Thai Journal of Mathematics
Volume 15 (2017) Number 1 : 91–105
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

On a Class of Nonlocal $p(x)$-Laplacian Neumann Problems

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Abstract : This paper is aiming at obtaining weak solutions for a Kirchhoff problem involving $p(x)-$Laplacian operator. The notion of Sobolev space with a variable exponent with combined conditions of nonlinearities lead us to generalize the case of $p-$Kirchhoff problem where p is a constant.

Keywords : $p(x)$-Kirchhoff type; Mountain Pass theorem; variable exponent; Sobolev spaces.
2010 Mathematics Subject Classification : 34B27; 35J60; 58E30.

1 Introduction

In these latest years, the study of the Kirchhoff equation and of Kirchhoff systems has been considered in the case involving the $p-$ Laplacian operator (see [1, 2, 3]). Recently, Autuori, Pucci and Salvatori [1] have investigated the Kirchhoff type equation involving the $p(x)-$Laplacian of the form

$$u_{tt} - M(\int_\Omega \frac{1}{p(x)} \mid \nabla u \mid^{p(x)} dx)\Delta_{p(x)}u + Q(t, x, u, u_t) + f(x, u) = 0.$$

They have introduced the asymptotic stability, as time tends to infinity. Now, the study of the stationary version of Kirchhoff type problems has also been the object

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of considerable study; see e.g. [4, 5, 6]. The operator $\Delta_{p(x)}u = div(\mid \nabla u \mid^{p(x)-2} \nabla u)$ is called $p(x)-$Laplacian. Here we point out that the $p(x)-$Laplacian operator possesses more complicated nonlinearities than $p-$Laplacian, for example, it is inhomogeneous and usually it does not have the so- called first eigenvalue, since the infimum of its principle eigenvalue is zero.

The problem studied in the present work involves a nonconstant exponent and thereby the Lebesgue and Sobolev spaces with variable exponents are suitable contexts in which problem (1.1) below can be studied. Many authors have studied $p(x)-$Kirchhoff type equations with Dirichlet boundary value problems e.g. we refer to Fan [7] and Chung [8].

The $p(x)-$Kirchhoff type equations with Neumann boundary value problems have been studied by several authors. One of the first result in this direction was obtained by Dai and Ma in [9], wherein they prove several properties of the $p(x)-$Kirchhoff-Laplace operator. Guo and Zhao [10] presented several sufficient conditions for the existence of solutions for a general problem by using Ambrosetti-Rabinowitz condition. In [11], Chung studied a class of nonlocal $p(x)-$Laplacian Neumann problem by assuming that $a:\ \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function and verifies

$$a(t) \geq a_0|t|^{\alpha-1},\ for\ t \geq 0,$$

where $a_0 > 0$ and $\alpha > 1$. Using Ekeland variational principle, Yucedag et al in [12] showed the existence of a weak solution for a $p(x)-$Kirchhoff system.

In this paper, we are concerned with the elliptic problems:

$$\begin{aligned} a(I_1(u))(-\Delta_{p(x)}u + \mu \mid u \mid^{p(x)-2} u) = f(x,u) \quad in \quad \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, ν is the outward normal vector on $\partial\Omega$, $1 < p^- = \inf_{x\in\Omega} p(x) \leq p(x) \leq \sup_{x\in\Omega} p(x) = p^+ < +\infty, p \in C(\overline{\Omega}),\quad \mu \geq 0$, $I_1(u) = \int_\Omega \frac{1}{p(x)}(\mid \nabla u \mid^{p(x)} + \mu \mid u \mid^{p(x)})dx$, $a(t)$ is a real valued, continuous function and $f : \Omega \times \mathbb{R} \to \mathbb{R}$, is a Carathéodory function with $F(x,t) = \int_0^t f(x,s)ds$.

(a_1) $a :]0,+\infty[\to]0,+\infty[$ continuous and $a_0 = \inf\limits_{s>0} a(s) > 0$.

(f_1) There exist two positives constants C_1, C_2 such that

$$\mid f(x,t) \mid \leq C_1 + C_2 \mid t \mid^{q(x)-1},\quad (x,t) \in \Omega \times \mathbb{R}, q \in C(\Omega), 1 < q(x) < p^*(x),$$

with

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & if\ \ p(x) < N, \\ +\infty & if\ \ p(x) \geq N. \end{cases}$$

(a_2) There is a positive constant α such that $\limsup_{t\to 0^+} \frac{\widehat{a}(t)}{t^\alpha} < +\infty$, with $\widehat{a}(t) = \int_0^t a(s)ds$.

(f_2) $\liminf_{t\to 0} \frac{F(x,t)}{|t|^{r_1}} > 0$ uniformly for a.e $x \in \Omega$, where r_1 is a positive constant such that $r_1 < \alpha p^-$.

(f_3) $\lim_{|t|\to+\infty}[F(x,t) - a_0\lambda_1 \frac{|t|^{p^-}}{p(x)}] = -\infty$, uniformly for almost every $x \in \Omega$, where

$$\lambda_1 = \inf_{W^{1,p(x)}(\Omega)\backslash\{0\}} \frac{\int_\Omega \frac{1}{p(x)}(| \nabla u |^{p(x)} + | u |^{p(x)})dx}{\int_\Omega \frac{|u|^{p(x)}}{p(x)}dx} > 0,$$

which is different from the first eigenvalue of the $p-$Laplace.

