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Maximum Terms, Maximum Moduli Related and Slowly Changing Functions Based Growth Measurement of Composition of Entire Functions

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Abstract : In the paper we prove some growth properties related to the maximum terms and maximum moduli of composite entire functions using generalised L^*-order and generalised L^*-lower order as compared to the growths of their corresponding left and right factors.

Keywords : entire function; maximum term; maximum modulus; composition; growth; generalised L^*-order; slowly changing function.

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1 Introduction and Preliminaries

Let $\mathbb{C}$ be the set of all finite complex numbers and f be entire defined in the open complex plane $\mathbb{C}$. The maximum term $\mu(r,f)$ of $f=\sum_{n=0}^{\infty}a_n z^n$ on $|z|=r$ is defined by $\mu(r,f)=\max(|a_n|\, r^n)$ and the maximum modulus $M(r,f)$ of $f=\sum_{n=0}^{\infty}a_n z^n$ on $|z|=r$ is defined by $M(r,f)=\max_{|z|=r}|f(z)|$. We use the standard notations and definitions in the theory of entire functions which are available in [1]. In the sequel the following notation is used:

$\log^{[k]} x=\log\left(\log^{[k-1]} x\right)$ for $k=1,2,3,...$ and $\log^{[0]} x=x$.

To start our paper we just recall the following definition:

Definition 1.1 ([1])**.** The *order* ρ_f and *lower order* λ_f of an entire function f are defined as

$$\rho_f=\limsup_{r\to\infty}\frac{\log^{[2]} M(r,f)}{\log r} \quad\text{and}\quad \lambda_f=\liminf_{r\to\infty}\frac{\log^{[2]} M(r,f)}{\log r}.$$

Definition 1.2 ([1])**.** The *type* σ_f of an entire function f is defined as

$$\sigma_f=\limsup_{r\to\infty}\frac{\log M(r,f)}{r^{\rho_f}},\quad 0<\rho_f<\infty.$$

Sato [2] defined the generalised order and generalised lower order of an entire function as follows:

Definition 1.3 ([2])**.** Let m be an integer such that $m\geq 2$. The *generalised order* $\rho_f^{[m]}$ and *generalised lower order* $\lambda_f^{[m]}$ of an entire function f are defined as

$$\rho_f^{[m]}=\limsup_{r\to\infty}\frac{\log^{[m]} M(r,f)}{\log r} \text{ and } \lambda_f^{[m]}=\liminf_{r\to\infty}\frac{\log^{[m]} M(r,f)}{\log r} \quad\text{respectively.}$$

For $m=2$, Definition 1.2 reduces to Definition 1.1.

If $\rho_f<\infty$ then f is of finite order. Also $\rho_f=0$ means that f is of order zero. In this connection Datta and Biswas [3] gave the following definition:

Definition 1.4 ([3])**.** Let f be an entire function of order zero. The *quantities* ρ_f^{**} and λ_f^{**} of f are defined by:

$$\rho_f^{**}=\limsup_{r\to\infty}\frac{\log M(r,f)}{\log r} \text{ and } \lambda_f^{**}=\liminf_{r\to\infty}\frac{\log M(r,f)}{\log r}\ .$$

Let $L\equiv L(r)$ be a positive continuous function increasing slowly, *i.e.*, $L(ar)\sim L(r)$ as $r\to\infty$ for every positive constant a. Somasundaram and Thamizharasi [4] introduced the notions of L-order and L-type for entire function where $L\equiv L(r)$ is a positive continuous function increasing slowly, *i.e.*, $L(ar)\sim L(r)$ as $r\to\infty$ for every positive constant 'a'. The more generalised concept for L-order and L-type for entire function are L^*-order and L^*-type. Their definitions are as follows:

Definition 1.5 ([4]). The L^**-order* $\rho_f^{L^*}$ and the L^**-lower order* $\lambda_f^{L^*}$ of an entire function f are defined as

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log\left[re^{L(r)}\right]}.$$

Definition 1.6. The L^**-type* $\sigma_f^{L^*}$ of an entire function f is defined as

$$\sigma_f^{L^*} = \limsup_{r\to\infty} \frac{\log M(r,f)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}}\ ,\ 0 < \rho_f^{L^*} < \infty.$$

In the line of Sato [2], Datta and Biswas [3] one can define the generalised L^*-order $\rho_f^{[m]L^*}$ and generalised L^*-lower order$\lambda_f^{[m]L^*}$ of an entire function f in the following manner:

Definition 1.7. Let m be an integer such that $m \geq 1$. The *generalised* L^**-order* $\rho_f^{[m]L^*}$ and *generalised* L^**-lower order* $\lambda_f^{[m]L^*}$ of an entire function f are defined as

$$\rho_f^{[m]L^*} = \limsup_{r\to\infty} \frac{\log^{[m]} M(r,f)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{[m]L^*} = \liminf_{r\to\infty} \frac{\log^{[m]} M(r,f)}{\log\left[re^{L(r)}\right]} \text{ respectively.}$$

Using the inequality $\mu(r,f) \leq M(r,f) \leq \frac{R}{R-r}\mu(R,f)$ $\{cf.\ [5]\}$ one can easily verify that

$$\rho_f^{[m]L^*} = \limsup_{r\to\infty} \frac{\log^{[m]} \mu(r,f)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{[m]L^*} = \liminf_{r\to\infty} \frac{\log^{[m]} \mu(r,f)}{\log\left[re^{L(r)}\right]}.$$

Similarly, in the line of Somasundaram and Thamizharasi [4] for any positive integer $m \geq 2$ one may define the generalised L^*-type $\sigma_f^{[m-1]L^*}$ in the following manner:

Definition 1.8. The *generalised* L^**-type* $\sigma_f^{[m-1]L^*}$ for $m \geq 2$ of an entire function f is defined as follows:

$$\sigma_f^{[m-1]L^*} = \limsup_{r\to\infty} \frac{\log^{[m]} M(r,f)}{\left[re^{L(r)}\right]^{\rho_f^{[m]L^*}}},\ 0 < \rho_f^{[m]L^*} < \infty.$$

Lakshminarasimhan [6] introduced the idea of the functions of L-bounded index. Later Lahiri and Bhattacharjee [7] worked on entire functions of L-bounded index and of non uniform L-bounded index. In the paper we study some growth properties related to the maximum terms and maximum moduli of composite entire functions using generalised L^*-order and generalised L^*-lower order as compared to the growths of their corresponding left and right factors.

