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Stability Properties of Positive Solutions to a Quasilinear Elliptic Equation Involving the p-Laplacian and Indefinite Weight

Ghasem A. Afrouzi[†,1] **and Saleh Shakeri**[‡]

[†]Faculty of Mathematical Sciences, University of Mazandaran,
Babolsar, Iran
e-mail : afrouzi@umz.ac.ir

[‡]Department of Mathematics, Islamic Azad University-
Ayatollah Amoli Branch, Amol, Iran
e-mail : s.shakeri@iauamol.ac.ir

Abstract : In this note, we discuss the stability and instability results of positive solutions for the following reaction-diffusion equation

$$\begin{cases} -\Delta_p u = m(x)u^{p-1} - u^{\gamma-1} - ch(x) & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $p > 1$, $\gamma < p$, Ω is a bounded domain in $R^N(N \geq 1)$ with smooth boundary, $Bu(x) = \alpha g(x)u(x) + (1-\alpha)\frac{\partial u(x)}{\partial n}$ where $\alpha \in [0,1]$ is a constant, $g : \partial\Omega \longrightarrow R^+$ with $g = 1$ when $\alpha = 1$, i.e., the boundary condition may be of Dirichlet, Neumann or mixed type, c is a positive constant and the weight functions $m(x)$ satisfies $m(x) \in C(\Omega)$ and $m(x) > 1$ for $x \in \Omega$ and $h : \overline{\Omega} \longrightarrow R$ is a $C^{1,\alpha}(\overline{\Omega})$ function satisfying $h(x) > 0$ for $x \in \Omega$, $\max h(x) = 1$ for $x \in \overline{\Omega}$ and $h(x) = 0$ for $x \in \partial\Omega$. Here u is the population density $m(x)u^{p-1} - u^{\gamma-1}$ represents the logistic growth and $ch(x)$ represents the constant yield harvesting rate. we shall establish that every positive solution is linearly unstable.

Keywords : p-Laplacian; diffusive logistic equation; linearized stability; harvesting.

[1]Corresponding author.

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1 Introduction

This paper deals with the stability properties of positive stationary solutions for nonlinear elliptic systems of the form

$$\begin{cases} -\Delta_p u = m(x)u^{p-1} - u^{\gamma-1} - ch(x) & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator $p > 1$, $\gamma < p$, Ω is a bounded domain in $R^N(N \geq 1)$ with smooth boundary, $Bu(x) = \alpha g(x)u(x) + (1-\alpha)\frac{\partial u(x)}{\partial n}$ where $\alpha \in [0,1]$ is a constant, $g : \partial\Omega \longrightarrow R^+$ with $g = 1$ when $\alpha = 1$, i.e., the boundary condition may be of Dirichlet, Neumann or mixed type, c is a positive constant and the weight functions $m(x)$ satisfies $m(x) \in C(\Omega)$ and $m(x) > 1$ for $x \in \Omega$ and $h : \overline{\Omega} \longrightarrow R$ is a $C^{1,\alpha}(\overline{\Omega})$ function satisfying $h(x) > 0$ for$x \in \Omega$, $\max h(x) = 1$ for $x \in \overline{\Omega}$ and $h(x) = 0$ for $x \in \partial\Omega$. Here u is the population density $m(x)u^{p-1} - u^{\gamma-1}$ represents the logistic growth and $ch(x)$ represents the constant yield harvesting rate (see [1, 2, 3, 4]). we shall prove the instability of solution u by showing that the principal eigenvalue μ_1 of the equation linearized about u is negative; the instability of u then follows from the well-known principle of linearized stability (see [5]). For existence results of positive solutions for Eq(1.1) (see [6]) We recall that if u be any non-negative solution of

$$\begin{cases} -\Delta_p u = g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

then the linearized equation about u is

$$\begin{cases} -(p-1)\text{div}(|\nabla u|^{p-2}\nabla\phi) - g_u(x,u)\phi = \mu\phi & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $g_u(x,u)$ denotes the partial derivative of $g(x,u)$ with respect to u and μ is eigenvalue of the operator Δ_p. Eq. (4) obtained from the formal derivative of the operator Δ_p.

Definition 1.1. we call a solution u of (1.1) a *linearly stable solution* if all eigenvalues of (1.1) are strictly positive, which can be *inferred* if the principal eigenvalue $\mu_1 > 0$. Otherwise, u is *linearly unstable.*

2 Main Results

These are the main results of the paper. Here we would establish instability (stability) of non-trivial stationary solution u of the equation (1.1) directly by showing that the principle eigenvalue μ_1 of its linearization is negative (positive). The instability (stability) of u then follows from the well-known principle of linearized stability (see [5]).

Our main result is formulated in the following theorem.