Hereinafter, we report the main results.

Theorem 1.1. *Assume that the conditions* $(f_1), (f_2), (f_3)$ *and* $(a_1), (a_2)$ *with* $\mu = 1$ *hold. Then (1.1) has a weak solution.*

Theorem 1.2. *Whether* $\mu = 0$. *Suppose* $(f_1), (a_1)$ *and the following assumptions:*

(a_3) $p^+\widehat{a}(t) \leq p^- a(t)t$ *for* $t > 0$, *with* $\widehat{a}(t) = \int_0^t a(s)ds$.

(f_4) $\lim_{|t|\to+\infty}[f(x,t)t - p^+F(x,t)] = -\infty$.

(f_5) $\lim_{|t|\to+\infty} \frac{F(x,t)}{|t|^{p^-}} = 0$.

(f_6) $f(x,t)t > 0$, *for all* $t \neq 0$.

Then (1.1) *has at least one weak solution.*

Theorem 1.3. *Suppose* (a_1) *and* (f_1) *with* $p^+ < q^-, \mu = 1$ *hold and* $a(t)$ *is bounded for* $t > 0$. *Assume the following hypotheses,*

(f_7) *The following limit hold uniformly for a.e* $x \in \Omega$

$$\lim_{|t|\to\infty} \frac{f(x,t)t}{| t |^{p^+}} = +\infty.$$

(f_8) $f(x,t) = o(t^{p(x)-1})$ *as* $t \to 0$ *uniformly for* x *in* Ω.

(f_9) *There exist two positive constants* c_1 *and* c_2 *such that*

$$\psi_1(x,t) \leq c_1\psi_1(x,s) \leq c_2\psi_2(x,s), \text{ for all } 0 \leq t \leq s.$$

Where,

$$\psi_1(x,t) = f(x,t)t - p^-F(x,t),$$
$$\psi_2(x,t) = f(x,t)t - p^+F(x,t).$$

Then, (1.1) admits at least one solution in $W^{1,p(x)}(\Omega)$.

Remark 1.4. *An example of functions satisfying the assumption of Theorem 1.3* $f(x,t) =| t |^{\alpha(x)-2} t$, *where* $p^+ < \alpha^- \leq \alpha(x) \leq \alpha^+ < p^*(x)$, *then*

$$F(x,t) = \frac{| t |^{\alpha(x)}}{\alpha(x)}, \; f(x,t)t =| t |^{\alpha(x)},$$

so we obtain

$$\psi_1(x,t) = (1 - \frac{p^-}{\alpha(x)}) | t |^{\alpha(x)}$$

and

$$\psi_2(x,t) = (1 - \frac{p^+}{\alpha(x)}) | t |^{\alpha(x)},$$

what means that (f_9) *is satisfied since we have* $\psi_1(x,t)$ *is nondecreasing in* $t \geq 0$ *and then* $\psi_1(x,t) \leq \psi_1(x,s)$ *when* $0 \leq t \leq s$*, so we take* $c_1 = 1$*. Accordingly, the fact that* $\psi_1, \psi_2 \geq 0$ *it follows that,*

$$\frac{\psi_1(x,t)}{\psi_2(x,t)} \leq \frac{\alpha^+ - p^-}{\alpha^- - p^+} = c_2.$$

Obviously the other assumptions are held.

For the assumptions of Theorem 1.2, we can see that $f(x,t) = arctan(t) + \frac{t}{1+t^2}$ *clearly verifies the hypotheses.*

The purpose of this work is to improve the results of the above-mentioned papers and many others. Without assuming the Ambrosetti-Rabinowitz type conditions (A-R), we prove the existence of solutions.

(A-R) there exist $\theta > p^+, M > 0$ such that for any $x \in \Omega$ and $t \geq M$ we have

$$0 \leq \theta F(x,t) \leq f(x,t)t.$$

For instance, it is known that (f_9) is much weaker than the (A-R) condition in the constant exponent case. In addition, the classical conditions of the coercivity of the energy functional ϕ associated to problem (1.1) are omitted here, so we extend them to the best.

This article is organized as follows. In Section 2, we give the necessary notations, we also include some useful results involving the variable exponent Lebesgue and Sobolev spaces in order to facilitate the reading of the paper. Finally, in Section 3, we prove the existence of nontrivial solution.

2 Preliminaries

We introduce the setting of our problem with some auxiliary results. For convenience, we only recall some basic facts which will be used later, we refer to [13] for more details. Set $C_+(\Omega) = \{h : \ h \in C(\overline{\Omega}), h(x) > 1 for \ \ all \ x \in \overline{\Omega}\}$. Define the variable exponent Lebesgue space $L^{p(x)}(\Omega)$,

$L^{p(x)}(\Omega) = \{u : \Omega \to \mathbb{R} \ mesurable \ : \ \int_\Omega | \ u \ |^{p(x)} \ dx < \infty\}$ then $L^{p(x)}(\Omega)$ endowed with the norm

$$| \ u \ |_{p(x)} = \inf\{\lambda > 0 \ : \ \int_\Omega | \ \frac{u}{\lambda} \ |^{p(x)} \ dx \leq 1\}$$

becomes a Banach space separable and reflexive space.

