Coincidence and Common Fixed Points for Hybrid Mappings Satisfying an Implicit Relation and Applications

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Abstract : In this paper a fixed point theorem for a pair of hybrid mappings involving altering distance and satisfying an implicit relation is proved, generalizing the main result from [1] (Theorem 3.1).

1 Introduction

Sessa [2] introduced the concept of weakly commuting mappings. Jungck [3] defined the notions of compatible mappings in order to generalize the concept of weak commutativity and showed that weak commuting mappings are compatible but the converse is not true. In recent years, a number of a fixed point theorems and coincidence theorems have been obtained by various authors utilizing this notion. Jungck further weakened the notion of weak compatibility [4] and in [5] Jungck and Rhoades further extended weak compatibility. Pant [6], [7], [8] initiated the study of noncompatible mappings. Itoh and Takahasi [9] and Sing and Mishra [10] introduced the notion of (I, T)-commutativity. More recently, Aamri

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and Moutawakil [11] defined property $(E.A)$ for all self mappings of a metric space (X, d) under strict contractive conditions. The class of $(E.A)$ mappings contain the class of noncompatible mappings. Recently, Kamran [12] extended the property $(E.A)$ for hybrid pair of single and multivalued mappings and generalize the notion of (I, T)-commutativity for such pairs. In [12] some coincidence and fixed point theorems for hybrid pairs are obtained which generalize the results from [11]. Quite recently, Sintunavarat and Kumam [1] established new coincidence and common fixed point theorems for hybrid strict contractions maps by dropping the assumption "f is T-weakly commuting for a hybrid pair (f, T) of single and multivalued maps" in Theorem 3.10 [12].

2 Preliminaries

Let (X, d) be a metric space. We denote by $CB(X)$ the family of non-empty closed and bounded subset of X and H the Hausdorff-Pompeiu distance on $CB(X)$

$$H(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\} \text{ for } A, B \in CB(X)$$

where $d(a, B) = \inf \{ d(a, b) : b \in B \}$.

Let $f : (X, d) \to (X, d)$ be and $T : (X, d) \to CB(X)$. A point $x \in X$ is said to be a *coincidence point* of f and T if $fx \in Tx$. The set of all coincidence points of f and T is denoted by $C(f, T)$.

The pair (f, T) is called *commuting* if $fTx = Tfx$ for all $x \in X$, weakly commuting [5] if f and T commute for all $x \in C(f, T)$, (I, T)-commuting [9] and [10] at $x \in X$ if $fTx \subset Tfx$, f is T weakly commuting at $x \in X$ if $ffx \subset Tfx$. Here we remark that hybrid pair (f, T), (I, T)-commuting at the coincidence points implies that f is T-weakly commuting, but the converse is not true in general ([12], Example 3.8). The mappings $f : (X, d) \to (X, d)$ and $T : (X, d) \to CB(X)$ are said to be compatible [13] if $fTx \in CB(X)$ for all $x \in X$ and $\lim H(fTx_n, Tfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim Tx_n = A \in CB(X)$ and $\lim fx_n = t \in A$. Therefore, the maps $f : (X, d) \to (X, d)$ and $T : (X, d) \to CB(X)$ are noncompatible if $fTx \in CB(X)$ for each $x \in X$ and there exists at least one sequence $\{x_n\}$ such that $\lim Tx_n = A \in CB(X)$ and $\lim fx_n = t \in A$ but $\lim H(fTx_n, Tfx_n) \neq 0$ or does not exist.

Definition 2.1. The mappings $f : (X, d) \to (X, d)$ and $T : (X, d) \to CB(X)$ are said to *satisfy property* $(E.A)$ [12] if there exists an sequence $\{x_n\}$ in X such that $\lim fx_n = t \in A = \lim Tx_n$.

Remark 2.1. *Every noncompatible hybrid pair* (f, T) *satisfy property* $(E.A)$.

