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Natural Partial Orders on the Semigroup of Binary Relations

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Abstract : Let $\mathscr{B}(X)$ denote the semigroup of binary relations on a set X under composition. We study two natural partial orders on $\mathscr{B}(X)$ and characterize when two elements of $\mathscr{B}(X)$ are related under these orders. The maximality, minimality, left compatibility and right compatibility of elements are considered with respect to each order.

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1 Introduction

In 1986, Mitsch [6] defined the partial order $\leq$ on any semigroup S as follows: for $a, b \in S$,

$$a \leq b \Longleftrightarrow a = xb = by \text{ and } a = ay \text{ for some } x, y \in S^1,$$

which is called the *natural partial order* on S.

Next, Kowol and Mitsch [4] studied the natural partial order on $\mathscr{T}(X)$, the semigroup of all transformations of a set X. Marques-Smith and Sullivan [5] extended that work to $\mathscr{P}\mathscr{T}(X)$, the semigroup of all partial transformations of X. They also determined when two elements in $\mathscr{P}\mathscr{T}(X)$ are related under $\leq$ and compared $\leq$ with another natural partial order $\subseteq$ on $\mathscr{P}\mathscr{T}(X)$. Moreover, they described the maximal, minimal, left compatible and right compatible elements of $\mathscr{P}\mathscr{T}(X)$ with respect to each order. In this paper, we study two natural partial orders $\leq$ and $\subseteq$ on $\mathscr{B}(X)$, the semigroup of binary relations on a set X under composition and characterize when two elements are related under these two orders. Furthermore, we determine the maximal, minimal, left compatible and right compatible elements of $\mathscr{B}(X)$ with respect to each order.

1.1 Preliminaries

Let S be a semigroup and $E(S)$ denote the set of all idempotents of S. The Green's relations $\mathcal{L}$ and $\mathcal{R}$ on a semigroup S are defined by $a\mathcal{L}b \Longleftrightarrow S^1a = S^1b$ and $a\mathcal{R}b \Longleftrightarrow aS^1 = bS^1$ for all $a, b \in S$.

Let $\leq$ be any partial order on a semigroup S. An element a of S is called *left [right] compatible with respect to* $\leq$ on S if for all $x, y \in S$, $x \leq y$ implies $ax \leq ay$ $[xa \leq ya]$.

In the remainder, the relation $\leq$ given on any semigroup S always means the natural partial order on S defined previously, that is, for any $a, b \in S$,

$$a \leq b \Longleftrightarrow a = xb = by \text{ and } a = ay \text{ for some } x, y \in S^1.$$

In this case, we have that $a = (xb)y = x(by) = xa$.

Lemma 1.1. ([2]). *Let S be a semigroup and $a, b \in S$. If $a \leq b$ and $(a, b) \in \mathcal{L} \cup \mathcal{R}$, then $a = b$.*

1.2 On the semigroup of binary relations

Let X be a set. From the definition of $\mathscr{B}(X)$, we have

$$\mathscr{B}(X) = \{\alpha \mid \alpha \subseteq X \times X\}$$

and for $\alpha, \beta \in \mathscr{B}(X)$,

$$\alpha\beta = \{(x, y) \in X \times X \mid (x, z) \in \alpha \text{ and } (z, y) \in \beta \text{ for some } z \in X\}.$$

Then the empty relation is the zero of $\mathscr{B}(X)$ which is denoted by 0. For $Y \subseteq X$, let

$$\Delta_Y = \{(y, y) \mid y \in Y\}$$

and

$$\nabla_Y = \{(x, y) \mid x, y \in Y\},$$

so Δ_X and ∇_X are the identity and universal relations on X, respectively.

In particular, for a finite set $X = \{a_1, a_2, \ldots, a_n\}$, we can represent a relation $\alpha \in \mathscr{B}(X)$ with the $n \times n$ Boolean matrix A defined by

$$A_{ij} = \begin{cases} 1 & \text{if } (a_i, a_j) \in \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

For example, if $\alpha = \{(a_1, a_1), (a_1, a_2), (a_2, a_1), (a_3, a_2), (a_3, a_3)\} \in \mathscr{B}(X)$ with $X = \{a_1, a_2, a_3\}$, then

$$\alpha = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Let $\alpha \in \mathscr{B}(X)$. For $x \in X$, let

$$x\alpha = \{y \in X \mid (x, y) \in \alpha\}$$

and

$$\alpha x = \{y \in X \mid (y, x) \in \alpha\}$$

and they are called a *row* and a *column* of α, respectively. For $A \subseteq X$, set

$$A\alpha = \bigcup_{x \in A} x\alpha \quad \text{and} \quad \alpha A = \bigcup_{x \in A} \alpha x.$$

Then $\alpha A = A\alpha^{-1}$. Let

$$V(\alpha) = \{A\alpha \mid A \subseteq X\}$$

and

$$W(\alpha) = \{\alpha A \mid A \subseteq X\}.$$

Lemma 1.2. ([8]). *Let $\alpha \in \mathscr{B}(X)$. Then $V(\alpha)$ and $W(\alpha)$ are anti-isomorphic lattices.*

We say that a relation $\alpha \in \mathscr{B}(X)$ is *row reduced* if for all $x \in X$ and $A \subseteq X$, $\emptyset \neq x\alpha = A\alpha$ implies that $x \in A$ and *column reduced* if for all $x \in X$ and $A \subseteq X$, $\emptyset \neq \alpha x = \alpha A$ implies that $x \in A$.

