



Subordination Results for New Subclasses of Analytic Univalent Functions

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Abstract : In this paper, we derive several subordination results of two classes of analytic and univalent functions. These two classes are defined by using the well-known fractional calculus operators. Moreover, we introduce some special results of some subclasses.

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1 Introduction

Let \mathcal{A} denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (1.1)$$

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that are analytic and univalent in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Also, let $g \in \mathcal{A}$, and be given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k. \quad (1.2)$$

Also, we denote by K the subclass of functions $f \in \mathcal{A}$ that are convex in U . It is well-known that:

$$f \in K \Leftrightarrow \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0.$$

Definition 1.1. (Hadamard Product or Convolution). Given two functions f and g in the class \mathcal{A} , where $f(z)$ is given by (1.1) and $g(z)$ is given by (1.2), then the *Hadamard product* (or *convolution*) of f and g is defined (as usual) by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \quad (1.3)$$

Various operators of fractional calculus (that is, fractional integral and fractional derivatives) have been studied in the literature rather extensively (cf., e.g., [1], [2], [3] and [4]; see also the various references cited therein). For our present investigation, we recall the following definitions. Various operators of fractional calculus (that is, fractional integral and fractional derivatives) have been studied in the literature rather extensively (cf., e.g., [1], [2], [3] and [4]; see also the various references cited therein). For our present investigation, we recall the following definitions.

Definition 1.2. The *fractional integral of order μ* is defined, for a function f by

$$D_z^{-\mu} f(z) = \frac{1}{\Gamma(\mu)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\mu}} d\zeta \quad (\mu > 0), \quad (1.4)$$

where the function f is analytic in a simply-connected domain of the complex z -plane containing the origin and the multiplicity of $(z-\zeta)^{\mu-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

Definition 1.3. The *fractional derivative of order μ* is defined for a function f by

$$D_z^{\mu} f(z) = \frac{1}{\Gamma(1-\mu)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\mu}} d\zeta \quad (0 \leq \mu < 1), \quad (1.5)$$

where the function f is constrained and the multiplicity of $(z-\zeta)^{-\mu}$ is similarly removed.

Under the hypotheses of Definition 1.3, the fractional derivative of order μ is defined for a function f by

$$D_z^{n+\mu} f(z) = \frac{d^n}{dz^n} \{D_z^\mu f(z)\} \quad (0 \leq \mu < 1; n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \tag{1.6}$$

As well as the fractional calculus operator D_z^μ for which it is well known that (see for details [2] and [5])

$$D_z^\mu \{z^\rho\} = \frac{\Gamma(\rho + 1)}{\Gamma(\rho + 1 - \mu)} z^{\rho - \mu} \quad (\rho > -1; \mu \in \mathbb{R}), \tag{1.7}$$

in terms of Gamma functions.

Definition 1.4. (Subordination Principle). For two functions f and g , analytic in U , we say that the function f is *subordinate* to g in U and write $f(z) \prec g(z)$ if there exists a Schwarz function w , which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))(z \in U)$.

Indeed it is known that

$$f(z) \prec g(z) \implies f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Furthermore, if the function g is univalent in U , then we have the following equivalence (see [6]):

$$f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Definition 1.5. Let $S^*(\alpha, \lambda)$ denote the subclass of \mathcal{A} consisting of functions f of the form (1.1) and satisfy the following:

$$\Re \left\{ \frac{\Gamma(1 - \lambda) z^{1+\lambda} D_z^{1+\lambda} f(z)}{f(z)} \right\} > \alpha \tag{1.8}$$

$$(0 \leq \alpha < 1; \lambda < 1; z \in U).$$

Also let $K(\alpha, \lambda)$ denote the subclass of \mathcal{A} consisting of functions f of the form (1.1) and satisfy the following:

$$\Gamma(1 - \lambda) z^{1+\lambda} D_z^{1+\lambda} f(z) \in S^*(\alpha, \lambda) \tag{1.9}$$

$$(0 \leq \alpha < 1; \lambda < 1; z \in U).$$

The classes $S^*(\alpha, \lambda)$ and $K(\alpha, \lambda)$ are introduced by Owa [7].