Theorem 2.2 ([12])**.** *Let* $f : (X, d) \to (X, d)$ *and* $T : (X, d) \to CB(X)$ *be such that*

(i) f *and* T *satisfy property* $(E.A)$;

(ii) for all $x, y \in X$

$$H(Tx, Ty) < \max \left\{ d(fx, fy), \frac{1}{2} \left(d(fx, Tx) + d(fy, Ty) \right), \right.$$
$$\left. \frac{1}{2} \left(d(fx, Ty) + d(fy, Tx) \right) \right\} \tag{2.1}$$

If $f(X)$ is a closed subset of X, then f and T have a coincidence point.

Theorem 2.3 ([12])**.** *Let $f : (X, d) \to (X, d)$ and $T : (X, d) \to CB(X)$ be such that f and T satisfies the conditions (i) and (ii) of Theorem 2.2 and (iii) f is T - weakly commuting at u and $ffu = fu$ for $u \in C(f, T)$. If $f(X)$ is a closed set of X, then f and T have a common fixed point.*

Sintunavarat and Kumam [1] proved the following theorem.

Theorem 2.4. *Let $f : (X, d) \to (X, d)$ and $T : (X, d) \to CB(X)$ be such that the following conditions are satisfied:*
(i) f and T satisfy property (E.A) and (2.1) holds;
(ii) $fv = ffv$ for $v \in C(f, T)$.
If $f(X)$ is a closed subset of X, then f and T have a common fixed point.

Definition 2.2. An *altering distance* is a mapping $\psi : [0, \infty) \to [0, \infty)$ which satisfies:
$(\psi_1) : \psi(t)$ is increasing and continuous,
$(\psi_2) : \psi(t) = 0$ if and only if $t = 0$.

Fixed point problem involving altering distance have been studied in [14], [15], [16], [17], [18] and other papers. The study of fixed points for mappings satisfying an implicit relation is initiated in [19] and [20]. In [16] some fixed point theorems for mappings involving altering distance and satisfying an implicit relation are proved.

In this paper a fixed point theorem for a pair of hybrid mappings involving altering distance and satisfying an implicit relation is proved, generalizing Theorem 2.4.

3   Implicit Relation

Let $\mathfrak{F}_a$ be the set of all continuous functions $F(t_1, ..., t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:
$(F_1) :$ F is nondecreasing in variable t_1,
$(F_2) :$ $F(t, 0, 0, t, t, 0) \leq 0$ implies $t = 0$.

Example 3.1. $F(t_1, ..., t_6) = t_1 - \max \{ t_2, (t_3 + t_4)/2, (t_5 + t_6)/2 \}$
(F_1): Obviously.
$(F_2) :$ $F(t, 0, 0, t, t, 0) = t/2 \leq 0$ implies $t = 0$.

Example 3.2. $F(t_1, ..., t_6) = t_1 - k \max\{t_2, t_3, ..., t_6\}$, where $0 < k < 1$.
(F_1): Obviously.
$(F_2): F(t, 0, 0, t, t, 0) = t(1 - k) \leq 0$ implies $t = 0$.

Example 3.3. $F(t_1, ..., t_6) = t_1 - at_2 - b(t_3 + t_4) - c(t_5 + t_6)$, where $a, b, c \geq 0$ and $b + c < 1$.
(F_1): Obviously.
$(F_2): F(t, 0, 0, t, t, 0) = t(1 - (b + c)) \leq 0$ implies $t = 0$.

Example 3.4. $F(t_1, ..., t_6) = t_1 - at_2 - b(t_3 + t_4) - c \min\{t_5, t_6\}$, where $a, c \geq 0$ and $0 < b < 1$.
(F_1): Obviously.
$(F_2): F(t, 0, 0, t, t, 0) = t(1 - b) \leq 0$ implies $t = 0$.

Example 3.5. $F(t_1, ..., t_6) = t_1 - at_2 - b(t_3 + t_4) - c\sqrt{t_5 t_6}$, where $0 < b < 1$ and $a, c \geq 0$.
(F_1): Obviously.
$(F_2): F(t, 0, 0, t, t, 0) = t(1 - b) \leq 0$ implies $t = 0$.