Define the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : \nabla u \in L^{p(x)}(\Omega)\}.$$

The space $W^{1,p(x)}(\Omega)$ with the norm $\| \ u \ \| = | \ u \ |_{p(x)} + | \ \nabla u \ |_{p(x)}$ is a Banach separable and reflexive space.

Let $X = W^{1,p(x)}(\Omega)$,

Definition 2.1. We say that $u \in W^{1,p(x)}(\Omega)$ is a *weak solution of* (1.1), if

$$a(I_1(u))\int_\Omega (\mid \nabla u \mid^{p(x)-2} \nabla u.\nabla v \quad + \quad \mu \mid u \mid^{p(x)-2} uv)dx - \int_\Omega f(x,u)vdx = 0,$$

$\forall v \in W^{1,p(x)}(\Omega)$.

Proposition 2.2. *Set,* $\rho(u) = \int_\Omega \mid u \mid^{p(x)} dx$, *if* $u \in L^{1,p(x)}(\Omega)$ *we have*

(1) $\| u \| \geq 1 \Rightarrow \mid u \mid_{p(x)}^{p^-} \leq \rho(u) \leq \mid u \mid_{p(x)}^{p^+}$.

(2) $\mid u \mid \leq 1 \Rightarrow \mid u \mid_{p(x)}{}^{p^+} \leq \rho(u) \leq \mid u \mid_{p(x)}{}^{p^-}$.

(3) For $u_n, u \in L^{p(x)}(\Omega)$,

$$\mid u_n \mid_{p(x)} \to 0 \Leftrightarrow \rho(u_n) \to 0 \ as \ n \to +\infty.$$

$$\mid u_n \mid_{p(x)} \to +\infty \Leftrightarrow \rho(u_n) \to +\infty \ as \ n \to +\infty.$$

Proposition 2.3. *For any* $u \in L^{p(x)}(\Omega)$, $v \in L^{p'(x)}(\Omega)$, *we have*

$$\mid \int_\Omega uv \ dx \mid \leq 2 \mid u \mid_{p(x)} \mid v \mid_{p'(x)},$$

with

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1.$$

Proposition 2.4. *Set,* $\varrho(u) = \int_\Omega \mid \nabla u \mid^{p(x)} + \mid u \mid^{p(x)} dx$, *if* $u \in W^{1,p(x)}(\Omega)$ *we have*

(1) $\| u \| \geq 1 \Rightarrow \| u \|^{p^-} \leq \varrho(u) \leq \| u \|^{p^+}$.

(2) $\| u \| \leq 1 \Rightarrow \| u \|^{p^+} \leq \varrho(u) \leq \| u \|^{p^-}$.

(3) For $u_n, u \in W^{1,p(x)}(\Omega)$,

$$\| u_n \| \to 0 \Leftrightarrow \varrho(u_n) \to 0 \ as \ n \to +\infty.$$

$$\| u_n \| \to +\infty \Leftrightarrow \varrho(u_n) \to +\infty \ as \ n \to +\infty.$$

Proposition 2.5. *If* $q \in C_+(\overline{\Omega})$ *and* $q(x) < p^*(x)$ *for any* $x \in \overline{\Omega}$, *then the imbedding from* $W^{1,p(x)}(\Omega)$ *to* $L^{q(x)}(\Omega)$ *is compact and continuous.*

Definition 2.6. We say that $\phi \in (X, \mathbb{R})$ satisfies the *Cerami condition (denoted by (C))* if any sequence $(u_n)_n \subset X$ for which $\phi(u_n)$ is bounded and $(1+ \| u_n \|)\phi'(u_n) \to 0$ as $n \to \infty$, possesses a convergent subsequence.

Remark 2.7. *In Theorems 1.1 and 1.3, we can suppose that the parameter* $\mu > 0$, *whith*

$$\lambda_1 = \inf_{W^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\int_\Omega \frac{1}{p(x)}(\mid \nabla u \mid^{p(x)} + \mu \mid u \mid^{p(x)})dx}{\int_\Omega \frac{|u|^{p(x)}}{p(x)} dx} > 0,$$

since $W^{1,p(x)}(\Omega)$ *with the norm*

$$\inf\{\nu > 0 \ : \ \int_\Omega (|\frac{\nabla u}{\nu}|^{p(x)} + \mu|\frac{u}{\nu}|^{p(x)})dx \leq 1\}$$

is a separable and reflexive Banach space. For the sake of simplicity, we take $\mu = 1$.

3 Proof of the Main Results

By means of a direct variational approach, we establish the existence of critical point of ϕ, where

$$\phi(u) = \widehat{a}(I_1(u)) - \int_\Omega F(x,u)dx$$

and

$$I_1(u) = \int_\Omega \frac{1}{p(x)}(\mid \nabla u \mid^{p(x)} + \mu \mid u \mid^{p(x)})dx.$$

So we need some results in form of some lemmas. Denote by $C, C_i, i = 1...$ positive constants which the exact value may change from line to line.

Lemma 3.1. *i)* $\widehat{a} \in C^1([0,+\infty[), \widehat{a}(0) = 0, \widehat{a}'(t) = a(t)$ *for any* $t > 0$.

ii) $\widehat{a}(I_1(u))$ *is sequentially weakly lower semi-continuous and* ϕ' *is bounded and* S_+ *type.*

The proof of lemma can be obtained easily in view of [7].