A relation $\alpha \in \mathscr{B}(X)$ is *row minimal* if for all $x \in X$ and $A \subseteq X$, $\emptyset \neq x\alpha = A\alpha$ implies that $\{x\} = A$ and *column minimal* if for all $x \in X$ and $A \subseteq X$, $\emptyset \neq \alpha x = \alpha A$ implies that $\{x\} = A$. Observe that if α is row [column] minimal, then it is row [column] reduced.

Example 1.3. Let

$$\alpha = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

Then α is column reduced but not column minimal.

Let $\alpha, \beta \in \mathscr{B}(X)$. We say that β is a *row descendant of* α if there exist $a, b \in X$ with $a \neq b$ and a nonempty subset A of X such that $A \subseteq a\alpha \subseteq b\alpha$, $b\beta = b\alpha \backslash A$ and $x\alpha = x\beta$ for all $x \neq b$. A *column descendant of* α is defined in a dual manner. Set

$$R^{\alpha} = \{\beta \mid \beta \text{ is a row descendant of } \alpha\}$$

and

$$C^{\alpha} = \{\beta \mid \beta \text{ is a column descendant of } \alpha\}.$$

It is easy to see that if α is row [column] minimal, then $R^{\alpha} [C^{\alpha}] = \emptyset$.

Example 1.4. Let

$$\alpha = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

Then

$$R^\alpha = \left\{ \begin{bmatrix} 0&1&0&1 \\ 1&0&1&0 \\ 0&0&1&0 \\ 0&1&1&0 \end{bmatrix}, \begin{bmatrix} 0&0&1&1 \\ 1&0&1&0 \\ 0&0&1&0 \\ 0&1&1&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&1 \\ 1&0&1&0 \\ 0&0&1&0 \\ 0&1&1&0 \end{bmatrix}, \begin{bmatrix} 0&1&1&1 \\ 1&0&0&0 \\ 0&0&1&0 \\ 0&1&1&0 \end{bmatrix}, \begin{bmatrix} 0&1&1&1 \\ 1&0&1&0 \\ 0&0&1&0 \\ 0&1&0&0 \end{bmatrix} \right\}$$

and

$$C^\alpha = \left\{ \begin{bmatrix} 0&1&0&1 \\ 1&0&1&0 \\ 0&0&1&0 \\ 0&1&1&0 \end{bmatrix}, \begin{bmatrix} 0&1&1&1 \\ 1&0&0&0 \\ 0&0&1&0 \\ 0&1&1&0 \end{bmatrix}, \begin{bmatrix} 0&1&1&1 \\ 1&0&1&0 \\ 0&0&1&0 \\ 0&1&0&0 \end{bmatrix}, \begin{bmatrix} 0&1&0&1 \\ 1&0&1&0 \\ 0&0&1&0 \\ 0&1&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&1&1 \\ 1&0&1&0 \\ 0&0&1&0 \\ 0&1&1&0 \end{bmatrix} \right\}.$$

The following lemma is easily shown.

Lemma 1.5. *Let $\alpha, \beta, \gamma \in \mathscr{B}(X)$. Then*
(1) $(\alpha')' = \alpha$,
(2) $(\alpha^{-1})^{-1} = \alpha$ *and* $(\alpha\beta)^{-1} = \beta^{-1}\alpha^{-1}$,
(3) $\alpha \subseteq \beta$ *implies* $\gamma\alpha \subseteq \gamma\beta$ *and* $\alpha\gamma \subseteq \beta\gamma$,
(4) $\alpha \subseteq \beta$ *if and only if* $\alpha^{-1} \subseteq \beta^{-1}$,
(5) $\alpha \subseteq \beta$ *if and only if* $\beta' \subseteq \alpha'$.

Lemma 1.6. ([7]). *Let $\alpha, \beta \in \mathscr{B}(X)$ and $A \subseteq X$. Then*
(1) $A(\alpha\beta) = (A\alpha)\beta$ *and* $(\alpha\beta)A = \alpha(\beta A)$,
(2) $V(\alpha\beta) \subseteq V(\beta)$,
(3) $W(\alpha\beta) \subseteq W(\alpha)$,
(4) $\alpha\mathcal{L}\beta$ *if and only if* $V(\alpha) = V(\beta)$,
(5) $\alpha\mathcal{R}\beta$ *if and only if* $W(\alpha) = W(\beta)$.