Example 3.6. $F(t_1, ..., t_6) = t_1 - at_2 - b \max\{t_3, t_4\} - c \max\{t_5, t_6\}$, where $a, b, c \geq 0$ and $0 < b + c < 1$.
(F_1): Obviously.
$(F_2): F(t, 0, 0, t, t, 0) = t(1 - (b + c)) \leq 0$ implies $t = 0$.

Example 3.7. $F(t_1, ..., t_6) = t_1^2 - t_2^2 - a(t_3^2 + t_4^2)/(1 + \min\{t_5, t_6\})$, where $0 < a < 1$.
(F_1): Obviously.
$(F_2): F(t, 0, 0, t, t, 0) = t^2(1 - a) \leq 0$ implies $t = 0$.

Example 3.8. $F(t_1, ..., t_6) = t_1 - \max\{t_2, t_3, t_4\} - (1 - \alpha)(at_5 + bt_6)$, where $0 \leq \alpha < 1$, $0 < a < 1$, $b \geq 0$.
(F_1): Obviously.
$(F_2): F(t, 0, 0, t, t, 0) = t(1 - \alpha)(1 - a) \leq 0$ implies $t = 0$.

Example 3.9. $F(t_1, ..., t_6) = t_1 - \max\{ct_2, ct_3, ct_4, at_5 + bt_6\}$, where $a, b, c \geq 0$ and $\max\{a, c\} < 1$.
(F_1): Obviously.
$(F_2): F(t, 0, 0, t, t, 0) = t(1 - \max\{a, c\}) \leq 0$ implies $t = 0$.

Example 3.10. $F(t_1, ..., t_6) = t_1 - \max\{t_2, k(t_3 + t_4)/2, (t_5 + t_6)/2\}$, where $0 < k < 1$.
(F_1): Obviously.
$(F_2): F(t, 0, 0, t, t, 0) = t(1 - k/2) \leq 0$ implies $t = 0$.

Example 3.11. $F(t_1, ..., t_6) = t_1 - \max\{k_1(t_2 + t_3 + t_4), k_2(t_5 + t_6)\}$, where $k_1, k_2 \geq 0$ and $\max\{k_1, k_2\} < 1$.
(F_1): Obviously.
$(F_2): F(t, 0, 0, t, t, 0) = t(1 - \max\{k_1, k_2\}) \leq 0$ implies $t = 0$.

Example 3.12. $F(t_1, ..., t_6) = t_1^2 - (t_3^2 t_4^2 + t_5^2 t_6^2)/(1 + t_2)$.
(F_1): Obviously.
$(F_2): F(t, 0, 0, t, t, 0) = t^2 \leq 0$ implies $t = 0$.

4 Main Results

Theorem 4.1. *Let $f : (X, d) \to (X, d)$ be and $T : (X, d) \to CB(X)$ such that*

$$F(\psi(H(Tx, Ty)), \psi(d(fx, fy)), \psi(d(fx, Tx)),$$
$$\psi(d(fy, Ty)), \psi(d(fx, Ty)), \psi(d(fy, Tx))) \leq 0 \qquad (4.1)$$

for all $x, y \in X$, where $F \in \mathfrak{F}_a$ and $\psi(t)$ is an altering distance. If $f(X)$ is a closed subset of X and (f, T) satisfy property $(E.A)$, then $C(f, T) \neq \phi$. Moreover, if $fv = ffv$ for $v \in C(f, T)$, then f and T have a common fixed point.