Theorem 2.1. *Every positive solution of (1.1) is linearly unstable.*

Proof. From (4) the linearized equation about u is

$$-(p-1)\mathrm{div}(|\nabla u|^{p-2}\nabla\phi)-[m(x)(p-1)u^{p-2}-(\gamma-1)u^{\gamma-2}]\phi=\mu\phi, \qquad (2.1)$$

$$B\phi=0. \qquad (2.2)$$

Let μ_1 be the principal eigenvalue and let $\psi(x)>0$ be a corresponding eigenfunction. Multiplying (1) by $(p-1)\psi(x)$ and (5) by u, then subtracting and integrating over Ω, we obtain

$$\begin{aligned}
&(p-1)\int_\Omega \big[u(x)\mathrm{div}(|\nabla u(x)|^{p-2}\nabla\psi(x))-\psi(x)\mathrm{div}(|\nabla u(x)|^{p-2}\nabla u(x))\big]dx\\
&+\int_\Omega \big[m(x)(p-1)u(x)^{p-1}-(\gamma-1)u(x)^{\gamma-1}\big]\psi(x)dx+\int_\Omega(p-1)ch(x)\psi(x)dx\\
&=-\mu_1\int_\Omega(u(x)\psi(x))dx \qquad (2.3)
\end{aligned}$$

But by using the Green's first identity we obtain

$$\begin{aligned}
\int_\Omega u\,\mathrm{div}(|\nabla u|^{p-2}\nabla\psi)dx &= -\int_\Omega|\nabla u|^{p-2}(u\Delta\psi)dx+\int_\Omega u\nabla(|\nabla u|^{p-2})\nabla\psi\\
&= -\int_\Omega\nabla(u|\nabla u|^{p-2})\nabla\psi(x)dx+\int_\Omega u\nabla(|\nabla u|^{p-2})\nabla\psi\\
&\quad+\int_{\partial\Omega}u|\nabla u|^{p-2}(\frac{\partial\psi}{\partial n})dS\\
&= \int_{\partial\Omega}u|\nabla u|^{p-2}(\frac{\partial\psi}{\partial n})dS-\int_\Omega\nabla u(|\nabla u|^{p-2})\nabla\psi \qquad (2.4)
\end{aligned}$$

and

$$\begin{aligned}
\int_\Omega \psi\,\mathrm{div}(|\nabla u|^{p-2}\nabla u)dx &= -\int_\Omega|\nabla u|^{p-2}(\psi\Delta u)dx+\int_\Omega \psi\nabla(|\nabla u|^{p-2})\nabla u\\
&= -\int_\Omega\nabla(\psi|\nabla u|^{p-2})\nabla u(x)dx+\int_\Omega \psi\nabla(|\nabla u|^{p-2})\nabla u\\
&\quad+\int_{\partial\Omega}\psi|\nabla u|^{p-2}(\frac{\partial u}{\partial n})dS\\
&= \int_{\partial\Omega}\psi|\nabla u|^{p-2}(\frac{\partial u}{\partial n})dS-\int_\Omega\nabla u(|\nabla u|^{p-2})\nabla\psi \qquad (2.5)
\end{aligned}$$

By using (2.4) and (2.5) in (2.3) and hypothesis we get

$$-\mu_1 \int_\Omega (u(x)\psi(x))dx = \int_\Omega \left[m(x)(p-1)u(x)^{p-1} - (\gamma-1)u(x)^{\gamma-1})\right]\psi(x)dx$$
$$+\int_\Omega (p-1)ch(x)\psi(x)dx + \int_{\partial\Omega} |\nabla u|^{p-2}[u(\frac{\partial\psi}{\partial n}) - \psi(s)(\frac{\partial u}{\partial n})\}dS$$

We notice that ($\alpha = 1$) then $h = 1$ we have $Bu = u = 0$ for $x \in \partial\Omega$ and also we have $\psi = 0$ for $x \in \partial\Omega$. Hence

$$\int_{\partial\Omega} |\nabla u|^{p-2}[u(\frac{\partial\psi}{\partial n}) - \psi(s)(\frac{\partial u}{\partial n})\}dS = 0$$

and when $\alpha \neq 1$ we have

$$\int_{\partial\Omega} |\nabla u|^{p-2}[u(\frac{\partial\psi}{\partial n}) - \psi(s)(\frac{\partial u}{\partial n})\}dS = \int_{\partial\Omega} \{\frac{\alpha h\psi}{(1-\alpha)}\}(u-u)dS = 0$$

then we get

$$-\mu_1 \int_\Omega (u(x)\psi(x))dx = \int_\Omega \left[m(x)(p-1)u(x)^{p-1} - (\gamma-1)u(x)^{\gamma-1})\right]\psi(x)dx$$
$$+\int_\Omega (p-1)ch(x)\psi(x)dx > 0$$

But $\psi > 0$ and $u > 0$ in Ω and Hence it is easy to see that $\mu_1 < 0$. Then the result follows (see [5]). □

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