Now, we present some lemmas which will be needed in the sequel.

Lemma 1.9 ([8]). *Let f and g be any two entire functions with $g(0) = 0$. Then for all sufficiently large values of r,*

$$\mu\left(r, f \circ g\right) \geq \frac{1}{2}\mu\left(\frac{1}{8}\mu\left(\frac{r}{4}, g\right) - |g(0)|, f\right).$$

Lemma 1.10 ([9]). *If f and g are two entire functions then for all sufficiently large values of r,*

$$M\left(\frac{1}{8}M\left(\frac{r}{2}, g\right) - |g(0)|, f\right) \leq M(r, f \circ g) \leq M\left(M\left(r, g\right), f\right).$$

2 Main Results

In this section we present the main results of the paper.

Theorem 2.1. *Let f and g be any two entire functions such that $\rho_f^{[m]L^*}$ and $\rho_g^{L^*}$ are both finite and positive where $m \geq 1$. Then for each $\alpha \in (-\infty, \infty)$,*

$$\liminf_{r\to\infty} \frac{\left\{\log^{[m]}\mu\left(r, f \circ g\right) + \log^{[m]}\mu\left(r, f\right)\right\}^{1+\alpha}}{\log^{[m]}\mu\left(\exp\left(r^{\beta}\right), f\right)} = 0 \text{ and}$$

$$\liminf_{r\to\infty} \frac{\left\{\log^{[m]}\mu\left(r, f \circ g\right) + \log^{[m]}\mu\left(r, f\right)\right\}^{1+\alpha}}{\log^{[2]}\mu\left(\exp\left(r^{\beta}\right), g\right)} = 0 \quad \text{where } \beta > (1+\alpha)\,\rho_g^{L^*}.$$

Proof. If $1 + \alpha < 0$, then the theorem is trivial. So we take $1 + \alpha > 0$. Now in view of Lemma 1.10 and the inequality $\mu\left(r, f\right) \leq M\left(r, f\right)$ $\{cf.$ [5]$\}$, we have for all sufficiently large values of r that

$$\begin{aligned} \mu(r, f \circ g) &\leq M(r, f \circ g) \leq M\left(M\left(r, g\right), f\right) \\ i.e.,\ \log^{[m]}\mu(r, f \circ g) &\leq \log^{[m]} M\left(M\left(r, g\right), f\right) \end{aligned}$$

$$i.e.,\ \log^{[m]}\mu(r, f \circ g) \leq \left(\rho_f^{[m]L^*} + \varepsilon\right)\left[\log M\left(r, g\right) e^{L(M(r,g))}\right]$$

$$\begin{aligned} &i.e.,\ \log^{[m]}\mu(r, f \circ g) \\ &\leq \left(\rho_f^{[m]L^*} + \varepsilon\right)\left[re^{L(r)}\right]^{\left(\rho_g^{L^*}+\varepsilon\right)} + \left(\rho_f^{[m]L^*} + \varepsilon\right) L\left(M\left(r, g\right)\right). \end{aligned} \tag{2.1}$$

Again in view of the inequality $\mu\left(r, f\right) \leq M\left(r, f\right)$ $\{cf.$ [5]$\}$, it follows for all sufficiently large values of r that

$$\log^{[m]}\mu(r, f) \leq \log^{[m]} M\left(r, f\right) \leq \left(\rho_f^{[m]L^*} + \varepsilon\right)\log\left[re^{L(r)}\right]. \tag{2.2}$$

Further we get for a sequence of r tending to infinity and for $\varepsilon(> 0)$ that

$$\log^{[m]}\mu\left(\exp\left(r^{\beta}\right), f\right) \geq \left(\rho_f^{[m]L^*} - \varepsilon\right)\log\left[\exp\left(r^{\beta}\right)\exp\left\{L\left(\exp\left(r^{\beta}\right)\right)\right\}\right]$$

$$i.e.,\ \log^{[m]} \mu\left(\exp\left(r^{\beta}\right), f\right) \geq \left(\rho_f^{[m]L^*} - \varepsilon\right)\left[r^{\beta} + L\left(\exp\left(r^{\beta}\right)\right)\right]. \qquad (2.3)$$

Now from (2.1) and (2.2) we obtain for for all sufficiently large values of r that

$$\left\{\log^{[m]} \mu\left(r, f \circ g\right) + \log^{[m]} \mu(r, f)\right\}^{1+\alpha} \leq \left[\left[re^{L(r)}\right]^{\left(\rho_g^{L^*}+\varepsilon\right)}\left(\rho_f^{[m]L^*} + \varepsilon\right)\right.$$
$$\left.\left(\rho_f^{[m]L^*} + \varepsilon\right)\left(L\left(M\left(r, g\right)\right) + \log\left[re^{L(r)}\right]\right)\right]^{1+\alpha}.$$