Proof. By Definition 2.1 there exists a sequence $\{x_n\}$ in X such that $\lim fx_n = x \in A = \lim Tx_n$ for some $x \in X$. Since $f(X)$ is closed in X we have

$$F(\psi(H(Tx_n, Tx)), \psi(d(fx_n, fx))), \psi(d(fx_n, Tx_n)),$$
$$\psi(d(fx, Tx)), \psi(d(fx_n, Tx)), \psi(d(fx, Tx_n))) \leq 0.$$

Letting n tend to infinity, we obtain

$$F(\psi(d(A, Tx)), 0, 0, \psi(d(fx, Tx)), \psi(d(fx, Tx)), 0) \leq 0.$$

Since $fx \in A$, then $d(fx, Tx) \leq H(A, Tx)$ which implies by (F_1) that

$$F(\psi(d(fx, Tx)), 0, 0, \psi(d(fx, Tx)), \psi(d(fx, Tx)), 0) \leq 0.$$

By (F_2), $\psi(d(fx, Tx)) = 0$ which implies $d(fx, Tx) = 0$ i.e. $fx \in Tx$ and $C(f, T) \neq \phi$.

Let $v \in C(f, T)$ be, hence $fv \in Tv$ and $z = fv = ffv = fz \in Tv$. By (4.1) we have successively

$$F(\psi(H(Tv, Tz)), \psi(d(fv, fz)), \psi(d(fv, Tv)),$$
$$\psi(d(fz, Tz)), \psi(d(fv, Tz)), \psi(d(fz, Tv))) \leq 0,$$

$$F(\psi(H(Tv, Tz)), 0, 0, \psi(d(fz, Tz)), \psi(d(fv, Tz)), 0) \leq 0.$$

Since $fz \in Tv$ we have that $d(fz, Tz) \leq H(Tv, Tz)$, which implies by (F_1)

$$F(\psi(d(fz, Tz)), 0, 0, \psi(d(fz, Tz)), \psi(d(fv, Tz)), 0) \leq 0.$$

By (F_2) we have $\psi(d(fz, Tz)) = 0$ which implies $d(fz, Tz) = 0$ i.e. $fz \in Tz$. Therefore $z = fz \in Tz$ and z is a common fixed point for f and T. $\square$

If $\psi(t) = t$ we obtain

Theorem 4.2. *Let $f : (X, d) \to (X, d)$ be and $T : (X, d) \to CB(X)$ such that*

$$F(H(Tx, Ty), d(fx, fy), d(fx, Tx), d(fy, Ty), d(fx, Ty), d(fy, Tx)) \leq 0 \quad (4.2)$$

for all $x, y \in X$, where $F \in \mathfrak{F}_a$. If $f(X)$ is closed and (f, T) satisfy property $(E.A)$, then $C(f, T) \neq \phi$. Moreover, if $fv = ffv$ for $v \in C(f, T)$, then f and T have a common fixed point.

Remark 4.3.

1. *By Example 3.1 and Theorem 4.2 we obtain a generalization of Theorem 2.4.*

2. *By Examples 3.2 - 3.12 we obtain new results.*

By Theorem 4.1 and Remark 4.3 we obtain

Corollary 4.4. *Let $f : (X, d) \to (X, d)$ be and $T : (X, d) \to CB(X)$ such that f and T are noncompatible and satisfy inequality (4.1) for all $x, y \in X$. If $f(X)$ is closed in X, then $C(f, T) \neq \phi$. Moreover, if $fv = ffv$ for $v \in C(f, T)$, then f and T have a common fixed point.*

Remark 4.5. *By Corollary 4.4 and Example 3.1 we obtain Corollary 3.7 [1].*

5   Applications

In [21], Branciari established the following result

Theorem 5.1. *Let (X, d) be a complete metric space and and $f : (X, d) \to (X, d)$ be a mapping such that for all $x, y \in X$*

$$\int_0^{d(fx,fy)} h(t)dt \le c \int_0^{d(x,y)} h(t)dt, \ 0 < c < 1 \tag{5.1}$$

where $h : [0, \infty) \to [0, \infty)$ is a Lebesgue measurable mapping (i.e., with a finite integral) on each compact subsets of $[0, \infty)$ such that for $\epsilon > 0$, $\int_0^\epsilon h(t)dt > 0$. Then, f has a unique point $z \in X$ such that for all $x \in X$, $\lim f^n x = z$.