Specializing the parameters α and λ , we obtain the following subclasses studied by various authors:

$$S^*(\alpha, 0) = S^*(\alpha) = \left\{ f \in \mathcal{A} : \Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (0 \leq \alpha < 1; z \in U) \right\},$$

$$K(\alpha, 0) = K(\alpha) = \left\{ f \in \mathcal{A} : \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha \quad (0 \leq \alpha < 1; z \in U) \right\}$$

(see Robertson [8], Schild [9] and MacGregor [10]).

Also we note that:

$$\begin{aligned} S^*(0, \lambda) &= S_\lambda^* = \left\{ f \in \mathcal{A} : \Re \left\{ \frac{\Gamma(1-\lambda)z^{1+\lambda}D_z^{1+\lambda}f(z)}{f(z)} \right\} > 0 \quad (\lambda < 1; z \in U) \right\}, \\ K(0, \lambda) &= K_\lambda = \left\{ f \in \mathcal{A} : \Gamma(1-\lambda)z^{1+\lambda}D_z^{1+\lambda}f(z) \in S_\lambda^* \quad (\lambda < 1; z \in U) \right\}. \end{aligned}$$

Definition 1.6. (Subordination Factor Sequence). A Sequence $\{c_k\}_{k=0}^\infty$ of complex numbers is said to be a *subordinating factor sequence* if, whenever f of the form (1.1) is analytic, univalent and convex in U we have the subordination given by

$$\sum_{k=1}^{\infty} a_k c_k z^k \prec f(z) (a_1 = 1; z \in U) \quad (1.10)$$

2 Main Results

Unless otherwise mentioned, we assume in the reminder of this paper that $0 \leq \alpha < 1$, $\lambda < 1$ and $z \in U$. To prove our main results we need the following lemmas.

Lemma 2.1. *The sequence $\{c_k\}_{k=0}^\infty$ is a subordinating factor sequence if and only if (see [11]):*

$$\Re \left\{ 1 + 2 \sum_{k=1}^{\infty} c_k z^k \right\} > 0 (z \in U). \quad (2.1)$$

Lemma 2.2. *A function f of the form (1.1) is in the class $S^*(\alpha, \lambda)$ if (see [7]):*

$$\sum_{k=2}^{\infty} \left[\frac{\Gamma(k+1)\Gamma(1-\lambda)}{\Gamma(k-\lambda)} - \alpha \right] |a_k| \leq 1 - \alpha. \quad (2.2)$$

Lemma 2.3. *A function f of the form (1.1) is in the class $K(\alpha, \lambda)$ if (see [7]):*

$$\sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(1-\lambda)}{\Gamma(k-\lambda)} \left[\frac{\Gamma(k+1)\Gamma(1-\lambda)}{\Gamma(k-\lambda)} - \alpha \right] |a_k| \leq 1 - \alpha. \quad (2.3)$$

Let $S^{**}(\alpha, \lambda)$ denote the class of functions $f \in \mathcal{A}$ whose coefficients satisfy the condition (2.2). We note that $S^{**}(\alpha, \lambda) \subseteq S^*(\alpha, \lambda)$. Also let $K^*(\alpha, \lambda)$ denote the class of functions $f \in \mathcal{A}$ whose coefficients satisfy the condition (2.3). We note that $K^*(\alpha, \lambda) \subseteq K(\alpha, \lambda)$. Employing the technique used earlier by Attiya [12] and Srivastava and Attiya [13], we introduce the following theorem:

Theorem 2.4. *Let $f \in S^{**}(\alpha, \lambda)$. Then*

$$\frac{2 - \alpha(1 - \lambda)}{2[2 + (1 - 2\alpha)(1 - \lambda)]} (f * h)(z) \prec h(z), \quad (2.4)$$

for every function $h \in K$, and

$$\Re \{f(z)\} > -\frac{2 + (1 - 2\alpha)(1 - \lambda)}{2 - \alpha(1 - \lambda)}. \quad (2.5)$$

The constant factor $\frac{2 - \alpha(1 - \lambda)}{2[2 + (1 - 2\alpha)(1 - \lambda)]}$ in the subordination result (2.4) cannot be replaced by a larger one.

