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The Order of Elements of the Semigroups $(W_\tau(X_2))^2$ and $(W_\tau(X_3))^3$

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Abstract : In this paper, we determine the order of all elements of the semigroups $(W_\tau(X_2))^2$ and $(W_\tau(X_3))^3$, respectively. We show that the order of an element of $(W_\tau(X_2))^2$ is $1, 2$ or infinite and the order of an element of $(W_\tau(X_3))^3$ is $1, 2, 3$ or infinite.

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1 Preliminaries

Using the operation which is called the *superposition* we can define an associative binary operation on the cartesian power of terms of a given type. Let $\tau = (n_i)_{i\in I}$ be a type of an algebra with operation symbols f_i, indexed by some set I, each having arity n_i ($n_i \geq 1$ a natural number). Let $X = \{x_1, x_2, x_3, \ldots\}$ be a countably infinite alphabet of variables, disjoint from the set of operation symbols of type τ. Let $X_n = \{x_1, x_2, \ldots, x_n\}$ be the n-element alphabet of variables. For each natural number $n \geq 1$, the *n-ary terms* [1] of type τ are inductively defined as follows.

(i) Every variable $x_i \in X_n$ is an n-ary term of type τ.

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(ii) If $t_1, \ldots, t_{n_i}$ are n-ary terms of type τ and f_i is an n_i-ary operation symbol, then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term of type τ.

Let $W_\tau(X_n)$ be the smallest set containing $x_1, \ldots, x_n$ which is closed under finite application of (ii). The set of all terms of type τ over the alphabet X is defined as the union $W_\tau(X) = \bigcup_{n=1}^{\infty} W_\tau(X_n)$.

There is a well known $(n+1)$-ary superposition operation

$$S^n : W_\tau(X_n)^{n+1} \to W_\tau(X_n)$$

which maps $(n+1)$ n-ary terms to a single n-ary term.

Define a binary operation $+$ on $(W_\tau(X_n))^n$ by

$$(t_1, \ldots, t_n) + (s_1, \ldots, s_n) = (S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_n, s_1, \ldots, s_n))$$

for all $(t_1, \ldots, t_n), (s_1, \ldots, s_n) \in (W_\tau(X_n))^n$.

By [2], $(W_\tau(X_n))^n$ satisfies the identity

$$\tilde{S}^n(U_0, S^n(V_1, W_1, \ldots, W_n), \ldots, \tilde{S}^n(V_n, W_1, \ldots, W_n)) \approx \tilde{S}^n(\tilde{S}^n(U_0, V_1, \ldots, V_n), W_1, \ldots, W_n).$$

Here, $\tilde{S}$ is an $(n+1)$-ary operation symbol, and $U_0, V_1, \ldots, V_n, W_1, \ldots, W_n$ are variables. Therefore, $((W_\tau(X_n))^n, +)$ is a semigroup.

Semigroup properties of $((W_\tau(X_n))^n, +)$ have been studied by many authors. In [3], the maximal regular subsemigroups and the maximal inverse subsemigroups of $(W_\tau(X_2))^2$ were determined. In [4], the regular elements of $(W_\tau(X_n))^n$ was characterized. The same authors also studied Green's relations on $(W_\tau(X_n))^n$.

Let S be a semigroup. If S is finite, then the *order* [5] of S is defined as the number of its elements, otherwise S is of *infinite order*. For an element a of S, the subsemigroup of S generated by a will be denoted by $\langle a \rangle$. The order of an element a of S is defined as the order of $\langle a \rangle$. An element e of S is called an *idempotent* of S if $ee = e$. Note that the order of an idempotent of S is 1.

The idempotent elements of $(W_\tau(X_n))^n$ were characterized by the authors in [4] as follows: An element $(t_1, t_2, \ldots, t_n) \in (W_\tau(X_n))^n$ is idempotent if and only if

$$x_j \in \bigcup_{i=1}^{n} var(t_i) \Rightarrow t_j = x_j$$

where $var(t_i)$ denotes the set of all variables occurring in t_i. For example: the terms $t_1 = (x_1, f(x_1, x_1))$ and $t_2 = (x_1, f(x_3, x_1, x_1), x_3)$ are idempotent elements in $(W_\tau(X_2))^2$ and $(W_\tau(X_3))^3$, respectively.

The purpose of this paper is to determine the order of elements of the semigroups $((W_\tau(X_2))^2, +)$ and $((W_\tau(X_3))^3, +)$ where τ is not unary type.