Lemma 1.7. *Let $\alpha \in \mathscr{B}(X)$. Then the following statements are equivalent.*
(1) $\alpha^2 = \alpha$.
(2) *For every* $A \in V(\alpha)$, $A\alpha = A$.
(3) *For every* $A \in W(\alpha)$, $\alpha A = A$.

Proof. (1) $\Longrightarrow$ (2) Assume that $\alpha^2 = \alpha$. Let $A \in V(\alpha)$, Then there exists $B \subseteq X$ such that $B\alpha = A$. Thus $A\alpha = B\alpha\alpha = B\alpha = A$.

(2) $\Longrightarrow$ (1) For $x \in X$, $x\alpha \in V(\alpha)$. By assumption, $x\alpha^2 = x\alpha\alpha = x\alpha$. Hence $\alpha^2 = \alpha$.

(1) $\Longleftrightarrow$ (3) can be proved similarly. □

Lemma 1.8. *Let A and B be nonempty subsets of X such that $A \cap B \neq \emptyset$. Define $\alpha = A \times B$. Then $V(\alpha) = \{\emptyset, B\}$ and $W(\alpha) = \{\emptyset, A\}$, hence α is an idempotent of $\mathscr{B}(X)$.*

Proof. By the definition of α, $a\alpha = B$ for all $a \in A$ and $a\alpha = \emptyset$ for all $a \in X\setminus A$. Then $V(\alpha) = \{\emptyset, B\}$. Since $A \cap B \neq \emptyset$, we have

$$B\alpha = \bigcup_{x\in B} x\alpha = \left(\bigcup_{x\in A\cap B} x\alpha\right) \bigcup \left(\bigcup_{x\in B\setminus A} x\alpha\right) = B \cup \emptyset = B.$$

By Lemma 1.7, α is an idempotent. □

2 Main results

2.1 Natural partial orders on $\mathscr{B}(X)$

Regarding elements of $\mathscr{B}(X)$ as subsets of $X \times X$, $\subseteq$ is a natural partial order of $\mathscr{B}(X)$, that is,

$$\alpha \subseteq \beta \Longleftrightarrow \text{ for every } (x, y) \in X \times X, (x, y) \in \alpha \text{ implies } (x, y) \in \beta.$$

The next proposition is evident.

Proposition 2.1. *Let $\alpha, \beta \in \mathscr{B}(X)$. Then the following statements are equivalent.*
(1) $\alpha \subseteq \beta$.
(2) $\alpha X \subseteq \beta X$ *and for every* $x \in \alpha X,\ x\alpha \subseteq x\beta$.
(3) $X\alpha \subseteq X\beta$ *and for every* $y \in X\alpha,\ \alpha y \subseteq \beta y$.

From Proposition 2.1, we have

Corollary 2.2. *Let $\alpha, \beta \in \mathscr{B}(X)$. Then*
(1) $\alpha = \beta$ *if and only if* $\alpha X = \beta X$ *and for every* $x \in \alpha X,\ x\alpha = x\beta$,
(2) $\alpha = \beta$ *if and only if* $X\alpha = X\beta$ *and for every* $x \in X\alpha,\ \alpha x = \beta x$.

Recall that the natural partial order $\leq$ defined on $\mathscr{B}(X)$ is as follows:

$$\alpha \leq \beta \Longleftrightarrow \alpha = \lambda\beta = \beta\mu \text{ and } \alpha = \alpha\mu \text{ for some } \lambda, \mu \in \mathscr{B}(X).$$

In this case, we also have $\alpha = \lambda\alpha$.

In the next theorem, we give a characterization when $\alpha, \beta \in \mathscr{B}(X)$ are comparable under $\leq$.

Theorem 2.3. *Let $\alpha, \beta \in \mathscr{B}(X)$. Then the following statements are equivalent:*
(1) $\alpha \leq \beta$.
(2) $\alpha^{-1} \leq \beta^{-1}$.
(3) $V(\alpha) \subseteq V(\beta), W(\alpha) \subseteq W(\beta)$ *and for all* $A, B \in X, A\alpha = B\beta$ *implies* $A\alpha = B\alpha$ *and* $\alpha A = \beta B$ *implies* $\alpha A = \alpha B$.

Proof. (1)$\Longrightarrow$(2) Assume that $\alpha \leq \beta$. Then there exist $\lambda, \mu \in \mathscr{B}(X)$ such that $\alpha = \lambda\beta = \beta\mu$ and $\alpha = \alpha\mu = \lambda\alpha$. By Lemma 1.5 (2), we have $\alpha^{-1} = \beta^{-1}\lambda^{-1} = \mu^{-1}\beta^{-1}$ and $\alpha^{-1} = \alpha^{-1}\lambda^{-1}$. Hence $\alpha^{-1} \leq \beta^{-1}$.

