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On the Properties of Certain Commuting Squares and Their Corresponding Limiting Algebras

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Abstract : In his celebrated article [1], V. Jones introduced Index theory of subfactors, which is called Jones Index theory to his honor. In this article, he showed that to any type Π_1 subfactors, $A \subset B$ corresponds a number $[A : B]$, which is independent from the Hilbert space on which the above subfactors act upon. He proved that for values of $[A : B]$ less than 4 the values of index are given by the following set of numbers, $4\cos^2(\pi/n)$, $n = 1, 2, ...$

For a given subfactors $B_1 \subset B_2$, Jones introduced a construction in term of extending the above inclusion into the tower of subfactors, $B_1 \subset B_2 \subset ... \subset B_k \subset ... \subset B_\infty$, which is called *Jones tower.* One of the main tools to construct subfactors is called *commuting square.* A commuting square consists of four finite dimensional C^* algebras that satisfy the following geometry,

$$\begin{array}{ll} B_{2,1} & \subset B_{2,2} \\ \bigcup & \ \ \bigcup \\ B_{1,1} & \subset B_{1,2} \end{array}$$

If the commuting square is non-degenerate and equipped with Markov trace then we can extend it vertically using Jones construction to get type Π_1 limiting algebras $B_1 \subset B_2$. Now using Jones construction on the subfactors $B_1 \subset B_2$, we get the tower $B_1 \subset B_2 \subset ... \subset B_k \subset ... \subset B_\infty$. Next let us define the following algebras, $D_i = (B_i)' \bigcap B_\infty = (x \in B_\infty, x \text{ commute with } B_i)$, $i = 1, 2$. then we say that the graph of the inclusion $B_1 \subset B_2$ is Ergodic if the algebras D_1 and D_2 are factors, i.e., have trivial centers. We say that the inclusion $B_1 \subset B_2$ is strongly

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amenable if the exist a von Neumann algebra isomorphism taking the subfactors $B_1 \subset B_2$ onto the subfactors $D_2 \subset D_1$. Also usually we represent D_1 by M^{st} and D_2 by N^{st}. Note that in any case we always have, $[D_2 : D_1]$=$[B_1 : B_2]$. Now keeping the same notations as in the above, given a non-degenerate commuting square, we are going to show that the corresponding induced Π_1 subfactors $B_2 \supset B_1$, have Ergodic principal and dual graphs. Furthermore, in [2] we showed that if the induced subfactors index fall within certain interval then their inclusion is either strongly amenable or the inclusion of corresponding derived subalgebras $D_2 \subset D_1$ is isomorphic to Jones subfactors. In the later case we show that the corresponding graphs to the higher relative commutants of the above inclusion, Γ_{B_1,B_2}, have norms equal to 2. This extends the results of U. Haagerup in [3], Scott Morrison and Noah Snyder in [4], stating that the only infinite depth principal graphs corresponding to subfactors with their indices located in the interval (4, 5) are A_∞ graphs.

In this article we are going to introduce certain class of commuting squares such that their corresponding limiting subfactors are strongly amenable. This class contains the set of all symmetric commuting squares. In particular in Corolary 2.14 we prove that the set of strongly amenable subfactors is very large. Finally in this article we introduce new and simple methods to solve some interesting problems of Jones index theory.

Keywords : subfactors, von Neumann algebras, Jones index, lattice, relative commutants.
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1 Introduction and Preliminaries

In [5] and [6], V. Jones used results from index theory to find new polynomial invariant for Link and knots. Later on Edward Witten in [7] applied Jones polynomials to Quantum field theory and introduced polynomial invariant for three manifold.

In [8] A. Ocneanu shows that there has been link from subfactors to Coxter ADE graphs, Lie algebras, Lie groups and quantum Lie groups. He explain that the study of group like invariants of finite depth subfactors and quantum groups is now in situation similar to the study of simple Lie algebras a century ago. Further study of these structures is motivated from physics of quantum field theory and quantum gravity and application to quantum physics.

Since the advances of Jones Index theory motivated the need to answer unknown questions regarding irreducible subfactors of finite index. For example finding the set $IRRH$ of values of index for hyperfinite irreducible subfactors, with index larger than 4 is still and open problem. In Corollary 4.5 [9], S. Popa showed that $IRRH$ contains a gap between 4 to 4.026. In [10], the authors prove that $IRRH \supset [37.0037]$. Scope of this field of research is immense and at this

article we try to answer few of the open problems.
As we mentioned in the above one of the main tools to analyze and to construct a pair of irreducible type II_1 subfactors is called *commuting square*. They were first introduced by S. Popa in [11]. In this article we are going to use commuting squares to show some important properties of the corresponding induced subfactors.