Proof of Theorem 1.1: For simplicity, we set $H(x,t) = F(x,t) - a_0 \frac{\lambda_1}{p(x)} \mid t \mid^{p^-}$. Then, according to (f_3) we can conclude that, for every $M > 0$, there is $R_M > 0$ such that

$$H(x,t) \leq -M, \quad \forall \mid t \mid \geq R_M, \quad \text{almost every } x \in \Omega. \tag{3.1}$$

We claim that ϕ is coercive, otherwise, there exist $K \in \mathbb{R}$ and $(u)_n \subset X$ such that

$$\| u_n \| \to \infty \quad and \quad \phi(u_n) \leq K.$$

Putting $v_n = \frac{u_n}{\|u_n\|}$ i.e $\| v_n \| = 1$. Then for a subsequence, we may assume that for $v \in X$, we have $v_n \rightharpoonup v$ in X, $v_n \to v$ strongly in $L^{p(x)}(\Omega)$, $v_n(x) \to v(x)$ for almost every $x \in \Omega$. Now, using (3.1), we obtain

$$\begin{aligned} K \geq \phi(u_n) &\geq \int_\Omega \frac{a_0}{p(x)}(\mid \nabla u_n \mid^{p(x)} + \mid u_n \mid^{p(x)})dx - \int_\Omega F(x,u_n)dx. \\ &\geq \int_\Omega \frac{a_0}{p(x)}[\mid \nabla u_n \mid^{p(x)} + \mid u_n \mid^{p(x)}]dx - \lambda_1 \int_\Omega \frac{a_0}{p(x)} \mid u_n \mid^{p^-} dx \\ &\quad - \int_\Omega H(x,u_n)dx \\ &\geq \frac{a_0}{p^+} \| u_n \|^{p^-} - a_0\lambda_1 \int_\Omega \frac{1}{p(x)} \mid u_n \mid^{p^-} dx + M_1, \end{aligned} \tag{3.2}$$

where $M_1 \in \mathbb{R}$. Dividing (3.2) by $\| u_n \|^{p^-}$ and passing to the limit, we conclude that

$$\frac{a_0}{p^+} - a_0\lambda_1 \int_\Omega \mid v \mid^{p^-} dx \leq 0,$$

hence, $v \not\equiv 0$. Therefore $\mid \Omega^{\prime} \mid > 0$ with $\Omega^{\prime} = \{x \in \Omega \ \setminus \ v(x) \neq 0\}$, then $\mid u_n(x) \mid \to +\infty$ for almost every $x \in \Omega^{\prime}$. On the other hand,

$$\begin{aligned}
\lambda_1 \int_{[|u|\geq 1]} \frac{\mid u \mid^{p^-}}{p(x)} dx &\leq \lambda_1 \int_{[|u|\geq 1]} \frac{\mid u \mid^{p(x)}}{p(x)} dx \\
&\leq \lambda_1 \int_{\Omega} \frac{\mid u \mid^{p(x)}}{p(x)} dx \\
&\leq \int_{\Omega} \frac{1}{p(x)} [\mid \nabla u \mid^{p(x)} + \mid u \mid^{p(x)}] dx, \qquad (3.3)
\end{aligned}$$

where,

$$[\mid u \mid \geq 1] = \{x \in \Omega \quad \setminus \quad \mid u \mid \geq 1\} \ \ ; \ \ [\mid u \mid < 1] = \{x \in \Omega \setminus \mid u \mid < 1\}.$$

It is clear that $\int_{[|u_n|<1]} \frac{1}{p(x)} \mid u_n \mid^{p(x)} dx$ is bounded. From (f_3) and the above inequalities (3.3) we deduce

$$\begin{aligned}
K &\geq \int_{\Omega} \frac{a_0}{p(x)} [\mid \nabla u_n \mid^{p(x)} + \mid u_n \mid^{p(x)}] dx - \int_{\Omega} F(x, u_n) dx \\
&= \int_{\Omega} \frac{a_0}{p(x)} [\mid \nabla u_n \mid^{p(x)} + \mid u_n \mid^{p(x)}] dx - a_0 \lambda_1 \int_{\Omega} \frac{\mid u_n \mid^{p^-}}{p(x)} dx - \int_{\Omega} H(x, u_n) dx \\
&= \int_{\Omega} \frac{a_0}{p(x)} [\mid \nabla u_n \mid^{p(x)} + \mid u_n \mid^{p(x)}] dx - a_0 \lambda_1 \int_{[|u_n|\geq 1]} \frac{\mid u_n \mid^{p^-}}{p(x)} dx \\
&\quad - a_0 \lambda_1 \int_{[|u_n|<1]} \frac{\mid u_n \mid^{p^-}}{p(x)} dx - \int_{\Omega} H(x, u_n) dx \\
&\geq -a_0 \lambda_1 \int_{[|u_n|<1]} \frac{\mid u_n \mid^{p^-}}{p(x)} dx - \int_{\Omega} H(x, u_n) dx \\
&= -a_0 \lambda_1 \int_{[|u_n|<1]} \frac{\mid u_n \mid^{p^-}}{p(x)} dx - \int_{[|u_n|\leq R_1]} H(x, u_n) dx \\
&\quad - \int_{[|u_n|>R_1]} H(x, u_n) dx \to +\infty,
\end{aligned}$$

with R_1 is large enough, which is a contradiction. Hence ϕ is coercive and has a global minimizer. Indeed, for $t > 0$ is small enough,

