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On Fixed Point Theory for Generalized Contractions in Cone Rectangular Metric Spaces via Scalarizing

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Abstract : In this paper, the equivalency between vectorial versions of fixed point theorems in cone rectangular metric spaces and scalar versions of fixed point theorems in rectangular metric spaces is presented. Moreover, some fixed point theorems for generalized contractions in cone rectangular metric spaces are provided. The results of this paper can be considered as the improvment and extension of [W.S. Du, A note on cone metric fixed point theory and its equivalence, Nonlinear Analysis, 72 (5) (2010), 2259-2261], [P. Ghosh, A. Deb Ray, A charactrization of completeness of generalized metric spaces using generalized Banach contraction principle, Demonstratio mathematica 45 (3) (2012) 717-724].

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1 Introduction and Preliminaries

Recently Branciari [1] introduced the concept of a rectangular metric space where the triangular inequality of a metric space has been replaced by a more general inequality involving four points instead of three.
In this section we recall some definitions and facts to set up our results in the next section.

Definition 1.1. $[2-5]$ Let E be a topological vector space with the zero vector θ. A subset P of E is called a *cone* if:

(i) P is closed, nonempty and nontrivial (i.e. $P \neq \{\theta\}$);

(ii) $ax + by \in P$, for all $x, y \in P$ and nonnegative real numbers a and b;

(iii) $P \bigcap (-P) = \{\theta\}$.

In addition to that, if the interior of P is nonempty, we say that P is a *solid cone*.

Definition 1.2. $[2, 3, 6]$ Let E be a topological vector space and $P \subseteq E$ be a cone. We define a partial ordering $\leq$ on E with respect to P by $x \leq y$ if and only if $y - x \in P$ and we write $x < y$ if $x \leq y$ and $x \neq y$. Likewise, we write $x \ll y$ if $y - x \in int(P)$, where $int(P)$ denotes the interior of P. If ambiguity is possible we can use the notations $\leq_P$, $<_P$ and $\ll_P$. The pair (E, P) consisting of a topological vector space E and a cone P of E is called a *partially ordered topological vector space*.

2 Some Properties of the Nonlinear Scalarization Mapping

The nonlinear scalarization mapping plays a key role in the paper. In this section we recall some useful properties of it that are needed in the next section.

Definition 2.1. $[7, 8]$ Let E be an ordered topological vector space with the solid cone P and $e \in intP$. The formula

$$\zeta_e(y) := inf\{t \in \mathbb{R} : y \leq te\},$$

where $y \in E$, defines a mapping from E into $\mathbb{R}$ (the real line) and is called *the nonlinear scalarization function on E (with respect to P and e).*

The following lemma characterizes some other properties of the non-linear scalarization mapping.

Lemma 2.2. [3, 7, 9] *Let (E, P) be an ordered topological vector space. For any $e \in int\ P$ and $r \in \mathbb{R}$, the mapping ζ_e has the following properties:*

(1) $\zeta_e(\theta) = 0$;

(2) $y \in P \Longrightarrow \zeta_e(y) \geq 0$;

(3) If $y_2 < y_1$, *then* $\zeta_e(y_2) < \zeta_e(y_1)$ *for any* $y_1, y_2 \in E$;

(4) $\zeta_e(y) \leq r \Longleftrightarrow y \in re - P \Longleftrightarrow y \leq re$;

(5) $\zeta_e(y) \geq r \Longleftrightarrow y \notin re - intP \Longleftrightarrow y \not\ll re$;

(6) $\zeta_e(y) < r \Longleftrightarrow y \in re - intP \Longleftrightarrow y \ll re$;

(7) $\zeta_e(y) > r \Longleftrightarrow y \notin re - P \Longleftrightarrow y \nleq re$;

(8) ζ_e *is subadditive on* E, *i.e.* $\zeta_e(x + y) \leq \zeta_e(x) + \zeta_e(y)$ *for all* $x, y \in E$;

(9) ζ_e *is positively homogeneous on* E, *i.e.* $\zeta_e(\beta x) = \beta\zeta_e(x)$ *for every* $x \in E$ *and positive real number* β, *and*

(10) ζ_e *is continuous on* E.