So from above and (2.3) we obtain for a sequence of r tending to infinity that

$$\frac{\left\{\log^{[m]} \mu\left(r, f \circ g\right) + \log^{[m]} \mu(r, f)\right\}^{1+\alpha}}{\log^{[m]} \mu\left(\exp\left(r^{\beta}\right), f\right)}$$
$$\leq \frac{\left[\left(\rho_f^{[m]L^*} + \varepsilon\right)\left[\left[re^{L(r)}\right]^{\left(\rho_g^{L^*}+\varepsilon\right)} + \left(L\left(M\left(r, g\right)\right) + \log\left[re^{L(r)}\right]\right)\right]\right]^{1+\alpha}}{\left(\rho_f^{[m]L^*} - \varepsilon\right)\left[r^{\beta} + L\left(\exp\left(r^{\beta}\right)\right)\right]}. \qquad (2.4)$$

Let

$$\begin{aligned}
\left[e^{L(r)}\right]^{\left(\rho_g^{L^*}+\varepsilon\right)}\left(\rho_f^{[m]L^*} + \varepsilon\right) &= k_1, \left(\rho_f^{[m]L^*} + \varepsilon\right) L\left(M\left(r, g\right)\right) = k_2, \\
\left(\rho_f^{[m]L^*} + \varepsilon\right)\log\left[re^{L(r)}\right] &= k_3, \left(\rho_f^{[m]L^*} - \varepsilon\right) = k_4, \\
\left(\rho_f^{[m]L^*} - \varepsilon\right) L\left(\exp\left(r^{\beta}\right)\right) &= k_5.
\end{aligned}$$

Then from (2.4) we obtain for a sequence of r tending to infinity that

$$\frac{\left\{\log^{[m]} \mu\left(r, f \circ g\right) + \log^{[m]} \mu(r, f)\right\}^{1+\alpha}}{\log^{[m]} \mu\left(\exp\left(r^{\beta}\right), f\right)} \leq \frac{\left[r^{\left(\rho_g^{L^*}+\varepsilon\right)} k_1 + k_2 + k_3\right]^{1+\alpha}}{k_4 r^{\beta} + k_5}$$

$$i.e., \frac{\left\{\log^{[m]} \mu\left(r, f \circ g\right) + \log^{[m]} \mu(r, f)\right\}^{1+\alpha}}{\log^{[m]} \mu\left(\exp\left(r^{\beta}\right), f\right)} \leq \frac{r^{\left(\rho_g^{L^*}+\varepsilon\right)(1+\alpha)}\left[k_1 + \frac{k_2+k_3}{r^{\left(\rho_g^{L^*}+\varepsilon\right)}}\right]^{1+\alpha}}{k_4 r^{\beta} + k_5}$$

where k_1, k_2,k_3 and k_4 are finite.

Since$\left(\rho_g^{L^*} + \varepsilon\right)(1 + \alpha) < \beta$, therefore

$$\liminf_{r\to\infty} \frac{\left\{\log^{[m]} \mu\left(r, f \circ g\right) + \log^{[m]} \mu(r, f)\right\}^{1+\alpha}}{\log^{[m]} \mu\left(\exp\left(r^{\beta}\right), f\right)} = 0$$

where we choose $\varepsilon(> 0)$ such that

$$0 < \varepsilon < \min\left\{\rho_f^{[m]L^*}, \frac{\beta}{1+\alpha} - \rho_g^{L^*}\right\},$$

which proves the first part of the theorem.

Similarly, the second part of the theorem follows from the following inequality in place of (2.3)

$$\log^{[2]} \mu\left(\exp\left(r^{\beta}\right)\right), g) \geq \left(\rho_g^{L^*} - \varepsilon\right)\left[r^{\beta} + L\left(\exp\left(r^{\beta}\right)\right)\right]$$

for a sequence of values of r tending to infinity. This proves the theorem. □

Remark 2.2. *In Theorem 2.1 if we take the condition "* $0 < \lambda_f^{[m]L^*} \leq \rho_f^{[m]L^*} < \infty$ *and* $0 < \lambda_g^{L^*} \leq \rho_g^{L^*} < \infty$ *" in place of "* $\rho_f^{[m]L^*}$ *and* $\rho_g^{L^*}$ *are both finite and positive " then the theorem remains true with "* $\lim$ *" replaced by "* $\liminf$ *".*

In the line of Theorem 2.1, the following theorem can be proved:

Theorem 2.3. *Let f and g be any two entire functions with finite and positive* $\rho_f^{[m]L^*}$ *and* $\rho_g^{L^*}$ *where* $m \geq 1$*.Then for each* $\alpha \in (-\infty, \infty)$,

$$\liminf_{r\to\infty} \frac{\left\{\log^{[m]} M\left(r, f \circ g\right) + \log^{[m]} M\left(r, f\right)\right\}^{1+\alpha}}{\log^{[m]} M\left(\exp\left(r^{\beta}\right), f\right)} = 0 \ \textit{and}$$

$$\liminf_{r\to\infty} \frac{\left\{\log^{[m]} M\left(r, f \circ g\right) + \log^{[m]} M\left(r, f\right)\right\}^{1+\alpha}}{\log^{[2]} M\left(\exp\left(r^{\beta}\right), g\right)} = 0 \quad \textit{where } \beta > (1+\alpha)\,\rho_g^{L^*}.$$

Remark 2.4. *Also in Theorem 2.3 if we take the condition "* $0 < \lambda_f^{[m]L^*} \leq \rho_f^{[m]L^*} < \infty$ *and* $0 < \lambda_g^{L^*} \leq \rho_g^{L^*} < \infty$ *" inplace of "* $\rho_f^{[m]L^*}$ *and* $\rho_g^{L^*}$ *are both finite and positive " then the theorem remains true with "* $\lim$ *" replaced by "*$\liminf$*".*