(2)$\Longrightarrow$(3) Assume that $\alpha^{-1} \leq \beta^{-1}$. Then there are $\lambda, \mu \in \mathscr{B}(X)$ such that $\alpha^{-1} = \lambda\beta^{-1} = \beta^{-1}\mu$ and $\alpha^{-1} = \alpha^{-1}\mu = \lambda\alpha^{-1}$, so $\alpha = \beta\lambda^{-1} = \mu^{-1}\beta$ and $\alpha = \mu^{-1}\alpha = \alpha\lambda^{-1}$. By Lemma 1.6 (2) and (3), we have

$$V(\alpha) = V(\mu^{-1}\beta) \subseteq V(\beta) \text{ and } W(\alpha) = W(\beta\lambda^{-1}) \subseteq W(\beta).$$

Let $A, B \subseteq X$. If $A\alpha = B\beta$, then $A\alpha = A\alpha\lambda^{-1} = B\beta\lambda^{-1} = B\alpha$. If $\alpha A = \beta B$, then $\alpha A = \mu^{-1}\alpha A = \mu^{-1}\beta B = \alpha B$. Hence (3) holds.

(3)$\Longrightarrow$(1) Assume that (3) holds. Then for each $x \in X$, $x\alpha \in V(\alpha) \subseteq V(\beta)$ and $\alpha x \in W(\alpha) \subseteq W(\beta)$. Then there exist $A_x, B_x \subseteq X$ such that $x\alpha = A_x\beta$ and $\alpha x = \beta B_x$. By assumption, we have $x\alpha = A_x\alpha$ and $\alpha x = \alpha B_x$. Define $\lambda, \mu \in \mathscr{B}(X)$ by $x\lambda = A_x$ for all $x \in X$ and $\mu x = B_x$ for all $x \in X$. Then for every $x \in X$,

$$x\lambda\beta = A_x\beta = x\alpha \quad \text{and}$$
$$\alpha\mu x = \alpha B_x = \alpha x = \beta B_x = \beta\mu x.$$

This shows that $\alpha = \lambda\beta$ and $\alpha = \alpha\mu = \beta\mu$, so $\alpha \leq \beta$. □

The following example shows that $\leq$ and $\subseteq$ are distinct.

Example 2.4. Suppose that $X = \{a_1, a_2, a_3\}$ and let

$$\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \beta = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } \gamma = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Clearly, $\alpha \subseteq \beta$. Since $a_1\alpha = \{a_1\} \in V(\alpha)$ and $\{a_1\} \notin V(\beta)$, by Theorem 2.3 $\alpha \nleq \beta$. We can check that $\gamma \leq \beta$ but $\gamma \nsubseteq \beta$.

2.2 Maximal and minimal elements

Since 0 and ∇_X are the minimum and maximum elements of $(\mathscr{B}(X), \subseteq)$, respectively. The following proposition determines the minimal elements in $\mathscr{B}(X)\backslash\{0\}$ and the maximal element in $\mathscr{B}(X)\backslash\{\nabla_X\}$ with respect to $\subseteq$ and the proof is obvious.

Proposition 2.5. *The following statements about $\mathscr{B}(X)$ hold.*

(1) *For $\alpha \in \mathscr{B}(X)\backslash\{0\}$, α is minimal in $(\mathscr{B}(X)\backslash\{0\}, \subseteq)$ if and only if $|\alpha| = 1$.*

(2) *For $\alpha \in \mathscr{B}(X)\backslash\{\nabla_X\}$, α is maximal in $(\mathscr{B}(X)\backslash\{\nabla_X\}, \subseteq)$ if and only if α' is minimal in $(\mathscr{B}(X)\backslash\{0\}, \subseteq)$.*

(3) *For $\alpha \in \mathscr{B}(X)\backslash\{\nabla_X\}$, α is maximal in $(\mathscr{B}(X)\backslash\{\nabla_X\}, \subseteq)$ if and only if α^{-1} is maximal in $(\mathscr{B}(X)\backslash\{\nabla_X\}, \subseteq)$.*

(4) *For $\alpha \in \mathscr{B}(X)\backslash\{0\}$, α is minimal in $(\mathscr{B}(X)\backslash\{0\}, \subseteq)$ if and only if α^{-1} is minimal in $(\mathscr{B}(X)\backslash\{0\}, \subseteq)$.*

To obtain the theorem concerning the maximal elements of $\mathscr{B}(X)$ with respect to $\leq$, the following series of lemmas will be used.

Lemma 2.6. *Let* $\alpha, \beta \in \mathscr{B}(X)$. *If* $\beta \in R^\alpha \cap C^\alpha$, *then the following statements hold.*

(1) *There exist* $a, b, c, d \in X$ *with* $a \neq b$ *and* $c \neq d$ *such that*

$$\{d\} \subseteq a\alpha \subseteq b\alpha,\ \{b\} \subseteq \alpha c \subseteq \alpha d \text{ and } \beta = \alpha\backslash\{(b,d)\}.$$

(2) $\alpha X = \beta X$ *and* $X\alpha = X\beta$.