The following result extends part (1) of Lemma 2.2 which will be needed in the next section.

Lemma 2.3. *Let E be a partially ordered topological vector space with solid cone P and $e \in intP$, then:*

$$\zeta_e(y) = r \text{ if and only if } y \in re - (P \setminus intP)$$

Proof. Suppose that $\zeta_e(y) \leq r$ and $\zeta_e(y) \geq r$, by note that before lemma (part (4) and (5)), $y \in re - P$ and $y \notin re - intP$, thus $y \in re - (P \setminus intP)$. Now, if $y \in re - (P \setminus intP)$, then $y \in re - P$ and $y \notin re - intP$, by parts (4) and (5) of lemma 2.2 , therefore $\zeta_e(y) = r$. □

3 Cone Rectangular Metric Space

Definition 3.1. [1] Let X be a nonempty set and $d : X \times X \to [0, +\infty)$ such that for all $x, y \in X$ and for all distinct points $u, v \in X$, each of them differernt from x and y, one has the following:

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$;

(3) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$.

Then, (X, d) is called a rectangular metric space (or shortly r.m.s.).
Note that in some of the papers about rectangular metric space, we can see generalized metric space (g.m.s.) instead of rectangular metric space (r.m.s.).

Any metric space is a rectangular metric space, while the following example shows that the converse may fail.

Example 3.2. [10] Let $X = A \bigcup B$, where $A = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}$, $B = [\frac{3}{4}, +\infty)$. Define the rectangular metric d on X as follows:

$$d(\frac{1}{2}, \frac{1}{3}) = d(\frac{1}{4}, \frac{1}{5}) = \frac{1}{5},\ d(\frac{1}{2}, \frac{1}{5}) = d(\frac{1}{3}, \frac{1}{4}) = \frac{1}{4},\ d(\frac{1}{2}, \frac{1}{4}) = d(\frac{1}{5}, \frac{1}{3}) = \frac{1}{2},$$

$$d(\frac{1}{2}, \frac{1}{2}) = d(\frac{1}{3}, \frac{1}{3}) = d(\frac{1}{4}, \frac{1}{4}) = d(\frac{1}{5}, \frac{1}{5}) = 0,$$

$$d(x, y) = d(y, x), \quad \text{for all } x, y \in A,$$

$$d(x, y) = \mid x - y \mid \quad \text{if} \quad \begin{array}{l} x \in B \ \text{ or } \ y \in A, \\ x \in A\ ,\ y \in B, \\ x, y \in B. \end{array}$$

It is clear that d does not satisfy the triangle inequality on A. Indeed,

$$\frac{1}{2} = d(\frac{1}{2}, \frac{1}{4}) > d(\frac{1}{2}, \frac{1}{3}) + d(\frac{1}{3}, \frac{1}{4}) = \frac{9}{20}$$

Notice that (3) holds, so d is a rectangular metric.

Definition 3.3. [1, 11] Let (X, d) be a r.m.s., $\{x_n\}$ be a sequence in X, and $x \in X$. We say that $\{x_n\}$ is r.m.s. convergent to x if and only if $d(x_n, x) \to 0$ as $n \to +\infty$. We denote this by $x_n \to x$.

Definition 3.4. [1,12] Let (X,d) be a r.m.s., $\{x_n\}$ be a sequence in X. We say that x_n is a *r.m.s. Cauchy sequence* if and only if for each $\epsilon > 0$, there exists a natural number N such that $d(x_n, x_m) < N$ for all $n > m > N$.

Definition 3.5. [1,13] Let (X,d) be a r.m.s. Then, (X,d) is called a *complete r.m.s.* if every r.m.s. Cauchy sequence is r.m.s convergent in X.

Remark 3.6. *Several papers attempting to generalize fixed point theorems in metric spaces to r.m.s. are plagued by the use of some false properties given in* [1] *(see, for example,* [14,15,16,17]*). This was observed first by Samet* [18,19] *and then by Sarma et al.*[20] *by assuming that the rectangular metric space is Hausdorff. We know that every metric space is Hausdorff, but in the case of a r.m.s it is false in general as it is shown in the example 1.1.* [20]

Definition 3.7. Let X be a nonempty set and E be a real topological vector space with cone P. A vector-valued function $d : X \times X \to E$ is said to be a *cone metric function* on X, if the the following conditions are satisfied:

(1) $\theta \leq d(x,y)$ for all $x, y \in X$ and $d(x,y) = \theta$ if and only if $x = y$;

(2) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(3) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$.