Suppose we are Given a following non-degenerate commuting square

$$\begin{array}{ll} B_{2,1} & \subset B_{2,2} \\ \bigcup & \quad \bigcup \\ B_{1,1} & \subset B_{1,2}. \end{array} \tag{1}$$

For a given inclusion of finite dimensional C^* algebras $A \subset B$, let T_A^B, be the matrix representation of the above inclusion with $\|T_A^B\|$, the norm of the linear operator T_A^B. Then the fact that (1) is non-degenerate implies that matrices $T = T_{B_{1,1}}^{B_{2,1}}$, $S = T_{B_{1,1}}^{B_{1,2}}$, $G = T_{B_{1,2}}^{B_{2,2}}$, $L = T_{B_{2,1}}^{B_{2,2}}$, are indecomposable with $\|T\| = \|G\|$ and $\|S\| = \|L\|$. Throughout this article the unique normal faithful normalized trace on a type II_1 factor M is represented by tr_M. In particular if M is a limiting algebra corresponding to periodic tower of finite dimensional algebras, $(B_i \subset B_{i+1})_{i=1}^{\infty}$ with $T_{B_1}^{B_2}$ indecomposable, then tr_M is the The Markov trace corresponding to the inclusion $B_1 \subset B_2$. For the definitions of commuting square, Markov trace, Jones tower, and other preliminaries see for instance [1, 12–15]. Now by the standard arguments in [13]. We can extend (1), upward using the basic construction on the pair $B_{1,1} \subset B_{2,1}$, to get the following tower of commuting squares

$$\begin{array}{ll} B_1 & \subset B_2 \\ \bigcup & \quad \bigcup \\ \vdots & \quad \vdots \\ \bigcup & \quad \bigcup \\ B_{k,1} & \subset B_{k,2} \\ \bigcup & \quad \bigcup \\ \vdots & \quad \vdots \\ \bigcup & \quad \bigcup \\ B_{3,1} & \subset B_{3,2} \\ \bigcup & \quad \bigcup \\ B_{2,1} & \subset B_{2,2} \\ \bigcup & \quad \bigcup \\ B_{1,1} & \subset B_{1,2} \end{array} \tag{2}$$

with $B_{3,2} = < B_{2,2},\, e_{B_{2,2}} >$, $B_{3,1} = < B_{2,1},\, e_{B_{2,2}} >$ proceeding inductively, for any integer k larger than 3, set, $B_{k,2} = < B_{k-1,2},\, e_{B_{k-2,2}} >$, $B_{k,1} = < B_{k-1,1},\, e_{B_{k-2,2}} >$, where $e_{B_{k-2,2}}$ is the projection corresponding to the action of $B_{k-2,2}$ on $L^2_{(B_{k-1,2},tr_{B_2})}$. Let us set new and more convenient names for the Jones projections in the above, $f_1 = e_{B_{1,2}}, f_2 = e_{B_{2,2}}, ..., f_k = e_{B_{k,2}}$, then we have $B_{k,2} =< B_{k-1,2}, f_{k-2} >$ for each integer k that is equal or larger than 3. Also note that

B_1 and B_2 are the limiting algebras of the towers $\{B_{k,1}\}$ and $\{B_{k,2}\}$ respectively. Next we can extend the inclusion $B_1 \subset B_2$ right ward using the basic construction to get the tower,

$$B_1 \subset B_2 \subset B_3 =< B_2, e_{B_1} >\subset \ldots \subset< B_k, e_{B_{k-1}} >= B_{k+1} \subset \ldots \subset B^{\infty}$$

where B^{∞} is the limiting algebra of the above tower. Furthermore for each $n \geq 1$, we have the following tower of finite dimensional algebras.

$$B_{n,1} \subset B_{n,2} \subset B_{n,3} =< B_{n,2}, e_{B_1} >\subset \ldots \subset B_{n,k} =< B_{n-1,k}, e_{B_{k-2}} >\subset \ldots \subset B^n$$

with B^n the limiting algebra of the above tower, which is isomorphic to the Jones tower induced from the finite algebras inclusion $B_{n,1} \subset B_{n,2}$ by standard arguments. Now in order to facilitate our notations, we rename the above Jones projections as in the following,

$$e_1 = e_{B_1}, e_2 = e_{B_2}, ..., e_k = e_{B_k}, ...$$

The above construction will provide us with the following Jones system of commuting squares i.e., each of the horizontal and vertical towers are isomorphic to Jones tower.