Theorem 2.5. *Let f and g be any two entire functions with* $0 < \lambda_f^{[m]L^*} \leq \rho_f^{[m]L^*} < \infty$ *where m is any positive integer and* $0 < \lambda_g^{L^*} \leq \rho_g^{L^*} < \infty$*. Then*

$$\limsup_{r\to\infty} \frac{\log^{[m+1]} \mu\left(r, f \circ g\right)}{\log^{[m]} \mu\left(r, f\right) + \log^{[3]} \mu\left(r, g\right) + L\left(\frac{1}{8}\mu\left(\frac{r}{4}, g\right) - |g(0)|\right)} \geq \frac{\rho_g^{L^*}}{\rho_f^{[m]L^*}}.$$

Proof. In view of Lemma 1.9, we have for all sufficiently large values of r

$$\log^{[m]} \mu\left(r, f \circ g\right) \geq o\left(1\right) + \log^{[m]} \mu\left(\frac{1}{8}\mu\left(\frac{r}{4}, g\right) - |g(0)|, f\right). \tag{2.5}$$

$$\textit{i.e.,} \ \log^{[m]} \mu\left(r, f \circ g\right) \geq o\left(1\right) +$$

$$\left(\lambda_f^{[m]L^*} - \varepsilon\right) \log\left\{\frac{1}{8}\mu\left(\frac{r}{4}, g\right) - |g(0)|\right\} + L\left(\frac{1}{8}\mu\left(\frac{r}{4}, g\right) - |g(0)|\right) \tag{2.6}$$

$$i.e.,\ \log^{[m]}\mu\left(r,f\circ g\right)\geq o\left(1\right)+\left(\lambda_f^{[m]L^*}-\varepsilon\right)$$
$$\left[\log\left\{\frac{1}{8}\mu\left(\frac{r}{4},g\right)\left(1-\frac{|g(0)|}{\frac{1}{8}\mu\left(\frac{r}{4},g\right)}\right)\right\}+L\left(\frac{1}{8}\mu\left(\frac{r}{4},g\right)-|g(0)|\right)\right]$$

$$i.e.,\ \log^{[m]}\mu\left(r,f\circ g\right)\geq\left(\lambda_f^{[m]L^*}-\varepsilon\right)\log\mu\left(\frac{r}{4},g\right).$$
$$\left\{\frac{\log\mu\left(\frac{r}{4},g\right)+\log\left(1-\frac{|g(0)|}{\frac{1}{8}\mu\left(\frac{r}{4},g\right)}\right)+L\left(\frac{1}{8}\mu\left(\frac{r}{4},g\right)-|g(0)|\right)}{\log\mu\left(\frac{r}{4},g\right)}\right\}$$

$$i.e.,\ \log^{[m+1]}\mu\left(r,f\circ g\right)\geq\log^{[2]}\mu\left(\frac{r}{4},g\right)$$
$$+\left(\frac{\lambda_g^{L^*}-\varepsilon}{\rho_f^{[m]L^*}+\varepsilon}\right)L\left(\frac{1}{8}\mu\left(\frac{r}{4},g\right)-|g(0)|\right)$$
$$-\log\left[\exp\left\{\left(\frac{\lambda_g^{L^*}-\varepsilon}{\rho_f^{[m]L^*}+\varepsilon}\right)L\left(\frac{1}{8}\mu\left(\frac{r}{4},g\right)-|g(0)|\right)\right\}\right]$$
$$+\log\left\{\frac{\left[\log\mu\left(\frac{r}{4},g\right)+L\left(\frac{1}{8}\mu\left(\frac{r}{4},g\right)-|g(0)|\right)\right]+o\left(1\right)}{\log\mu\left(\frac{r}{4},g\right)}\right\}$$

$$i.e.,\ \log^{[m+1]}\mu\left(r,f\circ g\right)\geq\log^{[2]}\mu\left(\frac{r}{4},g\right)$$
$$+\left(\frac{\lambda_g^{L^*}-\varepsilon}{\rho_f^{[m]L^*}+\varepsilon}\right)L\left(\frac{1}{8}\mu\left(\frac{r}{4},g\right)-|g(0)|\right)$$
$$+\log\left\{\frac{\left[\log\mu\left(\frac{r}{4},g\right)+L\left(\frac{1}{8}\mu\left(\frac{r}{4},g\right)-|g(0)|\right)\right]+o\left(1\right)}{\exp\left\{\left(\frac{\lambda_g^{L^*}-\varepsilon}{\rho_f^{[m]L^*}+\varepsilon}\right)L\left(\frac{1}{8}\mu\left(\frac{r}{4},g\right)-|g(0)|\right)\right\}\log\mu\left(\frac{r}{4},g\right)}\right\}$$

$$i.e.,\ \log^{[m+1]}\mu\left(r,f\circ g\right)\geq\log^{[2]}\mu\left(\frac{r}{4},g\right)+\left(\frac{\lambda_g^{L^*}-\varepsilon}{\rho_f^{[m]L^*}+\varepsilon}\right)L\left(\frac{1}{8}\mu\left(\frac{r}{4},g\right)-|g(0)|\right).\quad(2.7)$$

Now from (2.7) it follows for a sequence of values of r tending to infinity that

$$\log^{[m+1]}\mu\left(r,f\circ g\right)\geq\left(\rho_g^{L^*}-\varepsilon\right)\log\left\{\frac{r}{4}e^{L\left(\frac{r}{4}\right)}\right\}$$
$$+\left(\frac{\rho_g^{L^*}-\varepsilon}{\rho_f^{[m]L^*}+\varepsilon}\right)L\left(\frac{1}{8}\mu\left(\frac{r}{4},g\right)-|g(0)|\right).\qquad(2.8)$$