To prove the results, we need the following notion. For $(t_1, t_2, \ldots, t_n) \in (W_\tau(X_n))^n$, let

$$\overline{op}(t_1, t_2, \ldots, t_n) = \sum_{k=1}^{n} op(t_k)$$

where $op(t_k)$ is inductively defined by

(i) $op(t_k) = 0$, if t_k is a variable;

(ii) $op(t_k) = 1 + \sum_{j=1}^{n_i} op(t_{k_j})$, if $t_k = f_i(t_{k_1}, t_{k_2}, \ldots, t_{k_{n_i}})$ is a composite term.

For example, let $t_1 = f(x_1, f(x_2, f(x_2, x_1)))$ and $t_2 = f(x_1, x_1)$, then $op(t_1) = 3$, $op(t_2) = 1$ and $\overline{op}(t_1, t_2) = 4$.

2 The Order of Elements of $(W_\tau(X_2))^2$

We shall prove the main result in this section, Theorem 2.5, by considering the following lemmas.

Lemma 2.1.

(1) *The order of each of* $(x_1, x_1), (x_1, x_2), (x_2, x_2)$ *is* 1.

(2) *The order of* (x_2, x_1) *is* 2.

(3) *The order of* (x_1, t_2) *where* $t_2 \notin X_2$ *and* $var(t_2) = \{x_1\}$ *is* 1.

(4) *The order of* (t_1, x_2) *where* $t_1 \notin X_2$ *and* $var(t_1) = \{x_2\}$ *is* 1.

Proof. Since the order of an idempotent is 1, we have (1), (3) and (4). Since $\langle (x_2, x_1) \rangle = \{(x_2, x_1), (x_1, x_2)\}$, the order of (x_2, x_1) is 2. This proves (2). □

Lemma 2.2.

(1) *The order of* (x_1, t_2) *where* $t_2 \notin X_2$ *and* $var(t_2) \not\subseteq \{x_1\}$ *is infinite.*

(2) *The order of* (t_1, x_2) *where* $t_1 \notin X_2$ *and* $var(t_1) \not\subseteq \{x_2\}$ *is infinite.*

Proof. We will prove for the statement (1). For the statement (2), it can be proved similarly to (1). To show that $\overline{op}(x_1, t_2)^{n+1} > \overline{op}(x_1, t_2)^n$ for all positive integers n, let n be a positive integer. Then $(x_1, t_2)^n = (x_1, t)$ for some $t \in W_\tau(X_2) \setminus X_2$ such that $x_2 \in var(t)$. We have

$$\begin{aligned}
\overline{op}(x_1, t_2)^{n+1} &= \overline{op}((x_1, t_2)^n + (x_1, t_2)) \\
&= \overline{op}(x_1, t(x_1, t_2)) \\
&= op(t(x_1, t_2)) \\
&> op(t) \\
&= \overline{op}((x_1, t_2)^n).
\end{aligned}$$

Thus (1) holds. □

Lemma 2.3. *The order of* (t_1, t_2) *where* $t_1, t_2 \in W_\tau(X_2) \setminus X_2$ *is infinite.*

Proof. We claim that $\overline{op}(t_1,t_2)^{n+1} > \overline{op}(t_1,t_2)^n$ for all positive integers n. Let n be a positive integer. Then $(t_1,t_2)^n = (t', t'')$ for some $t', t'' \in W_\tau(X_2) \setminus X_2$. We have

$$\begin{aligned} \overline{op}(t_1,t_2)^{n+1} &= \overline{op}((t_1,t_2)^n + (t_1,t_2)) \\ &= \overline{op}(t'(t_1,t_2), t''(t_1,t_2)) \\ &> \overline{op}(t', t'') \\ &= \overline{op}(t_1,t_2)^n. \end{aligned}$$

Therefore, the order of (t_1,t_2) where $t_1,t_2 \in W_\tau(X_2) \setminus X_2$ is infinite. □

Lemma 2.4.

(1) *The order of* (x_2,t_2) *where* $t_2 \in W_\tau(X_2) \setminus X_2$ *is infinite.*

(2) *The order of* (t_1,x_1) *where* $t_1 \in W_\tau(X_2) \setminus X_2$ *is infinite.*

Proof. (1) We have $\langle (x_2,t_2)^2 \rangle$ is a subsemigroup of $\langle (x_2,t_2) \rangle$. Since $(x_2,t_2)^2 = (t',t'')$ for some $t',t'' \in W_\tau(X_2) \setminus X_2$, by Lemma 2.3, the order of $(x_2,t_2)^2$ is infinite. Therefore, the order of (x_2,t_2) is infinite.