$$\begin{aligned}
\phi(tv_0) &= \widehat{a}(I_1(tv_0)) - \int_{\Omega} F(x, tv_0) dx \\
&\leq C_1 \Big(\int_{\Omega} \frac{t^{p(x)}}{p(x)} \mid v_0 \mid^{p(x)} dx \Big)^{\alpha} - C_2 \int_{\Omega} t^{r_1} v_0^{r_1} dx \\
&\leq C_3 t^{\alpha p^-} - C_4 t^{r_1} < 0,
\end{aligned}$$

because $r_1 < \alpha p^-$. □

Proof of Theorem 1.2: Here, it assumed that $\mu = 0$, then

$$I_1(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx.$$

Our proof is based on the following result

Theorem 3.2. [14] *Let $X = X_1 \oplus X_2$, where X is a real Banach space and $X_2 \neq \{0\}$, and is finite dimensional. Suppose $\phi \in C^1(X, \mathbb{R})$ satisfies Cerami condition (C) with the following assertions:*

(i) There is a constant a and a bounded neighborhood D of 0 in X_2 such that $J|\partial D \leq \alpha$,

(ii) There is a constant $\beta > \alpha$ such that $J|X_1 \geq \beta$,

then ϕ possesses a critical value $c \geq \beta$, moreover, c can be characterized as

$$c = \inf_{h \in \Gamma} \max_{u \in \overline{D}} \phi(h(u)),$$

where $\Gamma = \{h \in C(D, X) | h = id \quad on \quad \partial D\}$.

We recall an important inequality ([11, 12]), which will be used later:

Lemma 3.3. (Poincaré-Wirtingers inequality) *There exists a positive constant C_0 such that for any $u \in W_0$ we have*

$$|u|_{p(x)} \leq |\nabla u|_{p(x)}.$$

Lemma 3.4. *Assume the conditions $(a_1), (a_3)$ and $(f_1), (f_4)$ and (f_5) are satisfied. Then, ϕ verifies the Cerami condition $(C)_c$.*

Proof. Let $K \in \mathbb{R}$ such that

$$|\phi(u_n)| \leq K$$

and

$$(1+ \| u_n \|)\phi'(u_n) \to 0 \quad in \quad X^*. \tag{3.4}$$

Assume that $\| u_n \| \to +\infty$ as $n \to +\infty$.

Putting $v_n = \frac{u_n}{\|u_n\|}$, so, $v_n \rightharpoonup v$ in X. Thus, $v_n(x) \to v(x) \quad a.e \ x \in \Omega$ and $v_n \to v$ in $L^{p(x)}(\Omega)$.

Let $h \in X$. It follows from (3.4) that,

$$\begin{aligned} &| a(I_1(u_n))(\int_\Omega | \nabla u_n |^{p(x)-2} \nabla u_n . \nabla h dx) - \int_\Omega f(x, u_n) h dx | \\ &\leq \frac{\varepsilon_n \| h \|}{1+ \| u_n \|}. \end{aligned} \tag{3.5}$$

Dividing (3.5) by $\| u_n \|^{p^- -1}$ we get

$$\begin{aligned} &| a(I_1(u_n))(\int_\Omega \frac{[| \nabla u_n |^{p(x)-2} \nabla u_n.\nabla h]}{\| u_n \|^{p^- -1}} dx) - \int_\Omega \frac{f(x,u_n)}{\| u_n \|^{p^- -1}} h dx | \\ &\leq \frac{\varepsilon_n}{\| u_n \|^{p^- -1}} \frac{\| h \|}{1+ \| u_n \|} \leq \frac{\varepsilon_n \| h \|}{1+ \| u_n \|}. \end{aligned} \tag{3.6}$$

On the other side, as $\| u_n \|^{p(x)-1} \geq \| u_n \|^{p^- -1} > 1$, we get

$$\begin{aligned} &| a(I_1(u_n))(\int_\Omega \frac{| \nabla u_n |^{p(x)-2} \nabla u_n.\nabla h}{\| u_n \|^{p^- -1}} dx) - \int_\Omega \frac{f(x,u_n)}{\| u_n \|^{p^- -1}} h dx | \\ &\geq | a(I_1(u_n))(\int_\Omega \frac{| \nabla u_n |^{p(x)-2} \nabla u_n.\nabla h}{\| u_n \|^{p^- -1}} dx) | - | \int_\Omega \frac{f(x,u_n)h}{\| u_n \|^{p^- -1}} dx | \\ &\geq |a(I_1(u_n))(\int_\Omega | \nabla v_n |^{p(x)-2} \nabla v_n.\nabla h dx)| - | \int_\Omega \frac{f(x,u_n)h}{\| u_n \|^{p^- -1}} dx |. \end{aligned}$$

Consequently,

$$| a(I_1(u_n))(\int_\Omega | \nabla v_n |^{p(x)-2} \nabla v_n.\nabla h dx) | - | \frac{\int_\Omega f(x,u_n)}{\| u_n \|^{p^- -1}} h dx | \leq \frac{\varepsilon_n \| h \|}{1+ \| u_n \|}, \tag{3.7}$$

with $\varepsilon_n \to 0$ and $h \in X$.