Proof. Assume that $\beta \in R^\alpha \cap C^\alpha$. Since $\beta \in R^\alpha$, there exist $a, b \in X$ with $a \neq b$ and a nonempty subset A of X such that

$$A \subseteq a\alpha \subseteq b\alpha,\ b\beta = b\alpha\backslash\ A \text{ and } x\alpha = x\beta \text{ for all } x \in X\backslash\{b\}.$$

Claim that $\beta = \alpha\backslash(\{b\} \times A)$. Let $(x, y) \in \beta$. Then $y \in x\beta$.

Case 1: $x = b$. Then $y \in b\beta$. Since $b\beta = b\alpha\backslash A$, we have $y \in b\alpha$ and $y \notin A$. So $(x, y) \in \alpha$, hence $(x, y) \in \alpha\backslash(\{b\} \times A)$.

Case 2: $x \neq b$. Since $x\beta = x\alpha$ for all $x \neq b$, $y \in x\alpha$. Thus $(x, y) \in \alpha\backslash(\{b\} \times A)$.

Hence $\beta \subseteq \alpha\backslash(\{b\} \times A)$.

For the reverse inclusion, let $(x, y) \in \alpha\backslash(\{b\} \times A)$. Then $y \in x\alpha$ and $(x, y) \notin \{b\} \times A$.

Case 1: $x = b$. Then $y \in b\alpha$ and $y \notin A$, so $y \in b\alpha\backslash A$. Since $b\alpha\backslash A = b\beta$, $y \in b\beta$. Hence $(x, y) \in \beta$.

Case 2: $x \neq b$. Then $y \in x\alpha = x\beta$, so $(x, y) \in \beta$.

Therefore $\alpha\backslash(\{b\} \times A) \subseteq \beta$. So we have the claim.

Since $\beta \in C^\alpha$, there are $c, d \in X$ with $c \neq d$ and a nonempty subset B of X such that

$$B \subseteq \alpha c \subseteq \alpha d,\ \beta d = \alpha d\backslash B \text{ and } \alpha x = \beta x \text{ for all } x \in X\backslash\{d\}.$$

We can prove similarly that $\beta = \alpha\backslash(B \times \{d\})$.

Since $A \subseteq b\alpha$ and $B \subseteq \alpha d$, $\{b\} \times A \subseteq \alpha$ and $B \times \{d\} \subseteq \alpha$, respectively. But $\beta \subseteq \alpha\backslash(\{b\} \times A)$ and $\beta = \alpha\backslash(B \times \{d\})$, so we have $B = \{b\}$ and $A = \{d\}$. Therefore $\beta = \alpha\backslash\{(b, d)\}$. But $\{b\} \subseteq \alpha d$, so we have $\alpha = \beta \cup \{(b, d)\}$. Since $\{b\} = B \subseteq \alpha c \subseteq \alpha d$, $\{(b, c), (b, d)\} \subseteq \alpha$. Then $\{(b, c)\} \subseteq \alpha\backslash\{(b, d)\} = \beta$ and so $b \in \beta X$. Thus

$$\alpha X = (\beta \cup \{(b, d)\})X = \beta X \cup \{b\} = \beta X.$$

Similarly, we can show that $X\alpha = X\beta$, as required. □

Lemma 2.7. *Let $\alpha \in \mathscr{B}(X)$. If α is a maximal element with respect to $\leq$, then $R^\alpha \cap C^\alpha = \emptyset$.*

Proof. Assume that $R^\alpha \cap C^\alpha \neq \emptyset$. Let $\beta \in R^\alpha \cap C^\alpha$. From the proof of Lemma 2.6, there exist $a, b, c, d \in X$ with $a \neq b$ and $c \neq d$ such that $\{d\} \subseteq a\alpha \subseteq b\alpha$, $\{b\} \subseteq \alpha c \subseteq \alpha d$, $b\beta = b\alpha \backslash \{d\}$, $\beta d = \alpha d \backslash \{b\}$, $x\beta = x\alpha$ for all $x \neq b$, $\beta y = \alpha y$ for all $y \neq d$, $\beta X = \alpha X$ and $X\beta = X\alpha$. Then

$$\{a, b\}\beta = a\beta \cup b\beta = a\alpha \cup b\alpha \backslash \{d\} = b\alpha$$

and

$$\beta\{c, d\} = \beta c \cup \beta d = \alpha c \cup \alpha d \backslash \{b\} = \alpha d.$$

Define $\lambda, \mu \in \mathscr{B}(X)$ by

$$u\lambda = \begin{cases} \{a, b\} & \text{if } u = b, \\ u & \text{if } u \in \alpha X \backslash \{b\} \end{cases}$$

and

$$\mu v = \begin{cases} \{c, d\} & \text{if } v = d, \\ v & \text{if } v \in X\alpha \backslash \{d\}. \end{cases}$$