(2) This can be proved similarly to (1). □

Theorem 2.5. *The order of an element of* $(W_\tau(X_2))^2$ *is* $1,2$ *or infinite.*

Proof. This follows from Lemma 2.1- Lemma 2.4. □

3 The Order of Elements of $(W_\tau(X_3))^3$

We begin with the following lemma obtained by calculations.

Lemma 3.1. *For* $t_1,t_2,t_3 \in X_3$, *the order of* (t_1,t_2,t_3) *is shown below:*

t_1	t_2	t_3	*the order of* (t_1,t_2,t_3)	t_1	t_2	t_3	*the order of* (t_1,t_2,t_3)
x_1	x_1	x_2	2	x_2	x_3	x_1	3
x_1	x_1	x_3	1	x_2	x_3	x_2	2
x_1	x_2	x_1	1	x_2	x_3	x_3	2
x_1	x_2	x_2	1	x_3	x_1	x_1	2
x_1	x_2	x_3	1	x_3	x_1	x_2	3
x_1	x_3	x_1	2	x_3	x_1	x_3	2
x_1	x_3	x_2	2	x_3	x_2	x_1	2
x_1	x_3	x_3	1	x_3	x_2	x_2	2
x_2	x_1	x_1	2	x_3	x_2	x_2	2
x_2	x_1	x_2	2	x_3	x_2	x_3	1
x_2	x_1	x_3	2	x_3	x_3	x_1	2
x_2	x_2	x_1	2	x_3	x_3	x_2	2
x_2	x_2	x_2	1	x_3	x_3	x_3	1
x_2	x_2	x_3	1				

Lemma 3.2. *The order of* (t_1, t_2, t_3) *where* $t_1, t_2, t_3 \in W_\tau(X_3) \setminus X_3$ *is infinite.*

Proof. Let $t_1, t_2, t_3 \in W_\tau(X_3) \setminus X_3$. We shall show that $\overline{op}(t_1, t_2, t_3)^{n+1} > \overline{op}(t_1, t_2, t_3)^n$ for all positive integers n. For any positive integer n, we have $(t_1, t_2, t_3)^n = (t', t'', t''')$ for some $t', t'', t''' \in W_\tau(X_3) \setminus X_3$. Then

$$\begin{aligned} \overline{op}(t_1, t_2, t_3)^{n+1} &= \overline{op}((t_1, t_2, t_3)^n + (t_1, t_2, t_3)) \\ &= \overline{op}(t'(t_1, t_2, t_3), t''(t_1, t_2, t_3), t'''(t_1, t_2, t_3)) \\ &> \overline{op}(t', t'', t'') \\ &= \overline{op}(t_1, t_2, t_3)^n. \end{aligned}$$

Hence the order of (t_1, t_2, t_3) where $t_1, t_2, t_3 \in W_\tau(X_3) \setminus X_3$ is infinite. □

Assume that $t_1 \in X_3$ and $t_2, t_3 \in W_\tau(X_3) \setminus X_3$. The following lemma shows that if (t_1, t_2, t_3) is not idempotent, then the order of (t_1, t_2, t_3) is infinite.

Lemma 3.3. *Let* $t_1 \in X_3$ *and* $t_2, t_3 \in W_\tau(X_3) \setminus X_3$.

(1) *If* $t_1 \neq x_1$, *then the order of* (t_1, t_2, t_3) *is infinite.*

(2) *If* $t_1 = x_1$ *and* $var(t_2) \nsubseteq \{x_1\}$, *then the order of* (t_1, t_2, t_3) *is infinite.*

(3) *If* $t_1 = x_1$, $var(t_2) \subseteq \{x_1\}$ *and* $var(t_3) \nsubseteq \{x_1\}$, *then the order of* (t_1, t_2, t_3) *is infinite.*

(4) *If* $t_1 = x_1$, $var(t_2) \subseteq \{x_1\}$ *and* $var(t_3) \subseteq \{x_1\}$, *then the order of* (t_1, t_2, t_3) *is* 1.