By $(f_1), (f_4)$ and (f_5) we entail that $\frac{f(x,u_n)}{\|u_n\|^{p^- -1}}$ is bounded in $(L^{p^-}(\Omega))^*$ which is separable and reflexive space, then up to a subsequence denoted also $(\frac{f(x,u_n)}{\|u_n\|^{p^- -1}})_n$, we have $\frac{f(x,u_n)}{\|u_n\|^{p^- -1}} \rightharpoonup \widetilde{f},\ in\ (L^{p^-}(\Omega))^*$. Since $\frac{f(x,u_n)}{\|u_n\|^{p^- -1}} \to 0$ a.e $x \in \Omega$ (which yields from (f_5)), hence

$$\frac{f(x,u_n)}{\| u_n \|^{p^- -1}} \rightharpoonup 0,\ in\ (L^{p^-}(\Omega))^*.$$

Therefore, taking $h = v_n - v \in X$, in (3.7),

$$\int_\Omega | \nabla v_n |^{p(x)-2} \nabla v_n \nabla (v_n - v) dx \to 0.$$

By S_+ type of ϕ' we have $v_n \to v$ in X, so $v \not\equiv 0$. Since $|\phi(u_n)| \leq K$, we obtain

$$p^+ \widehat{a}(I_1(u_n)) - p^+ \int_\Omega F(x,u_n) dx \geq -p^+ K. \tag{3.8}$$

Taking $h = u_n$, in (3.5) we obtain

$$-a(I_1(u_n)) \int_\Omega (| \nabla u_n |^{p(x)} + \int_\Omega f(x,u_n) u_n dx \geq -\varepsilon_n,$$

as $p(x) \geq p^-$ then we have

$$-p^- a(I_1(u_n)) \int_\Omega \frac{1}{p(x)} | \nabla u_n |^{p(x)} dx + \int_\Omega f(x,u_n) u_n dx \geq -\varepsilon_n. \tag{3.9}$$

Adding (3.8) to (3.9), then we get

$$\int_\Omega f(x,u_n)u_n dx - p^+ \int_\Omega F(x,u_n)dx \geq K_1. \tag{3.10}$$

Obviously, this is a contradiction and then we have the compactness condition of the Theorem 3.2 is satisfied. □

Next, we borrow the same idea from Chung [11], we may split $W^{1,p(x)}(\Omega)$ as follows.

Define $W_0 = \{u \in W^{1,p(x)}(\Omega) \ : \ \int_\Omega u dx = 0\}$. For $u \in W^{1,p(x)}(\Omega)$, denote, $\overline{u} = \frac{1}{\Omega}\int_\Omega u dx$ and $\widetilde{u} = u - \overline{u}$. Then, $u = \overline{u} + \widetilde{u}$, where $\widetilde{u} \in \mathbb{R}$, and $\widetilde{u} \in W_0$. So $W^{1,p(x)}(\Omega) = W_0 \oplus \mathbb{R}$. Noting that W_0 is a closed linear subspace of $W^{1,p(x)}(\Omega)$ with codimension 1.

Lemma 3.5. *Under the conditions* (f_4) *and* (f_6) *the functional* $\phi|\mathbb{R}$ *is anti-coercive.(i.e* $\phi(t) \to -\infty$ *when* $|t| \to \infty$*).*

Proof. From (f_4), for all $K > 0$ there exists $R > 0$ such that $p^+F(x,u) \geq f(x,u)u \geq K$, for a.e $x \in \Omega$, $\ |u| > R$, by (f_6) it yields $p^+F(x,u) \geq K - c$, for a.e $x \in \Omega, u \in \mathbb{R}$, and thus for all $u \in \mathbb{R}$,

$$\int_\Omega F(x,u)dx \geq \frac{1}{p^+}K|\Omega| + -c|\Omega|,$$

which implies that

$$\int_\Omega F(x,u)dx \to \infty \ \ when \ \ |u| \to \infty,$$

because K is arbitrary. Hence

$$\phi(u) = -\int_\Omega F(x,u)dx \to -\infty, \ \ when \ \ |u| \to \infty.$$

□

Lemma 3.6. *If* (f_5) *holds, then* $\inf\limits_{W_0} \phi > -\infty$.

Proof. Let $u \in W_0$ with $\| u \| > 1$, using the Proposition 2.1 we get

$$\int_\Omega \frac{\frac{1}{p(x)}|\nabla u|^{p(x)}}{p(x)} dx \geq \frac{1}{p^+}|\nabla u|^{p^i}_{p(x)},$$

in view of the Lemma 3.3, there is $C_5 > 0$ such that

$$\int_\Omega \frac{\frac{1}{p(x)}|\nabla u|^{p(x)}}{p(x)} dx \geq C_5 \parallel u \parallel^{p^i} dx,$$

with $i = \pm$, if $|\nabla u|_{p(x)} \leq 1$ we have $i = +$ and $i = -$ when $|\nabla u|_{p(x)} > 1$. By the continuous embeddings, there is $C_6 > 0$ such that

$$\int_\Omega |u|^{p^-} dx \leq C_6 \parallel u \parallel^{p^-} .$$

By virtue of the hypothesis (f_6), for $0 < \varepsilon < \frac{C_5}{C_6} a_0$, we may find $K(\epsilon) > 0$ such that $F(x,u) \leq \varepsilon |t|^{p^-} + K(\varepsilon)$, for a.e $x \in \Omega$ and for all $u \in \mathbb{R}$. Hence,