Therefore we get the following tower of commuting squares

$$\begin{array}{ccccccccc}
B_1 \subset & B_2 & \subset & \ldots & \subset & B_k & \subset & \ldots & \subset & B^{\infty} \\
\cup & \cup & & & & \cup & & & & \cup \\
\vdots & \vdots & & & & \vdots & & & & \vdots \\
\cup & \cup & & & & \cup & & & & \cup \\
B_{n,1} \subset & B_{n,2} & \subset & \ldots & \subset & B_{n,k} & \subset & \ldots & \subset & B^{n} \\
\cup & \cup & & & & \cup & & & & \cup \\
\vdots & \vdots & & & & \vdots & & & & \vdots \\
\cup & \cup & & & & \cup & & & & \cup \\
B_{2,1} \subset & B_{2,2} & \subset & \ldots & \subset & B_{2,k} & \subset & \ldots & \subset & B^{2} \\
\cup & \cup & & & & \cup & & & & \cup \\
B_{1,1} \subset & B_{1,2} & \subset & \ldots & \subset & B_{1,k} & \subset & \ldots & \subset & B^{1}
\end{array} \tag{3}$$

In particular B^1 is the limiting algebra of the tower, $B_{1,1} \subset B_{1,2} \subset \ldots B_{1,k} \subset \ldots \subset B^1$.

Finally, we have $B_{k,2} =< B_{k-1,2}, f_{k-2} >$ and $B_{n,k} =< B_{n,k-1}, e_{k-2} >$ In the process of writing this article we use perturbation technics frequently. These technics are mainly based on the results of E. Christensen [14], A. Ocneanu [16]. In the last section of this work some open problems have been addressed, with partial solution provided. The that we use here are simple based on S. Popa's work.

2 On Certain Properties of Commuting Squares

In this section we show some basic properties of non-degenerate commuting squares. In Lemma 2.2 we prove that if the commuting square satisfies certain conditions, the we can perform downward construction on the above commuting square. Next we will define a concept of symmetric and semi semi-symmetric commuting squares. We will show that for a given non-degenerate commuting square and we can construct a corresponding semi-symmetric commuting square.

Definition 2.1. Set $T = T_{B_{1,1}}^{B_{2,1}}$, $S = T_{B_{1,1}}^{B_{1,2}}$. We call a *commuting square* in diagram (1) symmetric if after certain permutation of vertices of $B_{2,1}$ or $B_{1,2}$. We get $S = T$. We call it *semi-symmetric* if the norm of S is equal the norm of T.

Lemma 2.2. *Considering the diagram (3), suppose for some $k_0 \geq 1$, B_{1,k_0} contains a subalgebra B_{0,k_0}, such that $T_{B_{0,k_0}}^{B_{1,k_0}} = T^*$. Then there exists a projection f in B_{2,k_0}, with $E_{B_{1,k_0}}(f) = \lambda = \frac{1}{[B^2:B^1]}$. Furthermore if $B_{0,k} = \{f\}' \cap B_{1,k}$, with $k \geq k_0$, then the Tower $\{B_{0,k}\}_{k=k_0}^{\infty}$ is a periodic Jones tower.*

Proof. Suppose B_{1,k_0} is acting on $H = L^2_{(B_{k-1,2},tr)}$, where tr is the canonical trace acting on B^∞. Let $e_{B_{0,k_0}}$ be the projection onto the the subspace of H that is generated by B_{0,k_0}. But the inclusions, $< B_{1,k}, e_{B_{0,k}} > \supset B_{1,k}$ and $B_{2,k_0} \supset B_{1,k_0}$ have the same Bratteli diagram. Therefore once we equip $< B_{1,k}, e_{B_{0,k}} >$, with the Markov trace corresponding to the inclusion, $< B_{1,k}, e_{B_{0,k}} > \supset B_{1,k}$ the arguments in [13] and [1] imply the existence of trace preserving isomorphism taking $B_{2,k}$ onto $< B_{1,k}, e_{B_{0,k}} >$. Hence there exists a projection f in B_{2,k_0} such that $E_{B_{1,k_0}}(f) = \lambda$ and $B_{0,k} = \{f\}' \cap B_{1,k}$. Now set $B^0 = \{f\}' \cap B^1$, then by Corollary 1.8 [12], B^0, is a result of downward construction on the pair $B^2 \supset B^1$. Hence it is easy to see that the following diagram,