Proof. (1) Assume that $t_1 \neq x_1$. Then $t_1 = x_2$ or $t_1 = x_3$. If $t_1 = x_2$, then by Lemma 3.2 the order of $(x_2, t_2, t_3)^2$ is infinite. Since $\langle (x_2, t_2, t_3)^2 \rangle$ is a subsemigroup of $\langle (x_2, t_2, t_3) \rangle$, so the order of (x_2, t_2, t_3) is infinite. Similarly, $t_1 = x_3$ implies that the order of $(x_2, t_2, t_3)^2$ is infinite. Hence the order of (x_3, t_2, t_3) is infinite.

(2) To show that $\overline{op}(x_1, t_2, t_3)^{n+1} > \overline{op}(x_1, t_2, t_3)^n$ for all positive integers n, let n be a positive integer. Assume that $x_2 \in var(t_2)$. We have $(x_1, t_2, t_3)^n = (x_1, t'', t''')$ for some $t'', t''' \in W_\tau(X_3) \setminus X_3$ such that $x_2 \in var(t'')$. Then

$$\begin{aligned} \overline{op}(x_1, t_2, t_3)^{n+1} &= \overline{op}((x_1, t_2, t_3)^n + (x_1, t_2, t_3)) \\ &= \overline{op}(x_1, t''(x_1, t_2, t_3), t'''(x_1, t_2, t_3)) \\ &= op(t''(x_1, t_2, t_3)) + op(t'''(x_1, t_2, t_3)) \\ &> op(t'') + op(t''') \\ &= \overline{op}(x_1, t_2, t_3)^n. \end{aligned}$$

Therefore, the order of (t_1, t_2, t_3) is infinite. Similarly, if $x_3 \in var(t_2)$, then the order of (t_1, t_2, t_3) is infinite.

(3) This can be proved similarly to (2).

(4) This follows since the order of an idempotent is 1. □

In the same manner as Lemma 3.3, we have the following two lemmas.

Lemma 3.4. *Let $t_2 \in X_3$ and $t_1, t_3 \in W_\tau(X_3) \setminus X_3$.*

(1) *If $t_2 \neq x_2$, then the order of (t_1, t_2, t_3) is infinite.*

(2) *If $t_2 = x_2$ and $var(t_1) \nsubseteq \{x_2\}$, then the order of (t_1, t_2, t_3) is infinite.*

(3) *If $t_2 = x_2$, $var(t_1) \subseteq \{x_2\}$ and $var(t_3) \nsubseteq \{x_2\}$, then the order of (t_1, t_2, t_3) is infinite.*

(4) *If $t_2 = x_2$, $var(t_1) \subseteq \{x_2\}$ and $var(t_3) \subseteq \{x_2\}$, then the order of (t_1, t_2, t_3) is 1.*

Lemma 3.5. *Let $t_3 \in X_3$ and $t_1, t_2 \in W_\tau(X_3) \setminus X_3$.*

(1) *If $t_3 \neq x_3$, then the order of (t_1, t_2, t_3) is infinite.*

(2) *If $t_3 = x_3$ and $var(t_1) \nsubseteq \{x_3\}$, then the order of (t_1, t_2, t_3) is infinite.*

(3) *If $t_3 = x_3$, $var(t_1) \subseteq \{x_3\}$ and $var(t_2) \nsubseteq \{x_3\}$, then the order of (t_1, t_2, t_3) is infinite.*

(4) *If $t_3 = x_3$, $var(t_1) \subseteq \{x_3\}$ and $var(t_2) \subseteq \{x_3\}$, then the order of (t_1, t_2, t_3) is 1.*

If $t_1, t_2 \in X_3$ such that $x_3 \in \{t_1, t_2\}$, then there are 5 cases to consider:

(i) $t_1 = x_1, t_2 = x_3$,

(ii) $t_1 = x_3, t_2 = x_2$,

(iii) $t_1 = x_3, t_2 = x_1$,

(iv) $t_1 = x_2, t_2 = x_3$,

(v) $t_1 = x_3, t_2 = x_3$.

We have the following.

Lemma 3.6.