$$\begin{aligned}\int_\Omega F(x,u)dx &\leq \varepsilon \int_\Omega |u|^{p^-} + K(\varepsilon)|\Omega| \\ &\leq C_6 \varepsilon \parallel u \parallel^{p^-} + K(\varepsilon)|\Omega|, \end{aligned} \tag{3.11}$$

where $C_6 > 0$. Thus,

$$\begin{aligned}\phi(u)\big|_{u \in W_0} &= \widehat{a}(\int_\Omega \frac{1}{p(x)} |\nabla|^{p(x)} dx) - \int_\Omega F(x,u)dx \\ &\geq a_0 C_5 \parallel u \parallel^{p^i} - \varepsilon C_6 \parallel u \parallel^{p^-} - K(\varepsilon)|\Omega| \\ &\geq -K(\varepsilon)|\Omega|, \end{aligned} \tag{3.12}$$

so we infer that $\inf_{W_0} \phi > -\infty$. □

By the Lemmas 3.6, 3.5 and 3.4, we see that the assumptions of the Theorem 3.2 are hold. Therefore, problem (1.1) has at least a solution in X. □

Proof of Theorem 1.3: Now, we consider the case when the energy functional ϕ possesses the Mountain Pass geometry and compactness condition [15], we check theses assumptions in form as the following lemmas.

Lemma 3.7. *Suppose that $(a_1), (a_3)$ and $(f_1), (f_7) - (f_9)$ hold. If $c \in \mathbb{R}$, then any sequence of Cerami $(C)_c$ of ϕ is bounded.*

Proof. Let $(u_n)_n$ be a $(C)_c$ sequence of ϕ. We claim that $(u_n)_n$ is bounded, otherwise, up to a subsequence we may assume that

$$\phi(u_n) \to c, \ \parallel u_n \parallel \to +\infty, \ \phi'(u_n) \to 0.$$

Putting $\omega_n = \frac{u_n}{\|u_n\|}$, up to a subsequence we have $\omega_n \rightharpoonup \omega$ in X, $\omega_n \to \omega$ in $L^{p(x)}(\Omega)$, $\omega_n(x) \to w(x)$, *a.e.* $x \in \Omega$.

Here, two cases appear, if $\omega \not\equiv 0$. Since $\phi'(u_n)u_n = 0$, that is,

$$a(I_1(u_n))(\int_\Omega \mid \nabla u_n \mid^{p(x)} + \mid u_n \mid^{p(x)} dx) - \int_\Omega f(x,u_n)u_n dx = 0. \tag{3.13}$$

As we know that a is bounded, dividing (3.13) by $\parallel u_n \parallel^{p^+}$, so

$$\int_\Omega \frac{f(x,u_n)u_n}{\parallel u_n \parallel^{p^+}} < \infty,$$

however, using (f_7) and lemma of Fatou we obtain

$$\int_\Omega \frac{f(x,u_n)u_n}{\| u_n \|^{p^+}} = \int_\Omega \frac{f(x,u_n)u_n \mid \omega \mid^{p^+}}{\mid u_n \mid^{p^+}} \to \infty.$$

Which is contradictory.

In the case when $\omega \equiv 0$, we choose a sequence $t_n \in [0,1]$ satisfying $\phi(t_n u_n) = \max_{t_n \in [0,1]} \phi(tu_n)$. If $w \equiv 0$, since $w_n \to 0$ in $L^{q(x)}(\Omega)$ and $|F(x,t)| \leq C(1+|t|^{q(x)})$, by the continuity of the Nemitskii operator, we see that $F(.,w_n) \to 0$ in $L^1(\Omega)$ as $n \to +\infty$, therefore

$$\lim_{n\to\infty} \int_\Omega F(x,w_n)dx = 0. \tag{3.14}$$

Given $m > 0$, since for n large enough we have $\|u_n\|^{-1}(2mp^+)^{\frac{1}{p^-}} \in (0,1)$, using (3.14) with $R = (2mp^+)^{\frac{1}{p^-}}$, it follows that

$$\begin{aligned}
\phi(t_n u_n) &\geq \phi(\frac{R}{\|u_{n\|}} u_n) = \phi(Rw_n) \\
&\geq a_0 \int_\Omega \frac{R^{p(x)}}{p(x)} (|\nabla w_n|^{p(x)} + |w_n|^{p(x)})dx - \int_\Omega F(x,Rw_n)dx \\
&\geq a_0 \frac{R^{p^-}}{p^+} - \int_\Omega F(x,Rw_n)dx \geq m.
\end{aligned}$$

Thereby, $\phi(t_n u_n) \to +\infty$, on the other hand, we know that $\phi(0) = 0$, $\phi(u_n) \to c$, so we can deduce that $t_n \in]0,1[$ and $< \phi'(t_n u_n), t_n u_n >= t_n \frac{d}{dt}|_{t=t_n} \phi(tu_n) = 0$.