$$\begin{array}{ll} B_{2,k} & \subset B^2 \\ \bigcup & \quad \bigcup \\ B_{1,k} & \subset B^1 \\ \bigcup & \quad \bigcup \\ B_{0,k} & \subset B^0 \end{array} \tag{4}$$

is a system of commuting square for each $k \geq k_0$. But, $\|T^*\|^2 = [B^2 : B^1]$. This implies that the following system

$$\begin{array}{ccccccc} B_{2,k_0} & \subset \ldots \subset & B_{2,k} & \subset \ldots \subset & B^2 \\ \bigcup & & \bigcup & & \bigcup \\ B_{1,k_0} & \subset \ldots \subset & B_{1,k} & \subset \ldots \subset & B^1 \\ \bigcup & & \bigcup & & \bigcup \\ B_{0,k_0} & \subset \ldots \subset & B_{0,k} & \subset \ldots \subset & B^0 \end{array} \tag{5}$$

is a periodic tower of commuting squares. Finally by Ocneanu's (5.7) [16], $B_{0,k}$ contains $e_{B_{k-2}}$, for $k \geq k_0 + 2$, which implies that system (5) is a periodic Jones Tower. $\square$

Lemma 2.3. *Without loss of generality we can assume that any non-degenerate system of commuting square is semi-symmetric.*

Proof. Consider the following graph

$$\begin{array}{ccc} B_{2,1} \bigotimes B_{1,3} & \subset B_{2,2} \bigotimes B_{2,3} \\ \bigcup & \bigcup \\ B_{1,1} \bigotimes B_{1,2} & \subset B_{1,2} \bigotimes B_{2,2} \end{array} \tag{6}$$

Let tr be the canonocal trace on B^∞ let $Tr = tr \bigotimes tr$ be the canonical trace acting on $B^\infty \bigotimes B^\infty$ then with respect to Tr, (6) becomes a commuting square and Tr, a normalized Markov trace acting on the commuting square. But it is easy to see that the inclusion matrix L of $B_{1,1} \bigotimes B_{1,2} \subset B_{1,2} \bigotimes B_{2,2}$ has the same norm as the inclusion matrix Q of $B_{1,1} \bigotimes B_{1,2} \subset B_{2,1} \bigotimes B_{1,3}$, which is equal to multiplication of the norms of S and T. This implies that the commuting square (6) is non-degenerate semi-symmetric commuting square. Extending the commuting square (6), upward and rightward using Jones basic construction, we get a Jones system of commuting square analog to system (3). It is easy to check that the $k'th$ horizontal limiting algebra(respectively, $k'th$ vertical limiting algebra) will be equal to $B^k \bigotimes B_k$(respectively will be equal to $B_k \bigotimes B^k$). □

Extending the commuting square (6) upward and rightward by Jones basic construction,we can assume without loss of generality that $T = T^{B_{2,1}}_{B_{1,1}}$ is symmetric and is equal to $T^{B_{2,2}}_{B_{1,2}}$, and $S = T^{B_{1,2}}_{B_{1,1}} = T^{B_{2,2}}_{B_{2,1}}$.

Lemma 2.4. *Keeping the same notations as in the above.Any commuting square that satisfies the above properties will associates to a symmetric commuting square.*