Now we get for all sufficiently large values of r that

$$\begin{aligned}\log^{[m]}\mu(r,f) &\leq \left(\rho_f^{[m]L^*}+\varepsilon\right)\log\left\{re^{L(r)}\right\}\\ i.e.,\ \log^{[m]}\mu(r,f) &\leq \left(\rho_f^{[m]L^*}+\varepsilon\right)\log\left\{\frac{r}{4}e^{L\left(\frac{r}{4}\right)}\right\}+\log 4. \qquad (2.9)\end{aligned}$$

Further it follows for all sufficiently large values of r that

$$\begin{aligned}\log^{[2]}\mu(r,g) &\leq \left(\rho_g^{L^*}+\varepsilon\right)\log\left\{re^{L(r)}\right\}\\ i.e.,\ \log^{[3]}\mu(r,g) &\leq \log^{[2]}\left\{re^{L(r)}\right\}+O(1). \qquad (2.10)\end{aligned}$$

Therefore from (2.9) and (2.10), we obtain for all sufficiently large values of r that

$$\log^{[m]}\mu(r,f)+\log^{[3]}\mu(r,g) \leq \left(\rho_f^{[m]L^*}+\varepsilon\right)\log\left\{\frac{r}{4}e^{L\left(\frac{r}{4}\right)}\right\}+ \log 4+\log^{[2]}\left\{re^{L(r)}\right\}+O(1). \qquad (2.11)$$

Hence from (2.8) and (2.11) it follows for all sufficiently large values of r that

$$\log^{[m+1]}\mu(r,f\circ g) \geq \left(\frac{\rho_g^{L^*}-\varepsilon}{\rho_f^{[m]L^*}+\varepsilon}\right)\left(\log^{[m]}\mu(r,f)+\log^{[3]}\mu(r,g)-\log 4-\log^{[2]}\left\{re^{L(r)}\right\}-O(1)\right) + \left(\frac{\rho_g^{L^*}-\varepsilon}{\rho_f^{[m]L^*}+\varepsilon}\right)L\left(\frac{1}{8}\mu\left(\frac{r}{4},g\right)-|g(0)|\right)$$

$$i.e.,\ \log^{[m+1]}\mu(r,f\circ g) \geq \left(\frac{\rho_g^{L^*}-\varepsilon}{\rho_f^{[m]L^*}+\varepsilon}\right)\left[\log^{[m]}\mu(r,f)+\log^{[3]}\mu(r,g)+L\left(\tfrac{1}{8}\mu\left(\tfrac{r}{4},g\right)-|g(0)|\right)\right] - \left(\frac{\rho_g^{L^*}-\varepsilon}{\rho_f^{[m]L^*}+\varepsilon}\right)\left(\log 4+\log^{[2]}\left\{re^{L(r)}\right\}+O(1)\right)$$

$$i.e.,\ \frac{\log^{[m+1]}\mu(r,f\circ g)}{\log^{[m]}\mu(r,f)+\log^{[3]}\mu(r,g)+L\left(\frac{1}{8}\mu\left(\frac{r}{4},g\right)-|g(0)|\right)} \geq \left(\frac{\rho_g^{L^*}-\varepsilon}{\rho_f^{[m]L^*}+\varepsilon}\right) - \frac{\left(\frac{\rho_g^{L^*}-\varepsilon}{\rho_f^{[m]L^*}+\varepsilon}\right)\left(\log 4+\log^{[2]}\left\{re^{L(r)}\right\}+O(1)\right)}{\log^{[m]}\mu(r,f)+L\left(\frac{1}{8}\mu\left(\frac{r}{4},g\right)-|g(0)|\right)}. \qquad (2.12)$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from (2.12) that

$$\limsup_{r\to\infty}\frac{\log^{[m+1]}\mu(r,f\circ g)}{\log^{[m]}\mu(r,f)+\log^{[3]}\mu(r,g)+L\left(\frac{1}{8}\mu\left(\frac{r}{4},g\right)-|g(0)|\right)} \geq \frac{\rho_g^{L^*}}{\rho_f^{[m]L^*}}\ .$$

Thus the theorem is established. □

In the line of Theorem 2.5, the following theorem can be proved:

Theorem 2.6. *Let f and g be any two entire functions with $0 < \lambda_f^{[m]L^*} \leq \rho_f^{[m]L^*} < \infty$ where $m \geq 1$ and $0 < \lambda_g^{L^*} \leq \rho_g^{L^*} < \infty$. Then*

$$\liminf_{r\to\infty} \frac{\log^{[m+1]} \mu\left(r, f\circ g\right)}{\log^{[m]} \mu\left(r, f\right) + \log^{[3]} \mu\left(r, g\right) + L\left(\frac{1}{8}\mu\left(\frac{r}{4}, g\right) - |g(0)|\right)} \geq \frac{\lambda_g^{L^*}}{\rho_f^{[m]L^*}}.$$

The proof is omitted.