The pair (X,d) is called a *cone metric space* (for short CMS).

Definition 3.8. [21] Let X be a nonempty set and E be a real topological vector space with cone P. Suppose the mapping $d : X \times X \to E$, satisfying

(1) $\theta \leq d(x,y)$ for all $x, y \in X$ and $d(x,y) = \theta$ if and only if $x = y$;

(2) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(3) $d(x,y) \leq d(x,w) + d(w,z) + d(z,y)$ for all $x, y \in X$ and for all distinct points $w, z \in X - \{x,y\}$ (rectangular property).

Then d is called a cone rectangular metric on X, and (X,d) is called a *cone rectangular metric space* (for short c.r.m.s.).

Let $\{x_n\}$ be a sequence in (X,d) and $x \in (X,d)$ If for every $c \in E$ with $\theta \ll c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x, x_n) \ll c$, then $\{x_n\}$ is said to be convergent to x and x is the limit of $\{x_n\}$. We denote this by

$lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$. If for every $c \in E$ with $\theta \ll c$ there is $n_0 \in \mathbb{N}$ such that for all $n, m > n_0$ and $n, m \in \mathbb{N}$ we have $d(x_n, x_m) \ll c$. Then $\{x_n\}$ is called a Cauchy sequence in (X, d). If every Cauchy sequence is convergent in (X, d), then (X, d) is called a complete cone rectangular metric space.

Example 3.9. [22, 23] Let $X = \mathbb{N}$ (The set of natural numbers), $E = \mathbb{R}^2$ and $P = \{(x, y) : x, y \geqslant 0\}$. Define $d : X \times X \to E$ as follows:

$$d(x,y) = \begin{cases} (0,0) & if\ x = y, \\ (3,9) & if\ x\ and\ y\ are\ in \{1, 2\ \},\ x \neq y \\ (1,3) & if\ x\ and\ y\ both\ can\ not\ be\ at\ a\ time\ in\ \{1,2\},\ x \neq y. \end{cases}$$

Now (X, d) is a cone rectangular metric space, but (X, d) is not a cone metric space because it lacks the triangular property

$$(3, 9) = d(1, 2) > d(1, 3) + d(3, 2) = (1, 3) + (1, 3) = (2, 6)$$

as $(3, 9) - (2, 6) = (1, 3) \in P$.

4 Main Results

In 2010, Du[3] investigated the equivalence of vectorial versions of fixed point theorems in cone metric spaces and scalar versions of fixed point theorems in (general) metric spaces (in usual sense). He showed that the Banach contraction principles in general metric spaces and in TVS-cone metric spaces are equivalent. His results also extended some results of [24] and [25]. In theorem 2.2 [3], the author has claimed that the conclusion (iii) is immediate from conditions (i) and (ii). This assertion is not true. Of course the complete proof given in [26] in the setting of locally convex spaces. In here by using Theorem 2.2, for the sake of reader, we establish t.v.s version of it in rectangular cone metric space. Also, by using the non-linear scalarization mapping, we present the Banach's contraction principle from rectangular metric space in to cone rectangular metric space and obtain the extension of the theorem 1.3 in [20]. The results of this section extend and improve and repair theorem 2.2 [3] and the equivalency between vectorial versions of fixed point theorems in cone rectangular metric spaces and scalar versions of fixed point theorems in rectangular metric spaces is presented. Now we are ready to present the first main result.