Proof. Let $B_{3,3}$ be as in diagram (3). Then $B_{3,3} \bigotimes B_{3,3}$ contains all algebras of diagram (6). Let $H = L^2(B_{3,3} \bigotimes B_{3,3}, Tr)$ with tr as defined in the above. And suppose $B_{3,3}$ acts standardly on H. Let h in H, be acyclic and separating vector for the algebra $B_{3,3} \bigotimes B_{3,3}$ acting on H. Let us define an equivalence relation EQ on H as in the following. For x, y in $B_{3,3}$, we identify $(x \bigotimes y)h$ with $(y \bigotimes x)h$. Let $K = H/EQ$ be a Banach space constructed using standard arguments. It is easy to check that K becomes a Hilbert space with the inner product inherited from H. Let us denote, $L_{1,1} = B_{1,1} \bigotimes B_{1,2}$, $L_{1,2} = B_{1,2} \bigotimes B_{2,2}$, $L_{2,1} = B_{2,1} \bigotimes B_{1,3}$ and $L_{2,2} = B_{2,2} \bigotimes B_{2,3}$. Then the following commuting square,

$$\begin{array}{cc} L_{2,1} & \subset L_{2,2} \\ \bigcup & \bigcup \\ L_{1,1} & \subset L_{1,2} \end{array} \tag{7}$$

equipped with $tr_1 = Tr/EQ$ is a non-degenerate symmetric commuting square. In particular it is easy to check the following properties, tr_1 is a Markov trace on the commuting square (7) and furthermore tr_1 is a perron Frobenius vector for T^*T, with the eigenvalue equal to $(\lambda_1)^{-1}(\lambda_2)^{-1}$, where $\lambda_1^{-1} = [B^2 : B^1]$ and $\lambda_2^{-1} = [B_2 : B_1]$. □

Now let $L_1 \subset L_2$ and $L^1 \subset L^2$ be the limiting algebras corresponding to the commuting square (7). Then the fact that the limiting algebras inclusions corresponding to commuting square (1) are irreducible so are the limiting algebras inclusions corresponding to the commuting square (7). We get the following corollary.

Corollary 2.5. *Keeping the same notations as in the above the inclusions* $L_1 \subset L_2$ *and* $L^1 \subset L^2$ *are irreducible with* $[L_2 : L_1] = [L^2 : L^1] = (\lambda_1)^{-1}(\lambda_2)^{-1}$.

To proceed we need to state the following definition.

Definition 2.6. The commuting square (1) is called the *source for the system of commuting squares* (3).

Now consider the system (3) and the corresponding limiting algebras $B^2 \supset B^1$. then by the results of (3.1.9) [1], there exists a projection e_0 in B^2 such that e_0 induces the expectation of B^1 onto $B^0 = (e_0)' \bigcap B^1$. Now using Ocneanu's compactness it is easy to show that there exists an integer k_0 such that for $k > k_0$, e_0 will be a member of $B_{2,1}$.

Definition 2.7. Given two vectors

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad and \;\; Y = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

set $X \geq Y$ if $x_i \geq y_i$, for all $1 \geq i \geq n$. As in Section (3.2) [1] we identify each finite dimensional C^*-algebra, C with an s-dimensional vector, $\vec{C}$, where s is equal to the dimension of the center of C. We have the following equalities. We get our limiting.

Definition 2.8. Consider the commuting square at diagram (1). Let T be the inclusion matrix of $B_{1,1}$ into $B_{1,2}$ and S be the inclusion matrix of $B_{1,1}$ into $B_{2,1}$. Suppose S and T are symmetric and all the corresponding C^* algebras in diagram (1) have s dimensional center. We say that the commuting square given by diagram (1) is *lower expandable* if we are given $V \in R^s$, with positive entries such that for each positive integer n, there exists a positive integer, m_n and an integer valued vector, $Y_n \in R^s$ with the following properties

(i) $Y_n \leq V$

(ii) $\vec{B}_{1,n} = S^n \cdot \vec{B}_{1,1} = T^{m_n} \cdot Y_n$.

In particular the conditions of Definition 2.8 hold if there are two integers r_1 and r_2, such that $T^{r_1} = S^{r_2}$. We say that the commuting square given by diagram (1) is *left lower expandable* if we are given $V \in R^s$, with positive entries such that for each positive integer n, There exists a positive integer, m_n and an integer valued vector, $Y_n \in R^s$ with the following properties

(i) $Y_n \leq V$

(ii) $\vec{B}_{n,1} = T^n \cdot \vec{B}_{1,1} = S^{m_n} \cdot Y_n$.

In particular the conditions of Definition 2.6 hold if there are two integers r_1 and r_2, such that $T^{r_1} = S^{r_2}$.

Let Y_n also represent the corresponding sub algebra, $Y_n \subseteq B_{1,n}$, $Y_n \subseteq B_{n,1}$.