Theorem 2.7. *Let f and g be any two entire functions with $0 < \lambda_f^{[m]L^*} \leq \rho_f^{[m]L^*} < \infty$ where m is any positive integer and $0 < \lambda_g^{L^*} \leq \rho_g^{L^*} < \infty$. Then*

$$\limsup_{r\to\infty} \frac{\log^{[m+1]} M\left(r, f\circ g\right)}{\log^{[m]} M\left(r, f\right) + \log^{[3]} M\left(r, g\right) + L\left(\frac{1}{8}M\left(\frac{r}{2}, g\right) - |g(0)|\right)} \geq \frac{\rho_g^{L^*}}{\rho_f^{[m]L^*}}.$$

Theorem 2.8. *Let f and g be any two entire functions such that $0 < \lambda_f^{[m]L^*} \leq \rho_f^{[m]L^*} < \infty$ where $m \geq 1$ and $0 < \lambda_g^{L^*} \leq \rho_g^{L^*} < \infty$. Then*

$$\liminf_{r\to\infty} \frac{\log^{[m+1]} M\left(r, f\circ g\right)}{\log^{[m]} M\left(r, f\right) + \log^{[3]} M\left(r, g\right) + L\left(\frac{1}{8}M\left(\frac{r}{2}, g\right) - |g(0)|\right)} \geq \frac{\lambda_g^{L^*}}{\rho_f^{[m]L^*}}.$$

We omit the proofs of Theorem 2.7 and Theorem 2.8 because those can be carried out in the line of Theorem 2.5 and Theorem 2.6 respectively and with the help of Lemma 1.10

Theorem 2.9. *Let f and g be any two entire functions with $\rho_f^{[n]L^*} < \infty$, $\rho_g^{[p]L^*} < \infty$ and $\lambda_{f\circ g}^{[m]L^*} = \infty$ where m, n and p are positive integers. Then*

$$\lim_{r\to\infty} \frac{\log^{[m]} \mu(r, f\circ g)}{\log^{[n]} \mu(r, f) + \log^{[p]} \mu(r, g)} = \infty.$$

Proof. Let us suppose that the conclusion of the theorem do not hold. Then we can find a constant $\beta > 0$ such that for a sequence of values of r tending to infinity

$$\log^{[m]} \mu(r, f\circ g) \leq \beta \left[\log^{[n]} \mu(r, f) + \log^{[p]} \mu(r, g)\right]. \tag{2.13}$$

Again from the definition of $\rho_f^{[n]L^*}$, it follows that for all sufficiently large values of r that

$$\log^{[n]} \mu(r, f) \leq \left(\rho_f^{[n]L^*} + \varepsilon\right) \log\left(re^{L(r)}\right), \tag{2.14}$$

and from the definition of $\rho_g^{[p]L^*}$, it follows that for all sufficiently large values of r that

$$\log^{[p]} \mu(r, g) \leq \left(\rho_g^{[p]L^*} + \varepsilon\right) \log\left(re^{L(r)}\right). \tag{2.15}$$

Thus from (2.13), (2.14) and (2.15) we have for a sequence of values of r tending to infinity that

$$\log^{[m]} \mu(r, f \circ g) \leq \beta \left[\left(\rho_f^{[n]L^*} + \varepsilon \right) \log \left(re^{L(r)} \right) + \left(\rho_g^{[p]L^*} + \varepsilon \right) \log \left(re^{L(r)} \right) \right]$$

$$i.e., \ \frac{\log^{[m]} \mu(r, f \circ g)}{\log \left(re^{L(r)} \right)} \leq \frac{\beta \left[\left(\rho_f^{[n]L^*} + \varepsilon \right) + \left(\rho_g^{[p]L^*} + \varepsilon \right) \right] \log \left(re^{L(r)} \right)}{\log \left(re^{L(r)} \right)}$$

$$i.e., \ \liminf_{r \to \infty} \frac{\log^{[m]} \mu(r, f \circ g)}{\log \left(re^{L(r)} \right)} = \lambda_{f \circ g}^{[m]L^*} < \infty.$$

This is a contradiction. Thus the theorem follows. □

In the line of Theorem 2.9, the following theorem may also be proved:

Remark 2.10. *Theorem 2.9 is also valid with "limit superior" instead of "limit" if* $\lambda_{f \circ g}^{[m]L^*} = \infty$ *is replaced by* $\rho_{f \circ g}^{[m]L^*} = \infty$ *and the other conditions remaining the same.*

Theorem 2.11. *Let f and g be any two entire functions with* $\rho_f^{[n]L^*} < \infty$, $\rho_g^{[p]L^*} < \infty$ *and* $\lambda_{f \circ g}^{[m]L^*} = \infty$ *where m, n and p are positive integers. Then*

$$\lim_{r \to \infty} \frac{\log^{[m]} M(r, f \circ g)}{\log^{[n]} M(r, f) + \log^{[p]} M(r, g)} = \infty.$$

Further if $\rho_{f \circ g}^{[m]L^*} = \infty$ *instead of* $\lambda_{f \circ g}^{[m]L^*} = \infty$ *then*

$$\liminf_{r \to \infty} \frac{\log^{[m]} M(r, f \circ g)}{\log^{[n]} M(r, f) + \log^{[p]} M(r, g)} = \infty.$$

Corollary 2.12. *Under the assumptions of Theorem 2.9 or Remark 2.10 and Theorem 2.11,*

$$\lim_{r \to \infty} \frac{\log^{[m-1]} \mu(r, f \circ g)}{\log^{[n-1]} \mu(r, f) \cdot \log^{[p-1]} \mu(r, g)} = \infty$$

and

$$\lim_{r \to \infty} \frac{\log^{[m-1]} M(r, f \circ g)}{\log^{[n-1]} M(r, f) \cdot \log^{[p-1]} M(r, g)} = \infty.$$

Proof. By Theorem 2.9 or Remark 2.10 we obtain for all sufficiently large values of r and for $K > 1$ that

$$\log^{[m]} \mu(r, f \circ g) > K \left[\log^{[n]} \mu(r, f) + \log^{[p]} \mu(r, g) \right]$$

$$i.e., \ \log^{[m-1]} \mu(r, f \circ g) > \left[\log^{[n-1]} \mu(r, f) \cdot \log^{[p-1]} \mu(r, g) \right]^K,$$

from which the first part of the corollary follows.