Proof. Let $f \in S^{**}(\alpha, \lambda)$ and let $h(z) = z + \sum_{k=2}^{\infty} d_k z^k \in K$. Then we have

$$\frac{2 - \alpha(1 - \lambda)}{2[2 + (1 - 2\alpha)(1 - \lambda)]} (f * h)(z) = \frac{\left(\frac{2\Gamma(1 - \lambda)}{\Gamma(2 - \lambda)} - \alpha\right)}{2\left[\left(\frac{2\Gamma(1 - \lambda)}{\Gamma(2 - \lambda)} - \alpha\right) + (1 - \alpha)\right]} \left(z + \sum_{k=2}^{\infty} a_k d_k z^k\right). \quad (2.6)$$

Thus, by using Definition 1.6, the subordination result (2.4) will hold true if the sequence

$$\left\{ \frac{\left(\frac{2\Gamma(1 - \lambda)}{\Gamma(2 - \lambda)} - \alpha\right)}{2\left[\left(\frac{2\Gamma(1 - \lambda)}{\Gamma(2 - \lambda)} - \alpha\right) + (1 - \alpha)\right]} a_k \right\}_{k=1}^{\infty}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 2.1, this is equivalent to the following inequality:

$$\Re \left\{ 1 + \sum_{k=1}^{\infty} \frac{\left(\frac{2\Gamma(1 - \lambda)}{\Gamma(2 - \lambda)} - \alpha\right)}{\left(\frac{2\Gamma(1 - \lambda)}{\Gamma(2 - \lambda)} - \alpha\right) + (1 - \alpha)} a_k z^k \right\} > 0. \quad (2.7)$$

Now, since

$$\Psi(k) = \left[\frac{\Gamma(k + 1)\Gamma(1 - \lambda)}{\Gamma(k - \lambda)} - \alpha \right]$$

is an increasing function of k ($k \geq 2$), we have

$$\begin{aligned} & \Re \left\{ 1 + \sum_{k=1}^{\infty} \frac{\left(\frac{2\Gamma(1 - \lambda)}{\Gamma(2 - \lambda)} - \alpha\right)}{\left(\frac{2\Gamma(1 - \lambda)}{\Gamma(2 - \lambda)} - \alpha\right) + (1 - \alpha)} a_k z^k \right\} \\ &= \Re \left\{ 1 + \frac{\left(\frac{2\Gamma(1 - \lambda)}{\Gamma(2 - \lambda)} - \alpha\right)}{\left(\frac{2\Gamma(1 - \lambda)}{\Gamma(2 - \lambda)} - \alpha\right) + (1 - \alpha)} z \right. \\ & \quad \left. + \frac{1}{\left(\frac{2\Gamma(1 - \lambda)}{\Gamma(2 - \lambda)} - \alpha\right) + (1 - \alpha)} \sum_{k=2}^{\infty} \left(\frac{2\Gamma(1 - \lambda)}{\Gamma(2 - \lambda)} - \alpha\right) a_k z^k \right\} \\ &\geq 1 - \frac{\left(\frac{2\Gamma(1 - \lambda)}{\Gamma(2 - \lambda)} - \alpha\right)}{\left(\frac{2\Gamma(1 - \lambda)}{\Gamma(2 - \lambda)} - \alpha\right) + (1 - \alpha)} r \end{aligned}$$

$$\begin{aligned}
& - \frac{1}{\left(\frac{2\Gamma(1-\lambda)}{\Gamma(2-\lambda)} - \alpha\right) + (1-\alpha)} \sum_{k=2}^{\infty} \left[\frac{\Gamma(k+1)\Gamma(1-\lambda)}{\Gamma(k-\lambda)} - \alpha \right] |a_k| r^k \\
> & 1 - \frac{\left(\frac{2\Gamma(1-\lambda)}{\Gamma(2-\lambda)} - \alpha\right)}{\left(\frac{2\Gamma(1-\lambda)}{\Gamma(2-\lambda)} - \alpha\right) + (1-\alpha)} r - \frac{(1-\alpha)}{\left(\frac{2\Gamma(1-\lambda)}{\Gamma(2-\lambda)} - \alpha\right) + (1-\alpha)} r \\
= & 1 - r > 0 \quad (|z| = r < 1),
\end{aligned}$$