Then

$$\beta\mu v = \beta v = \alpha v \quad \text{and} \quad \alpha\mu v = \alpha v \quad \text{for all } v \in X\alpha \backslash d$$

and

$$\beta\mu d = \beta\{c, d\} = \alpha d \quad \text{and} \quad \alpha\mu d = \alpha\{c, d\} = \alpha d.$$

We conclude that $\alpha = \beta\mu = \alpha\mu$. Also, we have that

$$u\lambda\beta = u\beta = u\alpha \text{ for all } u \in \alpha X \backslash b \quad \text{and} \quad b\lambda\beta = \{a, b\}\beta = b\alpha.$$

Thus $\alpha = \lambda\beta$. This proves that $\alpha \leq \beta$. From Lemma 2.6 (1), we have $\alpha \neq \beta$. Hence α is not maximal. □

Lemma 2.8. *Let $\alpha \in \mathscr{B}(X)$. If α is maximal with respect to $\leq$, then either α is row reduced and $\alpha X = X$ or α is column reduced and $X\alpha = X$.*

Proof. Assume that the converse condition is not true.

Case 1: $\alpha X \subsetneq X$ and $X\alpha \subsetneq X$. Then there exist $a, b \in X$ such that $a \notin \alpha X$ and $b \notin X\alpha$. Define $\beta, \lambda, \mu \in \mathscr{B}(X)$ by $\beta = \alpha \cup \{(a, b)\}$, $\lambda = \Delta_{\alpha X}$ and $\mu = \Delta_{X\alpha}$. Clearly that $\alpha = \alpha\mu$ and $\alpha \neq \beta$. Since $\beta X = \alpha X \cup \{a\}$ and $X\beta = X\alpha \cup \{b\}$, $\alpha = \Delta_{\alpha X}\beta = \lambda\beta$ and $\alpha = \beta\Delta_{X\alpha} = \beta\mu$. Hence $\alpha = \lambda\beta = \beta\mu$ and $\alpha = \alpha\mu$, we deduce that $\alpha < \beta$. Hence α is not maximal.

Case 2: $\alpha X \subsetneq X$ and α is not column reduced. Then there exist $a, b \in X$ and a nonempty subset B of X such that $a \notin \alpha X$, $\emptyset \neq \alpha b = \alpha B$ and $b \notin B$. Define

$\beta \in \mathscr{B}(X)$ by $\beta = \alpha \cup \{(a,b)\}$. Then $\beta X = \alpha X \cup \{a\}$, $X\beta = X\alpha$, $x\beta = x\alpha$ for all $x \neq a$ and $\beta y = \alpha y$ for all $y \neq b$. Thus $\beta B = \alpha B = \alpha b$.

Define $\lambda, \mu \in \mathscr{B}(X)$ by $\lambda = \Delta_{\alpha X}$ and

$$\mu y = \begin{cases} B & \text{if } \ y = b, \\ y & \text{if } \ y \in X\alpha \backslash \{b\}. \end{cases}$$

Then we have

$$\alpha = \Delta_{\alpha X}\beta = \lambda\beta,$$

$$\beta\mu y = \beta y = \alpha y = \alpha\mu y \quad \text{for all } y \in X\alpha\backslash\{b\}$$

and

$$\beta\mu b = \beta B = \alpha b = \alpha B = \alpha\mu b.$$

Thus $\alpha = \lambda\beta = \beta\mu$ and $\alpha = \alpha\mu$. Hence $\alpha < \beta$, so α is not maximal.

Case 3: α is not row reduced and $X\alpha \subsetneq X$. Its proof is similar to the Case 2.

Case 4: α is neither row nor column reduced. Then there are $a, b \in X$ and nonempty subsets A, B of X such that $a\alpha = A\alpha, \alpha b = \alpha B, a \notin A$ and $b \notin B$. Define $\beta \in \mathscr{B}(X)$ by

$$\beta = \begin{cases} \alpha \cup \{(a,b)\} & \text{if } (a,b) \notin \alpha, \\ \alpha\backslash\{(a,b)\} & \text{if } (a,b) \in \alpha. \end{cases}$$

Then $\beta X = \alpha X$, $X\beta = X\alpha$ and $\alpha \neq \beta$. By the definition of β, we get that $x\beta = x\alpha$ for all $x \neq a$ and $\beta y = \alpha y$ for all $y \neq b$. Define $\lambda, \mu \in \mathscr{B}(X)$ by

$$x\lambda = \begin{cases} A & \text{if } x = a, \\ x & \text{if } x \in \alpha X\backslash\{a\} \end{cases}$$

and

$$\mu y = \begin{cases} B & \text{if } y = b, \\ y & \text{if } y \in X\alpha\backslash\{b\}. \end{cases}$$

If $x \in \alpha X\backslash\{a\}$, then $x\lambda\beta = x\beta = x\alpha$. And $a\lambda\beta = A\beta = A\alpha = a\alpha$. Thus $\alpha = \lambda\beta$. If $y \in X\alpha\backslash\{b\}$, then $\beta\mu y = \beta y = \alpha y = \alpha\mu y$. Since $\beta\mu b = \beta B = \alpha B = \alpha b$ and $\alpha\mu b = \alpha B = \alpha b$, we deduce that $\alpha = \beta\mu = \alpha\mu$. These imply that $\alpha < \beta$, hence α is not maximal.