(1) *Let $t_1, t_2 \in X_3$ and $t_3 \in W_\tau(X_3) \setminus X_3$ such that $var(t_3) \subseteq \{t_1, t_2\}$.*

(1.1) *If $x_3 \notin \{t_1, t_2\}$, then the order of (t_1, t_2, t_3) is shown below:*

t_1	t_2	$var(t_3)$	*the order of* (t_1, t_2, t_3)
x_1	x_1	$\{x_1\}$	1
x_1	x_2	$\{x_1\}$	1
x_1	x_2	$\{x_2\}$	1
x_1	x_2	$\{x_1, x_2\}$	1
x_2	x_1	$\{x_1\}$	2
x_2	x_1	$\{x_2\}$	2
x_2	x_1	$\{x_1, x_2\}$	2
x_2	x_2	$\{x_2\}$	1

(1.2) *If* $x_3 \in \{t_1, t_2\}$, $t_1 = x_1, t_2 = x_3$ *and* $x_3 \notin var(t_3)$ (*i.e.* $var(t_3) = \{x_1\}$), *then the order of* (t_1, t_2, t_3) *is* 2.

(1.3) *If* $x_3 \in \{t_1, t_2\}$, $t_1 = x_1, t_2 = x_3$ *and* $x_3 \in var(t_3)$, *then the order of* (t_1, t_2, t_3) *is infinite.*

(1.4) *If* $x_3 \in \{t_1, t_2\}$, $t_1 = x_3, t_2 = x_2$ *and* $x_3 \notin var(t_3)$ (*i.e.* $var(t_3) = \{x_2\}$), *then the order of* (t_1, t_2, t_3) *is* 2.

(1.5) *If* $x_3 \in \{t_1, t_2\}$, $t_1 = x_3, t_2 = x_2$ *and* $x_3 \in var(t_3)$, *then the order of* (t_1, t_2, t_3) *is infinite.*

(1.6) *If* $x_3 \in \{t_1, t_2\}$ *and* $t_1 = x_3, t_2 = x_1$ *or* $t_1 = x_2, t_2 = x_3$, *then the order of* (t_1, t_2, t_3) *is infinite.*

(1.7) *If* $x_3 \in \{t_1, t_2\}$, $t_1 = x_3, t_2 = x_3$, *then the order of* (t_1, t_2, t_3) *is infinite.*

(2) *Let* $t_1, t_2 \in X_3$ *and* $t_3 \in W_\tau(X_3) \setminus X_3$ *such that* $var(t_3) \nsubseteq \{t_1, t_2\}$ *and* $x_3 \in var(t_3)$. *Then the order of* (t_1, t_2, t_3) *is infinite.*

(3) *Let* $t_1, t_2 \in X_3$ *and* $t_3 \in W_\tau(X_3) \setminus X_3$ *such that* $var(t_3) \nsubseteq \{t_1, t_2\}$ *and* $x_3 \notin var(t_3)$.

(3.1) *If* $var(t_3) = \{x_1\}$ *and* $(t_1, t_2) = (x_2, x_2)$, *then the order of* (t_1, t_2, t_3) *is* 2.

(3.2) *If* $var(t_3) = \{x_1\}$, *and* $(t_1, t_2) \in \{(x_2, x_3), (x_3, x_2), (x_3, x_3)\}$, *then the order of* (t_1, t_2, t_3) *is infinite.*

(3.3) *If* $var(t_3) = \{x_2\}$ *and* $(t_1, t_2) = (x_1, x_1)$, *then the order of* (t_1, t_2, t_3) *is* 2.

(3.4) *If* $var(t_3) = \{x_2\}$ *and* $(t_1, t_2) \in \{(x_1, x_3), (x_3, x_1), (x_3, x_3)\}$, *then the order of* (t_1, t_2, t_3) *is infinite.*

(3.5) *If* $var(t_3) = \{x_1, x_2\}$ *and* $(t_1, t_2) = (x_1, x_1)$, *then the order of* (t_1, t_2, t_3) *is* 2.

(3.6) *If* $var(t_3) = \{x_1, x_2\}$ *and*

$$(t_1, t_2) \in \{(x_2, x_2), (x_1, x_3), (x_3, x_1), (x_3, x_3), (x_2, x_3), (x_3, x_2)\},$$

then the order of (t_1, t_2, t_3) *is infinite.*

Proof. (1) It is easy to see that (1.1), (1.2) and (1.4) hold.