Theorem 4.1. *Let (X, d) be a cone rectangular metric spaces. Then $\rho : X \times X \to [0, \infty)$ defined by $\rho := \zeta_e o d$ is a rectangular metric.*

Proof. Since $d(x,y) \in P$ by part 2 of lemma 2.2, we can conclude that $\zeta_e(d(x,y)) \geq 0$, i.e. $\theta \leq \rho(x,y)$ for all $x, y \in X$. If $\rho(x,y) = 0$ then $\zeta_e(d(x,y)) = 0 = r$, by lemma 2.3 we have $d(x,y) \in -P \cap P = \{\theta\}$ which implies $x = y$. Conversely, if $x = y$, then from part 1 of Lemma 2.2, we have $\rho(x,y) = \zeta_e(\theta) = 0$. It is clear that $\rho(x,y) = \rho(y,x)$. By parts 3 and 8 of lemma 2.2, we have $\rho(x,y) \leq \rho(x,w) + \rho(w,z) + \rho(z,y)$ for all $x, y \in X$ and for all distinct points $w, z \in X - \{x,y\}$. □

The following lemma will be use in the proof of the Theorem 4.3.

Lemma 4.2. *Let (X,d) be a cone rectangular metric space with solid cone P and $e \in int\ P$. Then for every $c \in int\ P$, there exists $\epsilon_c > 0$ such that $\epsilon_c e \ll c$.*

Proof. Since $c \in int\ P$, there exists a convex, symmetric and absorbing neighborhood B of zero such that $B + c \in intP$. Thus there exists $\epsilon_c > 0$ such that for all ϵ with $|\epsilon| \leq \epsilon_c$, $\epsilon e \in B$. So $-\epsilon_c e + c \in intP$. Therefore $\epsilon_c e \ll c$. □

The following theorem plays a crucial rule in the next result. Also parts 1,2 and 3 of the following theorem repair and extend theorem 2.2 of [3].

Theorem 4.3. *Let (X,d) be a cone rectangular metric space, $x \in X$ and $\{x_n\}$ be a sequence in (X,d). Let ρ be the same as in before theorem. Then the following statements hold.*

(1) $\{x_n\}$ is a cone rectangular converges to x, if and only if $\rho(x_n, x) \to 0$ as $n \to \infty$;

(2) $\{x_n\}$ is a cone rectangular Cauchy sequence in (X,d), if and only if $\{x_n\}$ is a rectangular Cauchy sequence in (X,ρ);

(3) (X,d) is a complete cone rectangular, if and only if (X,ρ) is a complete rectangular metric space;

(4) $C \subseteq X$ is a closed set of (X,d), if and only if C is a closed set of (X,ρ);

(5) If (X,d) is Hausdorff cone rectangular metric space, then (X,ρ) is Hausdorff rectangular metric space.

Proof. (1) Let $\{x_n\}$ be a cone rectangular converges to x and $\epsilon > 0$ be given. Since $e \in\ int\ P$, $\epsilon e \in int\ P$. Hence there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \ll \epsilon e$. By part 6 of lemma 2.2, we have

$\zeta_e od(x_n, x) < \epsilon$ i.e. $\rho(x_n, x) \to 0$ as $n \to \infty$. Conversely, if $\rho(x_n, x) \to 0$ as $n \to \infty$ and $c \in int\ P$, by lemma 4.2 there exists $\epsilon_c > 0$ such that $\epsilon_c e \ll c$. Since $\rho(x_n, x) \to 0$ as $n \to \infty$ and $\epsilon_c > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $\rho(x_n, x) < \epsilon_c$. By part 6 of lemma 2.2, we have $\epsilon_c e - d(x_n, x) \in int\ P$, i.e. $d(x_n, x) \ll \epsilon_c e \ll c$. This completes the part of (1).
(2) Let $\{x_n\}$ be a cone rectangular Cauchy sequence in (X, d), and $\epsilon > 0$ be given, since $\epsilon e \in int\ P$. Hence there exists $n_0 \in \mathbb{N}$ such that for all $n, m > n_0$, $d(x_n, x_m) \ll \epsilon e$. By part 6 of lemma 2.2, we have $\zeta_e od(x_n, x_m) < \epsilon$ i.e. $\rho(x_n, x_m) < \epsilon$. Hence $\{x_n\}$ is a rectangular Cauchy sequence in (X, ρ). Conversely, let $c \in int\ P$, by lemma 4.2 there exists $\epsilon_c > 0$ such that $\epsilon_c e \ll c$, since $\{x_n\}$ is a rectangular Cauchy sequence in (X, ρ) and $\epsilon_c > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m > n_0$, $\rho(x_n, x_m) < \epsilon_c$. By part 6 of lemma 2.2, we have $\epsilon_c e - d(x_n, x_m) \in int\ P$, i.e. $d(x_n, x_m) \ll \epsilon_c e \ll c$. This completes the part of (2).
(3) It follows from part of (1) and (2).
(4) It is clear by part of (1).
(5) We show that if (X, d) is Hausdorff cone rectangular metric space, and $\{x_n\}$ be a sequence in (X, ρ), then $\{x_n\}$ converges to at most one point of (X, ρ) and conclude (X, ρ) is a Hausdorff rectangular metric space. Let $\{x_n\}$ rectangular converges to x, y in (X, ρ). By part (1) we have $\{x_n\}$ cone rectangular converges to x, y in (X, d), since (X, d) is a Hausdorff cone rectangular metric space, then $x = y$. This completes the proof. □