Remark 2.1. *In particular if the commuting square given by Diagram (1) is lower expandable(respectively left lower expandable) then there exists $\delta > 0$, such that the trace of any minimal projection in Y_n is larger than δ.*

Corollary 2.9. *Following the same notations as in Definition 2.8. Suppose the commuting square in Diagram (1)is lower expandable and let the system of commuting squares in Diagram (3) to have the commuting square represented by (1) as its source. The there exists a choice of tunnel with respect to the inclusion $B_2 \supset B_1$, i.e., $B_2 \supset B_1 \supset B_3 \supset \ldots \supset B_k \supset \ldots$ that satisfies the following conditions. For each integer k, there exists an integer m_k and a Jones sequence of finite dimensional C^* algebras, $B_{-k,m_k} \subset B_{-k,m_{k+1}} \subset B_{-k,m_{k+2}} \subset \ldots \subset B_{-k,m_{k+n}} \subset \ldots \subset B^{-k}$. converging to B^{-k}, where $B_{-k,m_k+n} = B_{-k} \cap B_{1,m_k+n}$ and $\vec{B}_{-k,m_k} \leq V$.*

Proof. The proof is an outcome of using induction on the results of Lemma 2.2 and the assumption that the commuting square at Diagram (1) is lower expandable. □

The follwing lemma is an immidiate result of the above definitions.

Lemma 2.10. *Keeping the same notations as in Definition 2.8, suppose that the following conditions hold. Then there exists a number $r > 0$, such that for every integer n there exists an integer m_n and a integer valued vector Y_n in R^s such that,*

(i) $\vec{B}_{1,n} = S^n \cdot \vec{B}_{1,1} = T^{m_n} \cdot Y_n$

(ii) $\|Y_n\|/\|\vec{B}_{1,n}\| \leq r$.

Then for the commuting square (1) to satisfy the above conditions is eqquivalent to being lower expandable.

As a result of Corollary 2.9, it is clear that if the source of the System (3) is lower expandable, then all the commuting squares in System (3) are lower expandable too. It is an interesting problem to find necessary and sufficient conditions for non-degenerate commuting square to be lower expandable. It is also clear that by choosing an appropriate finite dimensional C^* algebra $\vec{B}_{n,m}$ in Diagram (3) and reducing the System (3) by appropriate projections of $\vec{B}_{n,m}$, we can assume without loss of generality for $\vec{B}_{1,1}$ to be any integer valued vector in R^s. Suppose the System (3) is lower expandable.Then by the results of Corollary 2.9, there exist an infinite of integer couples $(n_i, m_i)_{i=1}^{\infty}$ with, $n_1 < n_2 < n_3 < ... < n_k < ... < \infty$ and such that $\vec{B}_{-n_k,m_k} < V$ for all integers $k < \infty$. Furthermore for each integer $k < \infty$ the following sequence, $B_{-n_k,m_k} \subset B_{-n_k,m_k+1} \subset ... \subset B^{-n_k}$ is a Jones sequence. This implies that we have extended System (3) downward and got the new version of System (3) which is still a Jones system.

Theorem 2.11. *Suppose the System (3) is lower expandable. Let C be the intersection of all the algebras $B_{n_k}, \infty > n_k < -\infty$. Then C is finite dimensional.*

Proof. If C is infinite dimensional, then there exists an infinite dimensional sequence of orthogonal projections $G = (f_i)_{i=1}^{\infty}$, with their sum equal to the identity. Note that all the commuting squares in System (3) are equipped with Markov trace. This and the fact that for each integer k, the dimension of B_{-n_k,m_k} is less than dimension of V implies the existence of a real positive number δ such that the trace of each of the minimal projections in B_{-n_k,m_k} is larger than δ. For a given projection f_i in G, let us choose ϵ_i very small positive number. Let us choose any one of the limiting algebras say, B_{-n_i,m_i}. Since it contains f_i, there exists a finite dimensional algebra B_{-n_i,m_i} belonging to the System (3) and a positive operator f_{i,ϵ_i} which is the expectation of f_i, in B_{-n_i,m_i}, where f_{i,ϵ_i} is ϵ_i close to f_i in trace norm. By the fact that System (3) is a Jones system for $k > i$, f_{i,ϵ_i} is in B_{-n_k,m_k}. Using the arguments in [2], for ϵ_i small enough, f_{i,ϵ_i} can be assumed to be a projection. Hence its trace will be larger than δ and this is a contradiction because by the assumption the trace of f_i can be taken to be arbitrary small. □

In the following theorem we prove an important property of the limiting albebras corresponding to the commuting square (1).