Which yields,

$$\phi(t_n u_n) - \frac{1}{p^-}\phi'(t_n u_n)(t_n u_n) \to +\infty.$$

Therefore,

$\mid a \mid_\infty \int_\Omega (\frac{1}{p(x)} - \frac{1}{p^-})(\mid t_n \nabla u_n \mid^{p(x)} + \mid t_n u_n \mid^{p(x)} + \int_\Omega \frac{1}{p^-} f(x,t_n u_n)(t_n u_n) - F(x,t_n u_n)dx \to +\infty$,

so we get,

$$\int_\Omega \frac{1}{p^-} f(x,t_n u_n)(t_n u_n) - F(x,t_n u_n)dx \to +\infty.$$

Moreover,

$$\begin{aligned}
\phi(u_n) &= \phi(u_n) - \frac{1}{p^+}\phi'(u_n)(u_n) \\
&\geq a_0 \int_\Omega (\frac{1}{p(x)} - \frac{1}{p^+})(\mid \nabla u_n \mid^{p(x)} \\
&\quad + \mid u_n \mid^{p(x)})dx + \int_\Omega \frac{1}{p^+}(f(x,u_n)u_n - F(x,u_n)dx \\
&\geq \int_\Omega \frac{1}{p^+}(f(x,u_n)u_n - F(x,u_n)dx.
\end{aligned}$$

From (f_9) we have

$$\begin{aligned}\phi(u_n) &\geq \int_\Omega \frac{1}{p^+}(f(x,u_n)u_n - F(x,u_n)dx \\ &\geq c_1 \int_\Omega \frac{1}{p^-} f(x,u_n)(u_n) - F(x,u_n)dx \\ &\geq c_1c_2 \int_\Omega \frac{1}{p^-} f(x,t_nu_n)(t_nu_n) - F(x,t_nu_n)dx. \qquad (3.15)\end{aligned}$$

Hence, $\phi(u_n) \to +\infty$ which is impossible. □

Lemma 3.8. *Under the condition of Theorem 1.3, ϕ verifies the following:*
(a) *There exist $\rho > 0$ and $\beta > 0$ such that $\phi(u) > \beta$ when $\| u \| = \rho$.*
(b) *There exists $v \in X$ such that $\| v \| < \rho$ and $\phi(v) < 0$.*

Proof. In view of $(f1)$ and (f_8), there exists $C_1 > 0$ such that

$$|F(x,t)| \leq \frac{a_0}{2p^+}|t|^{p(x)} + C_1|t|^{q(x)}, \quad \text{for } (x,t) \in \Omega \times \mathbb{R}.$$

Therefore, for $\|u\| \leq 1$ we have

$$\begin{aligned}\phi(u) &\geq \frac{a_0}{p^+}\int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})dx - \frac{a_0}{2p^+}\int_\Omega |u|^{p(x)}dx - C_1\int_\Omega |u|^{q(x)}dx \\ &\geq \frac{a_0}{2p^+}\int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})dx - C_1\int_\Omega |u|^{q(x)}dx \\ &\geq \frac{a_0}{2p^+}\|u\|^{p^+} - C_2\|u\|^{q^-} \\ &\geq \|u\|^{p^+}(\frac{a_0}{2p^+} - C_2\|u\|^{q^- - p^+}).\end{aligned}$$

Since $p^+ < q^-$, the function $t \mapsto (\frac{1}{2p^+} - C_2 t^{q^- - p^+})$ is strictly positive in a neighborhood of zero. It follows that there exist $\rho > 0$ and $\beta > 0$ such that $\phi(u) \geq \beta, \forall u \in X : \|u\| = \rho$. □

To apply the Mountain Pass Theorem, it suffices to show that

$$\phi(tu) \to -\infty \quad \text{as } t \to +\infty,$$

for a certain $u \in X$.

Let $u \in X \backslash \{0\}$, by (f_7), we may choose a constant $A > \frac{\int_\Omega \frac{|a|_\infty}{p(x)}[|\nabla u|^{p(x)} + |u|^{p(x)}]dx}{\int_\Omega |u|^{p^+}dx}$, such that

$$F(x,t) \geq A|t|^{p^+} \quad \text{uniformly in} \quad x \in \Omega.$$

Let $t > 1$ large enough, we have

$$\begin{aligned}
\phi(tu) &\leq | a |_\infty \int_\Omega \frac{t^{p(x)}}{p(x)}[|\nabla u|^{p(x)} + |u|^{p(x)}]dx - \int_\Omega F(x, tu)dx \\
&\leq | a |_\infty t^{p^+} \int_\Omega \frac{1}{p(x)}[|\nabla u|^{p(x)} + |u|^{p(x)}]dx - \int_{|tu|>C_A} F(x, tu)dx \\
&\quad - \int_{|tu|\leq C_A} F(x, tu)dx \\
&\leq | a |_\infty t^{p^+} \int_\Omega \frac{1}{p(x)}[|\nabla u|^{p(x)} + |u|^{p(x)}]dx - At^{p^+} \int_\Omega |u|^{p^+} dx \\
&\quad - \int_{|tu|\leq C_A} F(x, tu)dx + At^{p^+} \int_{|tu|\leq C_A} |u|^{p^+} dx \\
&\leq | a |_\infty t^{p^+} \int_\Omega \frac{1}{p(x)}[|\nabla u|^{p(x)} + |u|^{p(x)}]dx - At^{p^+} \int_\Omega |u|^{p^+} dx + C_1,
\end{aligned}$$

where $C_1 > 0$ is a constant, which implies that $\phi(tu) \to -\infty$as $t \to +\infty$. □

Acknowledgement : The authors would like to thank the referees for their suggestions and helpful comments which improved the presentation of the original manuscript.

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(Received 31 May 2013)
(Accepted 30 September 2013)