In 2009, Sarma, Rao in [20] established the following theorem that is a corrected version of the generalization Banach's contraction principle in metric spacees to rectangular metric space that presented by Branciari in [1].

Theorem 4.4. [20] *(Banach's Contraction principle in a r.m.s) Let (X, d) be a Hausdorff and complete rectangular metric space and let $f : X \to X$ be a mapping and $0 < \lambda < 1$ satisfying the inequality $d(fx, fy) \le \lambda d(x, y)$ for all $x, y \in X$ (such a mapping is called a contraction mapping on X and λ is called the contractive constant of f). Then there is a unique point $x \in X$ satisfying $f(x) = x$ (such a point is called a fixed point of f).*

Now by combing Banach's contraction principle in a r.m.s and Theorem 4.3, we conclude the following theorem, which is an extension of the Banachs Contraction principle from generalized metric space in to cone generalized metric space. It should be pointed out that theorem 4.5 extend some results L.G. Huang and X. Zhang in [24] and Sh. Rezapour, R. Hamal barani in[25].

Theorem 4.5. *(Banach's contraction principle in a c.r.m.s) Let* (X, d) *be a Hausdorff and complete cone rectangular metric space and let* $f : X \to X$ *be a mapping and* $0 < \lambda < 1$ *satisfying the inequality* $d(fx, fy) \leq \lambda d(x, y)$ *for all* $x, y \in X$*. Then there is a unique point* $x \in X$ *satisfying* $f(x) = x$*.*

Proof. By using the inequality $d(fx, fy) \leq \lambda d(x, y)$ for all $x, y \in X$ and parts (3),(9) of lemma 2.2, we can conclude the inequality $\zeta_e(d(fx, fy)) \leq \lambda \zeta_e(d(x, y))$ for all $x, y \in X$ which implies $\rho(fx, fy) \leq \lambda \rho(x, y)$ for all $x, y \in X$. Also by parts (3),(6) of theorem 4.3, (X, ρ) be a Hausdorff and complete g.m.s (r.m.s). Then by using theorem 4.4, there is a unique point $x \in X$ satisfying $f(x) = x$. This completes the proof. □

In the following some fixed point theorems for generalized contractions in cone rectangular metric spaces are provided.

In 2012 Ghosh and Ray [27] proved the following fixed point theorem for generalized contractions in complete rectangular metric spaces.

Theorem 4.6. *Let* (X, d) *be a complete r.m.s. and let* $T : X \to X$ *be a map. Define a nonincreasing function* $\theta : [0, 1)$ *onto* $(\frac{1}{2}, 1]$ *by*

$$\theta(r) = \begin{cases} 1, & 0 \leq r \leq \dfrac{1}{3}; \\ \frac{2(1-r)}{3r(r+1)}, & \dfrac{1}{3} \leq r < 1. \end{cases}$$

Assume that there exists $r \in [0, 1)$ *such that*

$$\theta(r) d(x, T(x)) \leq d(x, y) \quad \Rightarrow \quad d(T(x), T(y)) \preceq r d(x, y), \quad for\ all\ x, y \in X.$$

Then there exists a unique fixed point z *of* T *and* $\lim_{n\to\infty} T^n x = z$*, for all* $x \in X$*.*

One can inform from the proof of theorem 4.6 that the authors assume that every r.m.s is Hausdorff while it is not true by Remark 3.6 Thus we need Hausdorff condition in the following theorem that is the cone rectangular metric space version of Theorem 4.6 which generalizes Theorem 2.1 in [27] and Theorem 2.3 in [3].