where we have also made use of assertion (2.2) of Lemma 2.2. Thus (2.7) holds true in U , this proves the subordination (2.4). The inequality (2.5) follows from (2.4) by taking the convex function $h(z) = \frac{z}{1-z} = z + \sum_{k=2}^{\infty} z^k$. To prove the sharpness of the constant $\frac{2-\alpha(1-\lambda)}{2[2+(1-2\alpha)(1-\lambda)]}$, we consider the function $f_0(z) \in S^{**}(\alpha, \lambda)$ given by

$$f_0(z) = z - \frac{(1-\alpha)(1-\lambda)}{2-\alpha(1-\lambda)} z^2. \quad (2.8)$$

Thus from (2.4), we get

$$\frac{2-\alpha(1-\lambda)}{2[2+(1-2\alpha)(1-\lambda)]} f_0(z) \prec \frac{z}{1-z}. \quad (2.9)$$

Moreover, it can easily be verified for the function $f_0(z)$ given by (2.8) that

$$\min_{|z| \leq r} \left\{ \Re \frac{2-\alpha(1-\lambda)}{2[2+(1-2\alpha)(1-\lambda)]} f_0(z) \right\} = -\frac{1}{2}. \quad (2.10)$$

This shows that the constant $\frac{2-\alpha(1-\lambda)}{2[2+(1-2\alpha)(1-\lambda)]}$ is the best possible. This completes the proof of Theorem 2.4. \square

Remark 2.5. Taking $\lambda = 0$ in Theorem 2.4, we obtain the result obtained by Frasin ([14], Corollary 2.3).

Also, we establish subordination results for the subclass S_{λ}^{**} , whose coefficients satisfy the inequality (2.2) in the special case as mentioned.

Putting $\alpha = 0$ in Theorem 2.4, we have

Corollary 2.6. Let the function f defined by (1.1) be in the class S_{λ}^{**} and suppose that $h \in K$. Then

$$\frac{1}{3-\lambda} (f * h)(z) \prec h(z), \quad (2.11)$$

and

$$\Re \{f(z)\} > -\frac{3-\lambda}{2}. \quad (2.12)$$

The constant factor $\frac{1}{3-\lambda}$ in the subordination result (2.11) cannot be replaced by a larger one.

Using Lemmas 2.1 and 2.3 together with the same technique used in Theorem 2.4, we introduce the following theorem.

Theorem 2.7. *Let $f \in K^*(\alpha, \lambda)$. Then*

$$\frac{2 - \alpha(1 - \lambda)}{4 - 2\alpha(1 - \lambda) + (1 - \alpha)(1 - \lambda)^2} (f * h)(z) \prec h(z), \quad (2.13)$$

for every function h in K , and

$$\Re\{f(z)\} > -\frac{4 - 2\alpha(1 - \lambda) + (1 - \alpha)(1 - \lambda)^2}{4 - 2\alpha(1 - \lambda)}. \quad (2.14)$$

The constant factor $\frac{2 - \alpha(1 - \lambda)}{4 - 2\alpha(1 - \lambda) + (1 - \alpha)(1 - \lambda)^2}$ in the subordination result (2.13) cannot be replaced by a larger one.

Remark 2.8. *Taking $\lambda = 0$ in Theorem 2.7, we obtain the result obtained by Frasin ([14], Corollary 2.6).*

Also, we establish subordination results for the associated subclass K_λ^* , whose coefficient satisfy the inequality (2.3) in the special case as mentioned. Putting $\alpha = 0$ in Theorem 2.7, we get

Corollary 2.9. *Let the function f defined by (1.1) be in the class K_λ^* and suppose that $h \in K$. Then*

$$\frac{2}{4 + (1 - \lambda)^2} (f * h)(z) \prec h(z), \quad (2.15)$$

and

$$\Re\{f(z)\} > -\frac{4 + (1 - \lambda)^2}{4}. \quad (2.16)$$

The constant factor $\frac{2}{4 + (1 - \lambda)^2}$ in the subordination result (2.15) cannot be replaced by a larger one.

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