Therefore the lemma is proved. □

Lemma 2.9. *Let $\alpha \in \mathscr{B}(X)$. If $R^\alpha \cap C^\alpha = \emptyset$ and either α is row reduced and $\alpha X = X$ or α is column reduced and $X\alpha = X$, then α is maximal with respect to $\leq$.*

Proof. Let $\beta \in \mathscr{B}(X)$ be such that $\alpha \leq \beta$. To prove that $\beta \subseteq \alpha$, let $x \in X$. First, assume that α is row reduced and $\alpha X = X$. Since $\alpha X = X$, $x\alpha \neq \emptyset$. By Theorem 2.3, we have $x\alpha \in V(\alpha) \subseteq V(\beta)$. Then $x\alpha = A\beta$ for some $A \subseteq X$. Again by Theorem 2.3, $x\alpha = A\alpha$. Since α is row reduced, we have $x \in A$, and so $x\beta \subseteq A\beta = x\alpha$. Hence $\beta \subseteq \alpha$. If α is column reduced and $X\alpha = X$, we can show similarly that $\beta \subseteq \alpha$.

Next, suppose that $\beta \neq \alpha$. Since $\beta \subseteq \alpha$, there exists $a \in X$ such that $a\beta \subsetneq a\alpha$. Set $A = a\alpha \backslash a\beta$. Since $a\alpha \in V(\alpha)$, by Theorem 2.3, $a\alpha = B\beta$ for some $B \subseteq X$, hence $a\alpha = B\alpha$. Let $c \in A$. Then $c \in a\alpha$, so $c \in b\beta$ for some $b \in B$ since $a\alpha = B\beta$. By $\beta \subseteq \alpha$, we have

$$c \in b\beta \subseteq b\alpha \subseteq B\alpha = a\alpha.$$

Since $\{a, b\} \subseteq \alpha c$ and $c \notin a\beta$, $\beta c \subsetneq \alpha c$. Let $C = \alpha c \backslash \beta c$. Then $\alpha c = \beta D$ for some $D \subseteq X$ since $\alpha c \in W(\alpha) \subseteq W(\beta)$. So $\alpha c = \alpha D$ by Theorem 2.3. Since $a \notin \beta c$, $a \in C$. And since $a \in \alpha c = \beta D$, $a \in \beta d$ for some $d \in D$. Hence

$$a \in \beta d \subseteq \alpha d \subseteq \alpha D = \alpha c.$$

Define $\rho = \alpha \backslash \{(a, d)\}$. From the above proof, we have that $\rho \in R^{\alpha} \cap C^{\alpha}$ which is a contradiction. Hence $\beta = \alpha$. □

Theorem 2.10. *Let $\alpha \in \mathscr{B}(X)$. Then α is maximal with respect to $\leq$ if and only if α satisfies the following two conditions.*

(1) *α is row reduced and $\alpha X = X$ or α is column reduced and $X\alpha = X$.*

(2) *$R^{\alpha} \cap C^{\alpha} = \emptyset$.*

Proof. It follows directly from Lemma 2.7, Lemma 2.8 and Lemma 2.9. □

The following theorem determines the minimal elements of $\mathscr{B}(X)$ with respect to $\leq$.

Theorem 2.11. *Let $\alpha \in \mathscr{B}(X) \backslash \{0\}$. Then α is minimal in $\mathscr{B}(X) \backslash \{0\}$ with respect to $\leq$ if and only if $V(\alpha) = \{\emptyset, X\alpha\}$.*

Proof. Assume that $V(\alpha) = \{\emptyset, X\alpha\}$. Let $\beta \in \mathscr{B}(X) \backslash \{0\}$ be such that $\beta \leq \alpha$. By Theorem 2.3(3), $V(\beta) \subseteq V(\alpha)$. Since $\beta \neq 0$, $V(\beta) = V(\alpha)$. Hence $\alpha = \beta$ by Lemma 1.6(4) and Lemma 1.1.

Conversely, suppose that $|V(\alpha)| > 2$. Define $\beta \in \mathscr{B}(X)$ by $\beta = \alpha X \times X\alpha$. Then $0 \neq \beta$ and $V(\beta) = \{\emptyset, X\alpha\}$, so $\alpha \neq \beta$. Define $\lambda, \mu \in \mathscr{B}(X)$ by

$$\lambda = \alpha X \times \alpha X \text{ and } \mu = X\alpha \times X\alpha.$$

Then for each $x \in \alpha X$, $x\lambda\alpha = (\alpha X)\alpha = X\alpha = x\beta$, so $\lambda\alpha = \beta$, and for each $x \in X\alpha$, $\alpha\mu x = \alpha(X\alpha) = \alpha X = \beta x$ and $\beta\mu x = \beta(X\alpha) = \alpha X = \beta x$. Hence $\beta = \alpha\mu = \beta\mu$.

