambda\tau_\beta = \sinh\theta. \tag{4.2}$$

Since $\kappa_\beta(s^\star) = \left\langle T'_\beta, N_\beta\right\rangle_{G_3^1}$ is defined and continuous on $I^\star$ and $\sigma^{-1}(D^\star)$ is dense, it follows by continuity that (4.2) holds throughout D. If $\cosh\theta \neq 0$ then (4.2) implies that $\frac{ds^\star}{ds} \neq 0$ on D. Hence $Z=\emptyset$. Similarly $N=\emptyset$. □

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