Similary, from Theorem 2.11 the second part of the corollary is established. □

Theorem 2.13. *If f and g be any two entire functions such that* (i) $0<\rho_f^{[n]L^*}<\infty$, (ii) $0<\sigma_f^{[n-1]L^*}<\infty$, (iii) $\rho_{f\circ g}^{[m]L^*}=\rho_f^{[n]L^*}$ *and* (iv) $\sigma_{f\circ g}^{[m-1]L^*}<\infty$. *Then for any* $\beta>1$,

$$\liminf_{r\to\infty}\frac{\log^{[m-1]}\mu\left(r,f\circ g\right)+\log^{[n]}\mu(r,f)}{\log^{[n-1]}\mu\left(r,f\right)} \leq \frac{\beta^{\rho_f^{[n]L^*}}\sigma_{f\circ g}^{[m-1]L^*}}{\sigma_f^{[n-1]L^*}} \text{ and}$$

$$\frac{\sigma_{f\circ g}^{[m-1]L^*}}{\beta^{\rho_{f\circ g}^{[m]L^*}}\sigma_f^{[n-1]L^*}} \leq \limsup_{r\to\infty}\frac{\log^{[m-1]}\mu(r,f\circ g)}{\log^{[n-1]}\mu\left(r,f\right)+\log^{[n]}\mu(r,f)}.$$

Proof. From the definition of generalised L^*-type and in view of the inequality $\mu\left(r,f\right)\leq M\left(r,f\right)$ $\{cf.\ [5]\}$, we obtain for all sufficiently large values of r that

$$\log^{[m-1]}\mu\left(r,f\circ g\right) \leq \log^{[m-1]}M\left(r,f\circ g\right) \leq \left(\sigma_{f\circ g}^{[m-1]L^*}+\varepsilon\right)\left\{re^{L(r)}\right\}^{\rho_{f\circ g}^{[m]L^*}} \qquad (2.16)$$

and

$$\log^{[n-1]}\mu\left(r,f\right)\leq\left(\sigma_f^{[n-1]L^*}+\varepsilon\right)\left\{re^{L(r)}\right\}^{\rho_f^{[n]L^*}}. \qquad (2.17)$$

Also taking $R=\beta r$ in the inequality $M\left(r,f\right)\leq\frac{R}{R-r}\mu\left(R,f\right)$ $\{cf.\ [5]\}$ we obtain for a sequence of values of r tending to infinity that

$$\log^{[m-1]}\mu\left(r,f\circ g\right)\geq\log^{[m-1]}M\left(\frac{r}{\beta},f\circ g\right)+O(1) \geq\left(\sigma_{f\circ g}^{[m-1]L^*}-\varepsilon\right)\left\{\left(\frac{r}{\beta}\right)e^{L\left(\frac{r}{\beta}\right)}\right\}^{\rho_{f\circ g}^{[m]L^*}}$$

$$\text{i.e., }\log^{[m-1]}\mu\left(r,f\circ g\right)\geq\frac{\left(\sigma_{f\circ g}^{[m-1]L^*}-\varepsilon\right)}{\beta^{\rho_{f\circ g}^{[m]L^*}}}\left\{re^{L(r)}\right\}^{\rho_{f\circ g}^{[m]L^*}}+O(1) \qquad (2.18)$$

and

$$\log^{[n-1]}\mu\left(r,f\right)\geq\log^{[n-1]}M\left(\frac{r}{\beta},f\right)+O(1) \geq\left(\sigma_f^{[n-1]L^*}-\varepsilon\right)\left\{\left(\frac{r}{\beta}\right)e^{L\left(\frac{r}{\beta}\right)}\right\}^{\rho_f^{[n]L^*}}$$

$$\text{i.e., }\log^{[n-1]}\mu\left(r,f\right)\geq\frac{\left(\sigma_f^{[n-1]L^*}-\varepsilon\right)}{\beta^{\rho_f^{[n]L^*}}}\left\{re^{L(r)}\right\}^{\rho_f^{[n]L^*}}+O(1). \qquad (2.19)$$

Now from (2.14), (2.16) and (2.19) it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[m-1]}\mu\left(r,f\circ g\right)+\log^{[n]}\mu(r,f)}{\log^{[n-1]}\mu\left(r,f\right)}\leq$$
$$\frac{\beta^{\rho_f^{[n]L^*}}\left(\sigma_{f\circ g}^{[m-1]L^*}+\varepsilon\right)\left\{re^{L(r)}\right\}^{\rho_{f\circ g}^{[m]L^*}}+\left(\rho_f^{[n]L^*}+\varepsilon\right)\log\left(re^{L(r)}\right)}{\left(\sigma_f^{[n-1]L^*}-\varepsilon\right)\left\{re^{L(r)}\right\}^{\rho_f^{[n]L^*}}+O(1)}. \quad (2.20)$$

In view of the condition (iii) we get from (2.20) that

$$\liminf_{r\to\infty}\frac{\log^{[m-1]}\mu\left(r,f\circ g\right)+\log^{[n]}\mu(r,f)}{\log^{[n-1]}\mu\left(r,f\right)}\leq\frac{\beta^{\rho_f^{[n]L^*}}\left(\sigma_{f\circ g}^{[m-1]L^*}+\varepsilon\right)}{\left(\sigma_f^{[n-1]L^*}-\varepsilon\right)}.$$

As $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\liminf_{r\to\infty}\frac{\log^{[m-1]}\mu\left(r,f\circ g\right)+\log^{[n]}\mu(r,f)}{\log^{[n-1]}\mu\left(r,f\right)}\leq\frac{\beta^{\rho_f^{[n]L^*}}\cdot\sigma_{f\circ g}^{[m-1]L^*}}{\sigma_f^{[n-1]L^*}}. \quad (2.21)$$