To prove (1.3), let n be a positive integer. We have $(x_1, x_3, t_3)^n = (x_1, t'', t''')$ for some $t'', t''' \in W_\tau(X_3) \setminus X_3$ such that $x_3 \in var(t''')$. Then

$$\begin{aligned}
\overline{op}(x_1, x_3, t_3)^{n+1} &= \overline{op}((x_1, x_3, t_3)^n + (x_1, x_3, t_3)) \\
&= \overline{op}(x_1, t''(x_1, x_3, t_3), t'''(x_1, x_3, t_3)) \\
&= op(t''(x_1, x_3, t_3)) + op(t'''(x_1, x_3, t_3)) \\
&> op(t'') + op(t''') \\
&= \overline{op}(x_1, x_3, t_3)^n.
\end{aligned}$$

Therefore, the order of (t_1, t_2, t_3) is infinite. Similarly, we obtain (1.5).

Assume that $t_1 = x_3$ and $t_2 = x_1$. By Lemma 3.2, the order of $(x_3, x_1, t_3)^3$ is infinite. Since $\langle (x_3, x_1, t_3)^3 \rangle$ is a subsemigroup of $\langle (x_3, x_1, t_3) \rangle$, so the order of (t_1, t_2, t_3) is infinite. Similarly, if $t_1 = x_2$ and $t_2 = x_3$, then the order of (t_1, t_2, t_3) is infinite. Hence (1.6) holds.

For (1.7), let $t_1 = x_3$ and $t_2 = x_3$. By Lemma 3.2, the order of $(x_3, x_3, t_3)^2$ is infinite. Since $\langle (x_3, x_3, t_3)^2 \rangle$ is a subsemigroup of $\langle (x_3, x_3, t_3) \rangle$, we have the order of (t_1, t_2, t_3) is infinite.

(2) This can be proved similarly to (1.3).

(3) It is easy to see that (3.1), (3.3) and (3.5) hold.

By Lemma 3.2, $(t_1, t_2) \in \{(x_2, x_3), (x_3, x_3)\}$ implies that the order of (t_1, t_2, t_3) is infinite. In a similar way as (1.3), if $(t_1, t_2) = (x_3, x_2)$, then the order of (t_1, t_2, t_3) is infinite. This proves (3.2). Similarly, we obtain (3.4).

By Lemma 3.2, if $(t_1, t_2) \in \{(x_3, x_1), (x_3, x_3), (x_2, x_3)\}$, then the order of (t_1, t_2, t_3) is infinite. In a similar way as (1.3), if $(t_1, t_2) \in \{(x_2, x_2), (x_1, x_3)(x_3, x_2)\}$, then the order of (t_1, t_2, t_3) is infinite. This proves (3.6). □

Similarly, we state the following two lemmas without proofs.

Lemma 3.7.

(1) *Let* $t_1, t_3 \in X_3$ *and* $t_2 \in W_\tau(X_3) \setminus X_3$ *such that* $var(t_2) \subseteq \{t_1, t_3\}$.

(1.1) *If* $x_2 \notin \{t_1, t_3\}$, *then the order of* (t_1, t_2, t_3) *is shown below:*

t_1	$var(t_2)$	t_3	*the order of* (t_1, t_2, t_3)
x_1	$\{x_1\}$	x_1	1
x_1	$\{x_1\}$	x_3	1
x_1	$\{x_3\}$	x_3	1
x_1	$\{x_1, x_3\}$	x_3	1
x_3	$\{x_1\}$	x_1	2
x_3	$\{x_3\}$	x_1	2
x_3	$\{x_1, x_3\}$	x_1	2
x_3	$\{x_3\}$	x_3	1

(1.2) *If* $x_2 \in \{t_1, t_3\}$, $t_1 = x_1, t_3 = x_2$ *and* $x_2 \notin var(t_2)$ (*i.e.* $var(t_2) = \{x_1\}$), *then the order of* (t_1, t_2, t_3) *is* 2.

(1.3) *If* $x_2 \in \{t_1, t_3\}$, $t_1 = x_1, t_3 = x_2$ *and* $x_2 \in var(t_2)$, *then the order of* (t_1, t_2, t_3) *is infinite.*

(1.4) *If* $x_2 \in \{t_1, t_3\}$, $t_1 = x_2, t_3 = x_3$ *and* $x_2 \notin var(t_2)$ (*i.e.* $var(t_2) = \{x_3\}$), *then the order of* (t_1, t_2, t_3) *is* 2.