Theorem 2.12. *Considering the the tower of commuting squares in diagram (3), and set C to be the commutant of B^1 we are going to show that the algebra $C \bigcap B^{\infty}$ is a factor, hence the graph of the inclusion $B^1 \subset B^2$ is Ergodic.*

Proof. Let the sequence of Jones projections $(e_k)_{k=1}^{\infty}$ be as in the arguments in connection to diagram (3). Then using downward construction on the couple $B^1 \subset B^2$ we get the sequence of Jones projections $(e_k)_{-\infty}^{\infty}$ as in the above. In according to the results of S. Popa in [17], to complete the proof of Theorem 2.12, we only have to prove that $C \bigcap B^{\infty}$ is a factor. Let Let p be a projection in the center $C \bigcap B^{\infty}$. For a given small $\varepsilon > 0$, using the arguments in the Proposition 2.3 [14], we can choose, an integer n large enough depending on ε such that, $p_n = E_{B_n}(p)$, the expectation of p onto B^n be a positive operator with, $\|E_{B_n}(p) - p\|_2 < \varepsilon$. Also note that $p_n \in C \bigcap B^n$, hence using Ocneanu's compactness theorem in [16], $p_n \in B_{n,1}$. Furthermore by Lemma 2.1 [13], p_n is a positive operator, with $\|p_n\|$ close enough to identity. Now consider the projection

$e = e_n \cdot e_{n-1} \cdots e_3$ and a positive operator p_n, Since p commutes with all the projections (e_i) for i larger or equal to 3, we have, $tr(p \cdot e) = tr(p)tr(e) = tr(p_n \cdot e)$ By Proposition 3.1.5 [14], there exists a positive central operator r_n in $B_{1,1}$ such that $p_n \cdot e = r_n \cdot e$. So $tr(p \cdot e) = tr(r_n \cdot e)$. Since r_n commutes with all set of projections (e_i), for i larger or equal to 3, we have $tr(r_n \cdot e) = tr(r_n) \cdot tr(e)$. This implies that $tr(r_n) = tr(p)$. Note that by the arguments in the proof of Proposition 2.3 [14] there exists a number $\beta > 1$, which is close enough to one such that $(\beta)p_n$ is larger than an spectral projection corresponding to p_n. This implies that $\|p_n e\|$ is close to identity. Hence $\|r_n\|$ is close to identity. Therefore without loss of generality we can assume that r_n dominate a central projection in $B_{1,1}$. This implies that the center of $C \bigcap B^{\infty}$ is finite dimensional. Thus by the results of S. Popa in (1.4) [17], $C \bigcap B^{\infty}$ is a factor. □

Theorem 2.13. *Suppose that the commuting square given by diagram (1) is left lower expandable. then the inclusion $B^2 \subseteq B^1$ is strongly amenable.*

Proof. Let us set $C_l = (B^l)' \bigcap B^{\infty}, 1 < l \leq l$. Let p_l be a projection, $p_l \in (C_{l-1})' \bigcap C_l$. Denote $p = p_2 p_3 \cdots p_k$ Using the fact that the above commuting square is left lower expandable and by the remark at the end of Definition 2.6, there exist a fix number $\delta > 0$, independent from k, negative integer n_k, and a C^* algebra B_{k,n_k} which is a sub algebra of $B_{k,1}$, with $T^{n_k} B_{k,n_k} = B_{k,1}$, where the trace of each minimal projection in B_{k,n_k} is larger than δ. Now for given $\epsilon > 0$ small enough using Proposition 2.3 [14] there exists an integer n, such that the positive operator $h = E_{B^n}(p)$ is close enough to p in trace norm. Set $e = e_n e_{n-2} \cdots e_{k+2}$. As in the proof of Theorem 2.12, using Proposition 2.3 [14] and Ocneanu's compactness in [16], there exists a positive operator $r \in B_{k,n_k}$ majoring a projection in

B_{k,n_k} such that $he = re$ and $tr(r) = tr(p)$. Hence for ϵ small enough we can assume that $tr(p) \geq \delta$. But this is a contradiction because taking k large enough $tr(p)$ tends to zero. □

Note that the Theorem 2.12 is valid if we replace B^1 by B^n for any integer $n \in N$. Let us denote the set of all the indices of irreducible subfactors corresponding to the limiting subfactors of commuting squares by IRC.