Again from (2.14), (2.17) and (2.18) we get for a sequence of values of r tending to infinity that

$$\frac{\log^{[m-1]}\mu\left(r,f\circ g\right)}{\log^{[n-1]}\mu\left(r,f\right)+\log^{[n]}\mu(r,f)}\geq$$
$$\frac{\left(\sigma_{f\circ g}^{[m-1]L^*}-\varepsilon\right)\left\{re^{L(r)}\right\}^{\rho_{f\circ g}^{[m]L^*}}+O(1)}{\beta^{\rho_{f\circ g}^{[m]L^*}}\left(\sigma_f^{[n-1]L^*}+\varepsilon\right)\left\{re^{L(r)}\right\}^{\rho_f^{[n]L^*}}+\left(\rho_f^{[n]L^*}+\varepsilon\right)\log\left(re^{L(r)}\right)}. \quad (2.22)$$

Since $\rho_{f\circ g}^{[m]L^*}=\rho_f^{[n]L^*}$, we obtain from (2.22) that

$$\limsup_{r\to\infty}\frac{\log^{[m-1]}\mu\left(r,f\circ g\right)}{\log^{[n-1]}\mu\left(r,f\right)+\log^{[n]}\mu(r,f)}\geq$$
$$\frac{\left(\sigma_{f\circ g}^{[m-1]L^*}-\varepsilon\right)}{\beta^{\rho_{f\circ g}^{[m]L^*}}\left(\sigma_f^{[n-1]L^*}+\varepsilon\right)\left(1+\frac{\left(\rho_f^{[n]L^*}+\varepsilon\right)\log\left(re^{L(r)}\right)}{\beta^{\rho_{f\circ g}^{[m]L^*}}\left(\sigma_f^{[n-1]L^*}+\varepsilon\right)\left\{re^{L(r)}\right\}^{\rho_f^{[n]L^*}}}\right)}.$$

As $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty}\frac{\log^{[m-1]}\mu\left(r,f\circ g\right)}{\log^{[n-1]}\mu\left(r,f\right)+\log^{[n]}\mu(r,f)}\geq\frac{\sigma_{f\circ g}^{[m-1]L^*}}{\beta^{\rho_{f\circ g}^{[m]L^*}}\cdot\sigma_f^{[n-1]L^*}}. \quad (2.23)$$

Thus the theorem follows from (2.21) and (2.23). □

In the line of Theorem 2.13, we may state the following theorem without proof:

Theorem 2.14. *If f and g be any two entire functions with (i) $0 < \rho_g^{[n]L^*} < \infty$, (ii) $0 < \sigma_g^{[n-1]L^*} < \infty$, (iii) $\rho_{f\circ g}^{[m]L^*} = \rho_g^{[n]L^*}$ and (iv) $\sigma_{f\circ g}^{[m-1]L^*} < \infty$. Then for any $\beta > 1$,*

$$\liminf_{r\to\infty} \frac{\log^{[m-1]} \mu(r, f\circ g) + \log^{[n]} \mu(r,g)}{\log^{[n-1]} \mu(r,g)} \leq \frac{\beta^{\rho_g^{[n]L^*}} \sigma_{f\circ g}^{[m-1]L^*}}{\sigma_g^{[n-1]L^*}} \text{ and}$$

$$\frac{\sigma_{f\circ g}^{[m-1]L^*}}{\beta^{\rho_{f\circ g}^{[m]L^*}} \sigma_g^{[n-1]L^*}} \leq \limsup_{r\to\infty} \frac{\log^{[m-1]} \mu(r, f\circ g)}{\log^{[n-1]} \mu(r,g) + \log^{[n]} \mu(r,g)}.$$

Theorem 2.15. *If f and g be any two entire functions such that (i) $0 < \rho_f^{[n]L^*} < \infty$, (ii) $0 < \sigma_f^{[n-1]L^*} < \infty$, (iii) $\rho_{f\circ g}^{[m]L^*} = \rho_f^{[n]L^*}$ and (iv) $\sigma_{f\circ g}^{[m-1]L^*} < \infty$. Then*

$$\liminf_{r\to\infty} \frac{\log^{[m-1]} M(r, f\circ g) + \log^{[n]} \mu(r,f)}{\log^{[n-1]} M(r,f)} \leq \left(\frac{\sigma_{f\circ g}^{[m-1]L^*}}{\sigma_f^{[n-1]L^*}}\right)$$

$$\leq \limsup_{r\to\infty} \frac{\log^{[m-1]} M(r, f\circ g)}{\log^{[n-1]} M(r,f) + \log^{[n]} \mu(r,f)}.$$

Theorem 2.16. *If f and g be any two entire fucntions such that (i) $0 < \rho_g^{[n]L^*} < \infty$, (ii) $0 < \sigma_g^{[n-1]L^*} < \infty$, (iii) $\rho_{f\circ g}^{[m]L^*} = \rho_g^{[n]L^*}$ and (iv) $\sigma_{f\circ g}^{[m-1]L^*} < \infty$. Then*

$$\liminf_{r\to\infty} \frac{\log^{[m-1]} M(r, f\circ g) + \log^{[n]} \mu(r,g)}{\log^{[n-1]} M(r,g)} \leq \left(\frac{\sigma_{f\circ g}^{[m-1]L^*}}{\sigma_g^{[n-1]L^*}}\right)$$

$$\leq \limsup_{r\to\infty} \frac{\log^{[m-1]} M(r, f\circ g)}{\log^{[n-1]} M(r,g) + \log^{[n]} \mu(r,g)}.$$

The proof of Theorem 2.15 and Theorem 2.16 are omitted because those can be carried out in the line of Theorem 2.13 and Theorem 2.14 respectively.

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