Some fixed point theorems for mappings satisfying contractive condition of integral type are proved in [22], [23], [24], [25], [26] and other papers.

Lemma 5.2 (Popa and Mocanu [16])**.** *The function $\psi(t) = \int_0^t h(x)dx$, where $h(x)$ is as in Theorem 5.1, is an altering distance.*

Theorem 5.3. *Let $f : (X, d) \to (X, d)$ be and $T : (X, d) \to CB(X)$ such that*

$$F\left(\int_0^{H(Tx,Ty)} h(t)dt, \int_0^{d(fx,fy)} h(t)dt, \int_0^{d(fx,Tx)} h(t)dt, \right.$$
$$\left. \int_0^{d(fy,Ty)} h(t)dt, \int_0^{d(fx,Ty)} h(t)dt, \int_0^{d(fy,Tx)} h(t)dt\right) \le 0 \tag{5.2}$$

for all $x, y \in X$, where $F \in \mathfrak{F}_a$ and $h(t)$ is as in Theorem 5.1. If $f(X)$ is a closed set of X and f and T satisfy property (E.A), then $C(f, T) \neq \phi$. Moreover, if $fv = ffv$ for $v \in C(f, T)$, then f and T have a common fixed point.

Proof. As in Lemma 5.2 we have

$$\psi(H(Tx,Ty)) = \int_0^{H(Tx,Ty)} h(t)dt, \ \psi(d(fx,fy)) = \int_0^{d(fx,fy)} h(t)dt,$$

$$\psi(d(fx,Tx)) = \int_0^{d(fx,Tx)} h(t)dt, \ \psi(d(fy,Ty)) = \int_0^{d(fy,Ty)} h(t)dt,$$

$$\psi(d(fx,Ty)) = \int_0^{d(fx,Ty)} h(t)dt, \ \psi(d(fy,Tx)) = \int_0^{d(fy,Tx)} h(t)dt.$$

By Lemma 5.2 $\psi(t)$ is an altering distance. Hence the conditions of Theorem 4.1 are satisfied and the results of Theorem 5.3 follows by Theorem 4.1. $\square$

Remark 5.4. *1)   If $h(t) = 1$ by Theorem 5.3 we obtain Theorem 4.2.*
2)   By Remark 2.1 it follows that Theorem 5.3 is true if f and T are non-compatible instead of (f,T) satisfy property $(E.A)$.

Corollary 5.5. *Let $f : (X,d) \to (X,d)$ be and $T : (X,d) \to CB(X)$ such that*

$$\int_0^{H(Tx,Ty)} h(t)dt \le \max \left\{ \int_0^{d(fx,fy)} h(t)dt, \ \frac{1}{2} \left[\int_0^{d(fx,Tx)} h(t)dt + \int_0^{d(fy,Ty)} h(t)dt \right], \right.$$
$$\left. \frac{1}{2} \left[\int_0^{d(fx,Ty)} h(t)dt + \int_0^{d(fy,Tx)} h(t)dt \right] \right\}$$

(5.3)

for all $x, y \in X$ and $h(t)$ is as in Theorem 5.1. If $f(X)$ is a closed subset of X and f and T satisfy property $(E.A)$, then $C(f,T) \ne \phi$. Moreover, if $fv = ffv$ for $v \in C(f,T)$, then f and T have a common fixed point.

Remark 5.6. *For $h(t) = 1$, by Corollary 5.5 we obtain Theorem 2.4.*

References

[1] W. Sintunavarat, P. Kumam, Concidence and fixed points for hybrid strict contractions without the weakly commuting condition, Appl. Math. Letters. 22 (2009) 1877-1881.

[2] S. Sessa, On weak comuutativity condition in fixed point considerations, Pub. Inst. Math. 32 (1982) 149-153.

[3] G. Jungck, Compatible mappings, common fixed points, Intern. J. Math. Math. Sci. 9 (1986) 771-779.