(1.5) *If* $x_2 \in \{t_1, t_3\}$, $t_1 = x_2, t_3 = x_3$ *and* $x_2 \in var(t_2)$, *then the order of* (t_1, t_2, t_3) *is infinite.*

(1.6) *If* $x_2 \in \{t_1, t_3\}$ *and* $t_1 = x_2, t_3 = x_1$ *or* $t_1 = x_3, t_3 = x_2$, *then the order of* (t_1, t_2, t_3) *is infinite.*

(1.7) *If* $x_2 \in \{t_1, t_3\}$ *and* $t_1 = x_2, t_3 = x_2$, *then the order of* (t_1, t_2, t_3) *is infinite.*

(2) *Let* $t_1, t_3 \in X_3$ *and* $t_2 \in W_\tau(X_3) \setminus X_3$ *such that* $var(t_2) \nsubseteq \{t_1, t_3\}$ *and* $x_2 \in var(t_2)$. *Then the order of* (t_1, t_2, t_3) *is infinite.*

(3) *Let* $t_1, t_3 \in X_3$ *and* $t_2 \in W_\tau(X_3) \setminus X_3$ *such that* $var(t_2) \nsubseteq \{t_1, t_3\}$ *and* $x_2 \notin var(t_2)$.

(3.1) *If* $var(t_2) = \{x_1\}$ *and* $(t_1, t_3) = (x_2, x_2)$, *then the order of* (t_1, t_2, t_3) *is* 2.

(3.2) *If* $var(t_2) = \{x_1\}$, *and* $(t_1, t_3) \in \{(x_2, x_3), (x_3, x_2), (x_3, x_3)\}$, *then the order of* (t_1, t_2, t_3) *is infinite.*

(3.3) *If* $var(t_2) = \{x_3\}$ *and* $(t_1, t_3) = (x_1, x_1)$, *then the order of* (t_1, t_2, t_3) *is* 2.

(3.4) *If* $var(t_2) = \{x_3\}$ *and* $(t_1, t_3) \in \{(x_1, x_2), (x_2, x_1), (x_2, x_2)\}$, *then the order of* (t_1, t_2, t_3) *is infinite.*

(3.5) *If* $var(t_2) = \{x_1, x_3\}$ *and* $(t_1, t_3) = (x_1, x_1)$, *then the order of* (t_1, t_2, t_3) *is* 2.

(3.6) *If* $var(t_2) = \{x_1, x_3\}$ *and*

$$(t_1, t_3) \in \{(x_1, x_2), (x_2, x_1), (x_2, x_2), (x_2, x_3), (x_3, x_2), (x_3, x_3)\},$$

then the order of (t_1, t_2, t_3) *is infinite.*

Lemma 3.8.

(1) *Let* $t_2, t_3 \in X_3$ *and* $t_1 \in W_\tau(X_3) \setminus X_3$ *such that* $var(t_1) \subseteq \{t_2, t_3\}$.

(1.1) *If* $x_1 \notin \{t_2, t_3\}$, *then the order of* (t_1, t_2, t_3) *is shown below:*

$var(t_1)$	t_2	t_3	*the order of* (t_1, t_2, t_3)
$\{x_2\}$	x_2	x_2	1
$\{x_2\}$	x_2	x_3	1
$\{x_3\}$	x_2	x_3	2
$\{x_2, x_3\}$	x_2	x_3	1
$\{x_2\}$	x_3	x_2	2
$\{x_3\}$	x_3	x_2	2
$\{x_2, x_3\}$	x_3	x_2	2
$\{x_3\}$	x_3	x_3	1

(1.2) *If* $x_1 \in \{t_2, t_3\}$, $t_2 = x_1, t_3 = x_3$ *and* $x_1 \notin var(t_1)$ *(i.e.* $var(t_1) = \{x_3\}$*), then the order of* (t_1, t_2, t_3) *is* 2.

(1.3) *If* $x_1 \in \{t_2, t_3\}$, $t_2 = x_1, t_3 = x_3$ *and* $x_1 \in var(t_1)$, *then the order of* (t_1, t_2, t_3) *is infinite.*

(1.4) *If* $x_1 \in \{t_2, t_3\}$, $t_2 = x_2, t_3 = x_1$ *and* $x_1 \notin var(t_1)$ *(i.e.* $var(t_1) = \{x_2\}$*), then the order of* (t_1, t_2, t_3) *is* 2.