Corollary 2.14. *As a result of Theorem 2.13 and Lemma 2.4 we get that if s_1 and s_2 are members of the set IRC, the $s_1 s_2$ correspond to index of a pair of strongly amenable subfactors. This and the fact that limiting subfactors of symmetric commuting squares are strongly amenable implies that the set of strongly amenable subfactors is very large.*

3 On Certain Properties of Limiting Algebras

In this section we represents some limiting algebras corresponding to a commuting square. Some of their properties are demonstrated and the calculation of indices are left as an exercise to the readers.

Lemma 3.1. *Let us set $M = < \bigcup_n (B_{n\in N})' \bigcap B^\infty >$. Then M is a factor.*

Proof. By Theorem 2.12, for each $n \in N$, $(B_n)' \bigcap B^\infty$ is a factor hence the union of the above sets is a factor which implies that their closer that means the Von Neumann algebra generated by the union is a factor too. □

Now consider the tower $(B_n), n = 1, 2, ...,$ as defined in section 1. Then by the results of Theorem 4.1.2 [17], if for some integer n, $(B_n \bigcap B^\infty)' \bigcap B^\infty$ $= B_n$, then the inclusion $B_1 \subset B_2$ is strongly amenable. Let us set as before,

$$M = < \bigcup_{n=-\infty}^{\infty} ((B_n)' \bigcap B^\infty) > .$$

Let C be as defined in Theorem 2.11.

Lemma 3.2. *Suppose the inclusion $B_1 \subset B_2$ is strongly amenable then $(M)' \bigcap B^\infty = C$.*

Proof. Suppose $x \in (M)' \bigcap B^\infty$. Then $x \in ((B_n)' \bigcap B^\infty)'$ for each integer n. This implies that x is an element of B^∞ and is in the intersection of all the sets $((B_n)' \bigcap B^\infty)'$ for all integers $n \in N$. But

$$(\bigcap_{n\in N}((B_n)'\bigcap B^\infty)')\bigcap B^\infty = \bigcap_{n\in N}((((B_n)'\bigcap B^\infty)')\bigcap B^\infty) = \bigcap_{n\in N} B_n = C.$$

This implies that $(M)'\bigcap B^\infty$ is subset of C. On the other hand if $x \in C$, then by the assumption of the inclusion being strongly amenable, for each integer n, $x \in ((B_n)'\bigcap B^\infty)'\bigcap B^\infty$, hence $x \in \bigcap_{n\in N}((B_n)'\bigcap B^\infty)'\bigcap B^\infty$, Thus $x \in (M)'\bigcap B^\infty$. □

Given a commuting square with corresponding diagram as given by (1). Let G be the general index corresponding to the above commuting square as defined in Definition 2.3 [18]. One can calculate the the value of $[B^\infty : M]$ using usual techniques and find out that if the commuting square induces subfactors of finite depth then $[B^\infty : M]$ can be formulated in terms of G and $[B_2 : B_1]$, otherwise $[B^\infty : M] = \infty$.

For each integer n, let us define the following Von Neumann algebras $G_n =< B_n, (B_n)'\bigcap B^\infty >$, $G^n =< B^n, (B^n)'\bigcap B^\infty >$, and $Q_n =< B_n, (B_{n+1})'\bigcap B^\infty >$. It is clear that the relative commutant of G_n and G^n with respect to B^∞ is trivial. Let us define $Q_n =< B_n, (B_{n+1})'\bigcap B^\infty >$. Note that $E_{G_{n+1}}(G_n) = Q_n$. Therefore the four Π_1 sub factors G_n, B_∞, G_{n+1} and Q_n form a commuting square. Furthermore $[G_n : Q_n] = [G_{n+1} : Q_n] = [B_2 : B_1]$. Also it is easy to see that $< G_n, e_{n+1} >= B^\infty$. The next question is whether $[B^\infty : G_n]$ is a finite number and to calculate it. First of all, it is easy to show that the above number is independent from n. In the case that $B_1 \subset B_2$ is of finite depth one can show this value is finite and can be formulated in term of G and $[B_2 : B_1]$. Otherwise the value of $[B^\infty : G_n]$ is equal to ∞.

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