[4] G. Jungck, Common fixed points for noncommuting nonself mappings on nonnumeric spaces, Far East. J. Math. Sci. 42 (1996) 191-221.

[5] G. Jungck, B.E. Rhoades, Fixed points for set valued functions without continuity, Indian Pure Appl. Math. 29 (1998) 227-238.

[6] R.P. Pant, Common fixed point theorems for noncommuting mappings, J. Math. Anal. Appl. 188 (1994) 436-440.

[7] R.P. Pant, Common fixed point theorems for noncontractive maps, J. Math. Anal. Appl. 226 (1998) 251-258.

[8] R.P. Pant, R-weak commutability and common fixed points for noncompatible mappings, Ganita. 49 (1998) 19-27.

[9] S. Itoh, W. Takahashi, Single valued mappings, multivalued mappings and fixed point theorems, J. Math. Anal. Appl. 59 (1997) 514-521.

[10] S. L. Singh, S.N. Mishra, Concidence and fixed points for non - self hybrid contractions, J. Math. Anal. Appl. 266 (2001) 486-497.

[11] M. Aamri, D. El-Moutawakill, Some new common fixed point theorems under strict contractive conditions, J. Math. Anal. Appl. 270 (2002) 181-188.

[12] H. Kamran, Coincidence, fixed points for hybrid contractions, J. Math. Anal. Appl. 229 (2004) 235-241.

[13] T. Kaneko, S. Sessa, Fixed point theorems for compatible multivalued single valued mappings, Intern. J. Math. Math. Sci. 12 (1989) 257-262.

[14] M.S. Khan, M. Swaleh, S. Sessa, Fixed point theorems by altering distance between two points, Bull. Austral. Math. Sci. 30 (1984) 1-9.

[15] V. Popa, Some fixed point theorems for compatible mappings in compact metric spaces, U. P. B. Sci. Bull. Ser. A. 63 (4) (2001) 43-46.

[16] V. Popa, M. Mocanu, Altering distance and common fixed points under implicit relations, Hacettepe J. Math. Stat. 33 (3) (2009) 329-337.

[17] K.P. Sastri, G.V.R. Babu, Fixed point theorems in metric spaces by altering distances, Bull. Calcutta Math. Soc. 90 (1998) 175-182.

[18] K.P. Sastri, G.V.R. Babu, Some fixed point theorems by altering distances between two points, Indian J. Pure Appl. Math. 30 (1999) 641-647.

[19] V. Popa, Fixed point theorems for implicit contractive mappings, Stud. Cerc. St. Ser. Math. Univ. Bacău 7 (1997) 127-133.

[20] V. Popa, Some fixed point theorems for compatible mappings satisfying an implicit relation, Demonstratio Math. 33 (1999) 157-163.

[21] A. Branciari, A fixed point theorem for mappings satisfying a general contractive condition of integral type, Intern. J. Math. Math. Sci. 29 (9) (2002) 531-536.

[22] A. Alliouche, A common fixed point theorems for weakly compatible mappings in symmetric spaces satisfying a contractive condition of integral type, J. Math. Anal. Appl. 332 (2006) 796-802.

[23] J.K. Kohli, S. Washistha, Common fixed point theorems for compatible and weak compatible mappings satisfying general contractive conditions, Stud. Cerc. St. Ser. Math. Univ. Bacău 16 (2006) 33-41.

[24] S. Kumar, R. Chung, R. Kumar, Fixed point theorems for compatible mappings satisfying contractive conditions of integral type, Soochow J. Math. 33 (3) (2007) 181-186.

[25] M. Mocanu, V. Popa, Some fixed point theorems for mappings satisfying implicit relations in symmetric spaces, Libertas Math. 33 (2008) 1-13.

[26] V. Popa, M. Mocanu, A new view points in the study of fixed points for mappings satisfying a contractive condition of integral type, Bull. Inst. Politehn. Iaşi, Ser. Mat. Mec. Teor. Fiz. 53 (57) 5 (2007) 269-286.