Theorem 4.7. *Let* (X, d) *be a Hausdorff and complete cone rectangular metric space,* θ *as given in Theorem 4.6,* $T : X \to X$ *be a map. Assume that there exists* $r \in [0, 1)$ *such that*

$$d(x, y) \not\ll \theta(r) d(x, T(x)) \Rightarrow d(T(x), T(y)) \preceq r d(x, y), \quad \forall x, y \in X. \qquad (*)$$

Then there exists a unique fixed point z of T. Moreover, $\lim_{n\to\infty} T^n x = z$, *for all* $x \in X$.

Proof. It is clear by part 3,5 of Theorem 4.3 $(X, \xi_e od)$ is a Hausdorff and complete metric space. Let $\theta(r)\xi_e od(x, T(x)) \leq \xi_e od(x, y)$. Then, by the properties of ξ_e (see Lemma 2.2) we have

$$0 \leq \xi_e od(x, y) - \theta(r)\xi_e od(x, T(x)) =$$

$$\xi_e(d(x, y)) - \xi_e(\theta(r)d(x, T(x))) \leq \xi_e(d(x, y) - \theta(r)d(x, T(x))).$$

So it follows from Lemma 2.2 (5) that

$$d(x, y) - \theta(r)d(x, T(x)) \not\ll \theta,$$

and then

$$d(x, y) \not\ll \theta(r)d(x, T(x)),$$

and so by using our assumption $(*)$ we get

$$d(T(x), T(y)) \preceq rd(x, y).$$

Hence it follows from Lemma 2.2 (3) that

$$\xi_e(d(T(x), T(y))) \preceq \xi_e(rd(x, y)) = r\xi_e(d(x, y)).$$

Now the result follows from Theorem 4.6. □

The following example satisfies in Theorem 4.7.

Example 4.8. Define a Hausdorff and complete cone rectangular metric space X with cone $P = \{(x, y) \in \mathbb{R}^2 : x, y \geqslant 0\}$ by

$$X = \{(0, 0), (4, 0), (0, 4)(4, 5), (5, 4)\}$$

and its rectangular metric d by

$$d((x_1, x_2), (y_1, y_2)) = (|x_1 - y_1| + |x_2 - y_2|, |x_1 - y_1| + |x_2 - y_2|)$$

Let $X = \mathbb{N}$ (The set of natural numbers), $E = \mathbb{R}^2$ and $P = \{(x, y) : x, y \geqslant 0\}$. Define a mapping T on X by

$$T(x_1, x_2) = \begin{cases} (x_1, 0) & if\ x_1 \leqslant x_2 \\ (0, x_2) & if\ x_1 > x_2. \end{cases}$$

Then T satisfies the assumption in Theorem 4.7 and then there exists a uniq fixed point. Moreover, $lim_{n\to\infty} T^n x = z$ for all $x \in X$. It is clear that in this example $z = (0, 0)$.

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References

[1] A. Branciari, A fixed point theorem of Banach Caccioppoli type on a class of generalized metric spaces, Publ. Math. Debrecen 57 (2000) 31-37.

[2] I. Beg, A. Azam, M. Arshad, Common fixed points for maps on topological vector space valud cone metric spaces, Internat. J. Math. Sciences 2009 (2009).

[3] W.S. Du, A note on cone metric fixed point theory and its equivalence, Nonlinear Analysis 72 (5) (2010) 2259-2261.

[4] P. Kumam, H. Rahimi, G. Soleimani Rad, The existence of fixed and periodic point theorems in cone metric type spaces, J. Nonlinear Sci. Appl. 7 (4) (2014) 255-263.

[5] N. Mehmood, A. Azam, L.D.R. Kocinac, Multivalued fixed point results in cone metric spaces, Topology Appl. 179 (2015) 156-170.

[6] C. Vetro, S. Chauhan, E. Karapinar, W. Shatanawi, Fixed points of weakly compatible mappings satisfying generalized ϕ-weak Contractions, Bull. Malays. Math. Sci. Soc. (2014-2015), in press. DOI 10.1007/s40840-014-0074-0.