Hence $\beta < \alpha$, so α is not minimal. □

By Lemma 1.2, we have the following corollary.

Corollary 2.12. *Let $\alpha \in \mathscr{B}(X)\setminus\{0\}$. Then α is minimal in $\mathscr{B}(X)\setminus\{0\}$ with respect to $\leq$ if and only if $W(\alpha) = \{\emptyset, \alpha X\}$.*

We know that if $\alpha \in \mathscr{B}(X)$ is row [column] minimal, then α is row [column] reduced and R^α $[C^\alpha] = \emptyset$. Thus from Theorem 2.10, we have the following corollary.

Corollary 2.13. *Let $\alpha \in \mathscr{B}(X)$.*
(1) *If α is row minimal and $\alpha X = X$, then α is maximal with respect to $\leq$.*
(2) *If α is column minimal and $X\alpha = X$, then α is maximal with respect to $\leq$.*

The next corollary is obtained directly from Theorem 2.3.

Corollary 2.14. *The following statements about $\mathscr{B}(X)$ hold.*
(1) *For $\alpha \in \mathscr{B}(X)$, α is maximal with respect to $\leq$ if and only if α^{-1} is maximal with respect to $\leq$.*
(2) *For $\alpha \in \mathscr{B}(X)\setminus\{0\}$, α is minimal in $\mathscr{B}(X)\setminus\{0\}$ with respect to $\leq$ if and only if α^{-1} is minimal in $\mathscr{B}(X)\setminus\{0\}$ with respect to $\leq$.*

2.3 Left and right compatible elements

By Lemma 1.5(3), we have that every element of $\mathscr{B}(X)$ is both left and right compatible with respect to $\subseteq$.

In the following two proposition, we provide necessary and sufficient conditions for elements in $\mathscr{B}(X)$ to be left compatible and right compatible with respect to $\leq$.

Proposition 2.15. *Let $\alpha \in \mathscr{B}(X)$. Then α is left compatible with respect to $\leq$ on $\mathscr{B}(X)$ if and only if $V(\alpha) = \mathscr{P}(X)$ where $\mathscr{P}(X)$ is the power set of X.*

Proof. Assume that $V(\alpha) = \mathscr{P}(X)$. Since $V(\Delta_X) = \mathscr{P}(X)$, by Lemma 1.6 (4), $\gamma\alpha = \Delta_X$ for some $\gamma \in \mathscr{B}(X)$. Let $\sigma, \beta \in \mathscr{B}(X)$ be such that $\sigma \leq \beta$. Then $\sigma = \lambda\beta = \beta\mu$ and $\sigma = \sigma\mu$ for some $\lambda, \mu \in \mathscr{B}(X)$. Thus $\alpha\sigma = (\alpha\beta)\mu, \alpha\sigma = (\alpha\sigma)\mu$ and $\alpha\sigma = \alpha\lambda\beta = \alpha\lambda\Delta_X\beta = \alpha\lambda\gamma(\alpha\beta)$ which imply that $\alpha\sigma \leq \alpha\beta$. Hence α is left compatible.

On the other hand, suppose that α is left compatible. To show that $V(\alpha) = \mathscr{P}(X)$, it suffices to prove that $\{a\} \in V(\alpha)$ for all $a \in X$. Let $a \in X$. Define $\sigma \in \mathscr{B}(X)$ by $\sigma = X \times \{a\}$. By Lemma 1.8, σ is an idempotent of $\mathscr{B}(X)$, so $\sigma \leq \Delta_X$. By assumption, $\alpha\sigma \leq \alpha\Delta_X = \alpha$. Then $\alpha\sigma = \lambda\alpha = \alpha\mu$ and $\alpha\sigma = \alpha\sigma\mu$ for some $\lambda, \mu \in \mathscr{B}(X)$. Since $\alpha\sigma \neq 0$ and $V(\alpha\sigma) \subseteq V(\sigma) = \{\emptyset, \{a\}\}$, $V(\alpha\sigma) = \{\emptyset, \{a\}\}$. But $V(\alpha\sigma) = V(\lambda\alpha) \subseteq V(\alpha)$, thus $\{a\} \in V(\alpha)$. □

Proposition 2.16. *Let $\alpha \in \mathscr{B}(X)$. Then α is right compatible with respect to $\leq$ on $\mathscr{B}(X)$ if and only if $W(\alpha) = \mathscr{P}(X)$.*

Proof. The proof can be given similarly to that of Proposition 2.15. □

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