(1.5) *If* $x_1 \in \{t_2, t_3\}$*,* $t_2 = x_2, t_3 = x_1$ *and* $x_1 \in var(t_1)$*, then the order of* (t_1, t_2, t_3) *is infinite.*

(1.6) *If* $x_1 \in \{t_2, t_3\}$ *and* $t_2 = x_3, t_3 = x_1$ *or* $t_2 = x_1, t_3 = x_2$*, then the order of* (t_1, t_2, t_3) *is infinite.*

(1.7) *If* $x_1 \in \{t_2, t_3\}$ *and* $t_2 = x_1, t_3 = x_1$*, then the order of* (t_1, t_2, t_3) *is infinite.*

(2) *Let* $t_2, t_3 \in X_3$ *and* $t_1 \in W_\tau(X_3) \setminus X_3$ *such that* $var(t_1) \nsubseteq \{t_2, t_3\}$ *and* $x_1 \in var(t_1)$*. Then the order of* (t_1, t_2, t_3) *is infinite.*

(3) *Let* $t_2, t_3 \in X_3$ *and* $t_1 \in W_\tau(X_3) \setminus X_3$ *such that* $var(t_1) \nsubseteq \{t_2, t_3\}$ *and* $x_1 \notin var(t_1)$*.*

(3.1) *If* $var(t_1) = \{x_2\}$ *and* $(t_2, t_3) = (x_2, x_2)$*, then the order of* (t_1, t_2, t_3) *is* 2*.*

(3.2) *If* $var(t_1) = \{x_2\}$*, and* $(t_2, t_3) \in \{(x_2, x_3), (x_3, x_2), (x_3, x_3)\}$*, then the order of* (t_1, t_2, t_3) *is infinite.*

(3.3) *If* $var(t_1) = \{x_3\}$ *and* $(t_2, t_3) = (x_1, x_1)$*, then the order of* (t_1, t_2, t_3) *is* 2*.*

(3.4) *If* $var(t_1) = \{x_3\}$ *and* $(t_2, t_3) \in \{(x_1, x_3), (x_3, x_1), (x_3, x_3)\}$*, then the order of* (t_1, t_2, t_3) *is infinite.*

(3.5) *If* $var(t_1) = \{x_2, x_3\}$ *and* $(t_2, t_3) = (x_1, x_1)$*, then the order of* (t_1, t_2, t_3) *is* 2*.*

(3.6) *If* $var(t_1) = \{x_2, x_3\}$ *and*

$$(t_2, t_3) \in \{(x_2, x_2), (x_1, x_3), (x_3, x_1), (x_3, x_3), (x_2, x_1), (x_1, x_2)\},$$

then the order of (t_1, t_2, t_3) *is infinite.*

Using Lemma 3.1 - Lemma 3.8, we obtain the main result of this section.

Theorem 3.9. *The order of an element of* $(W_\tau(X_3))^3$ *is* $1, 2, 3$ *or infinite.*

Remark. Lemma 2.3 and Lemma 3.2 can be generalized to the semigroup $(W_\tau(X_n))^n$. Indeed: the order of $(t_1, t_2, \ldots, t_n)$ where $t_1, t_2, \ldots, t_n \in W_\tau(X_n) \setminus X_n$ is infinite. In fact: let $t_1, t_2, \ldots, t_n \in W_\tau(X_n) \setminus X_n$. We shall show that $\overline{op}(t_1, t_2, \ldots, t_n)^{m+1} > \overline{op}(t_1, t_2, \ldots, t_n)^m$ for all positive integers m. For a positive integer m, we have $(t_1, t_2, \ldots, t_n)^n = (t'_1, t'_2, \ldots, t'_n)$ for some $t'_1, t'_2, \ldots, t'_n \in W_\tau(X_n) \setminus X_n$. Then

$$\begin{aligned}
\overline{op}(t_1, t_2, \ldots, t_n)^{m+1} &= \overline{op}((t_1, t_2, \ldots, t_n)^m + (t_1, t_2, \ldots, t_n)) \\
&= \overline{op}(t'_1(t_1, t_2, \ldots, t_n), t'_2(t_1, t_2, \ldots, t_n), \ldots, t'_n(t_1, t_2, \ldots, t_n)) \\
&> \overline{op}(t'_1, t'_2, \ldots, t'_n) \\
&= \overline{op}(t_1, t_2, \ldots, t_n)^m.
\end{aligned}$$

Therefore, the order of $(t_1, t_2, \ldots, t_n)$ where $t_1, t_2, \ldots, t_n \in W_\tau(X_n) \setminus X_n$ is infinite.

Question. Determine the order of the semigroup $(W_\tau(X_n))^n$ for a positive integer $n \geq 4$.

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