[7] I. D. Arandelovic, D. J. Keckic, TVS-Cone Metric Spaces as a Special case of Metric Spaces, arXiv: 1202.5930vl [math.FA], (2012).

[8] P. Zangenehmehr, A.P. Farajzadeh, S.M. Vaezpour, On fixed point theorems for monotone increasing vector valued mappings via scalarizing, J. Positivity 19 (2) (2015) 333-340.

[9] P. Zangenehmehr, A.P. Farajzadeh, Ar. Karamian, On generalized vector quasi equilibrium inequality problem via scalarization, Journal of Nonlinear Analysis and Optimization 5 (2) (2014) 119-125.

[10] H. Aydi, E. Karapinar, H. Lakzian, Fixed point results on a class of generalized metric spaces, Aydi et al. Mathematical Sciences (2012) 6-46.

[11] M. A. Khamsi, Remarks on cone metric spaces and fixed point theorems of contractive mappings. Fixed Point Theory Appl. 2010, Art. ID 315398.

[12] B. Rakesh, V. Sachin, K. Rajesh, Coupled coincidence point theorems for mappings without mixed monotone property under c-distance in cone metric spaces, J. Nonlinear Sci. Appl. 7 (5) (2014) 345-358.

[13] M. Imdad, S. Chauhan, A.H. Soliman, M.A. Ahmed, Hybrid fixed point theorems in symmetric spaces via common limit range property, Demonstr, Math. 47 (2014), 62.

[14] M. Akram, A.A. Siddiqui, A fixed point theorem for A-contractions on a class of generalized metric spaces, Korean J. Math. Sciences. 10 (2) (2003).

[15] A. Azam, M. Arshad, Kannan fixed point theorem on generalized metric spaces, J. Nonlinear Sci. Appl. 1 (1) (2008) 45-48.

[16] P. Das, A fixed point theorem on a class of generalized metric spaces, Korean J. Math. Sci. 9 (2002) 29-33.

[17] P. Das, B.K. Lahiri, Fixed point of a Ljubomir Cirics quasi-contraction mapping in a generalized metric space, Publ. Math. Debrecen. 61 (2002) 589-594.

[18] B. Samet, Discussion on "A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces" by A. Branciari, Publ. Math. Debrecen 76 (4) (2010) 493-494.

[19] H. Lakzian, B. Samet, Fixed point for $(\psi;\varphi)$-weakly contractive mappings in generalized metric spaces, Appl. Math. Lett. 25 (5) (2012) 902-906.

[20] I.R. Sarma, J.M. Rao, S.S. Rao, Contraction over generalized metric spaces, J. Nonlinear Sci. Appl. 2 (3) (2009) 180182.

[21] A. Azam , M. Arshad, I. Beg, Banach contraction principle in cone rectangular metric spaces, Appl. Anal. Discrete Math. 3 (2009) 236-241.

[22] S. Chauhan, E. Karapinar, Some integral type common fixed point theorems satisfying ϕ-contractive conditions, Bull. Belg. Math. Soc. Simon Stevin 21 (4) (2014) 593-612.

[23] S.K. Malhotra, S. Shukla, R. Sen, Some fixed point theorems for ordered Reich type contractions in cone rectangular metric spaces, Acta Math. Univ. Comenianae. 2 (2013) 165-175.

[24] L.G. Huang, X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings. J. Math. Anal. Appl. (2007) 323:14681476.

[25] Sh. Rezapour, R. Hamal Barani, Some notes on the paper Cone metric spaces and Fixed Point Theorems of Contractive mappings, J. Math. Anal. Appl. 345 (2) (2008) 719-724.

[26] T. Abdeljawad, E. Karapinar, A gap in the paper " A note on cone metric fixed point theory and its equivalent, Nonlinear Analysis, 72 (5) (2010) 2259-22611" Gaziuniversity Journal of Science 24 (2) (2011) 233-234.

[27] P. Ghosh, A.D. Ray, A charactrization of completeness of generalized metric spaces using generalized Banach contraction principle Demonstratio mathematica 45 (3) (2012) 717-724.

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