



A New Improved Hat Function for Numerical Solution of Linear Fredholm Integral Equations

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Abstract : A new and efficient numerical approach is developed for solving linear Fredholm integral equations. The fundamental structure of the presented method is based on the modification of hat functions (MHFs) in which a new operational matrix of integration is introduced. After implementation of our scheme, the solution of the main problem would be transformed into the solution of a system of linear algebraic equations. Also, an error analysis is provided under several mild conditions. In addition, examples that illustrate the pertinent features of the method are presented, and the results are discussed.

Keywords : modified hat functions; Fredholm integral equation; vector forms; operational matrices; error analysis

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1 Introduction

Integral equations are often involved in the mathematical formulation of physical phenomena. Integral equations can also be encountered in various fields such as physics, biology and engineering. Hence, the research on their numerical solutions is of interest both theoretically and practically.

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Integral equations have been developed to solve boundary value problems for both ordinary and partial differential equations [1]. The theory and numerical solutions of integral equations have been studied comprehensively [2]. In recent years, several authors for the integral and Fredholm integral equations of the second kind have worked semi-analytical methods such as the Taylor-series expansion method [3], the homotopy perturbation method [4], the quasi-interpolation method [5], the fast collocation method [6], the Adomian decomposition method [7, 8], the iteration method [9] and so on. Recently, Babolian et al. [10] have given a new method for solving systems of linear or nonlinear Fredholm integral equations of the second kind by hat basis functions.

In the present paper, we use MHFs to solve the linear equation

$$f - \mathcal{K}f = g, \quad (1.1)$$

where \mathcal{K} is a compact linear operator on the Banach space \mathcal{X} . The operator $(I - \mathcal{K})$ is assumed to be invertible, so that the equation has a unique solution $f \in \mathcal{X}$ for any given $g \in \mathcal{X}$. Let \mathcal{K} be the compact linear integral operator defined by

$$\mathcal{K}f(x) = \int_0^1 k(x, y)f(y)dy, \quad x \in D = [0, 1],$$

where $\mathcal{X} = C^4(D)$ and the kernel function $k \in L^2(D \times D)$. A standard technique to solve (1.1) approximately is to replace \mathcal{K} by a finite rank operator. The approximate solution is then obtained by solving a system of linear equations. For the integral equation (1.1), consider the iteration

$$f^{(n+1)} = g + \mathcal{K}f^{(n)}, \quad n = 0, 1, \dots$$

From the geometric series theorem, it can be shown that this iteration converges to the solution f if $\|\mathcal{K}\|_\infty < 1$, and in that case

$$\|f - f^{(n+1)}\|_\infty \leq \|\mathcal{K}\|_\infty \|f - f^{(n)}\|_\infty.$$

Sloan [11] showed that once such iteration is always a good idea if the initial guess is the solution obtained by the Galerkin method, regardless of the size of \mathcal{K} .

2 MHFs and Their Properties

In this section, we first give some basic definitions and then present properties of MHFs.

Definition 2.1. An $(m + 1)$ -set of MHFs consists of $(m + 1)$ functions which are defined over district D as follows:

$$h_0(x) = \begin{cases} \frac{1}{2h^2}(x - h)(x - 2h) & 0 \leq x \leq 2h, \\ 0 & \text{otherwise,} \end{cases}$$

if i is odd and $1 \leq i \leq m - 1$,

$$h_i(x) = \begin{cases} \frac{-1}{h^2}(x - (i - 1)h)(x - (i + 1)h) & (i - 1)h \leq x \leq (i + 1)h, \\ 0 & \text{otherwise,} \end{cases}$$

if i is even and $2 \leq i \leq m - 2$,

$$h_i(x) = \begin{cases} \frac{1}{2h^2}(x - (i - 1)h)(x - (i - 2)h) & (i - 2)h \leq x \leq ih, \\ \frac{1}{2h^2}(x - (i + 1)h)(x - (i + 2)h) & ih \leq x \leq (i + 2)h, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$h_m(x) = \begin{cases} \frac{1}{2h^2}(x - (1 - h))(x - (1 - 2h)) & 1 - 2h \leq x \leq 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $m \geq 2$ is an even integer and $h = \frac{1}{m}$.

According to definition of MHFs, we have:

$$h_i(jh) = \begin{cases} 1 & i = j, \\ 0 & i \neq j, \end{cases}$$

if i is even,

$$h_i(x)h_j(x) = 0, \quad |i - j| \geq 3, \quad (2.1)$$

if i is odd,

$$h_i(x)h_j(x) = 0, \quad |i - j| \geq 2, \quad (2.2)$$

and

$$\sum_{i=0}^m h_i(x) = 1.$$

In Fig. 1, the behavior of a set of hat functions with $m = 6$ is depicted.

Let us write the MHFs vector $H(x)$ as follows

$$H(x) = [h_0(x), h_1(x), \dots, h_m(x)]^T; \quad x \in D. \quad (2.3)$$

Form Eq.s (2.1) and (2.2) we have

$$H(x)H^T(x) = \tilde{H}(x) = [\tilde{h}_{ij}(x)]_{(n+1) \times (n+1)},$$

where

$$\tilde{h}_{ij}(x) = \begin{cases} h_i(x)h_j(x) & \begin{array}{l} i \text{ is even and } |i - j| < 3 \\ i \text{ is odd and } |i - j| < 2 \end{array} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$f_i = f(ih), \quad i = 0, \dots, m.$$

Similarly an arbitrary function of two variables, $k(x, y)$ on district $L^2(D \times D)$ may be approximated with respect to MHFs such as

$$k(x, y) \simeq H^T(x)KH(y),$$

where $H(x)$ and $H(y)$ are MHFs vector of dimension $(m + 1)$ and K is the $(m + 1) \times (m + 1)$ MHFs coefficients matrix.

3 Method of Solution

In this section, we convert the model (1.1) to linear systems of matrix equations which can be easily solved.

First, we approximate functions $f(x)$, $g(x)$ and $k(x, y)$ with respect to MHFs as

$$\begin{cases} f(x) \simeq F^T H(x) = H^T(x)F, \\ g(x) \simeq G^T H(x) = H^T(x)G, \\ k(x, y) \simeq H^T(x)KH(y), \end{cases} \quad (3.1)$$

where $H(x)$ is defined in Eq. (2.3) and the vectors F, G and matrix K are MHFs coefficients of $f(x)$, $g(x)$ and $k(x, y)$, respectively.

Substituting Eq. (3.1) in Eq. (1.1) yields

$$\begin{aligned} H^T(x)F &= H^T(x)G + \int_0^1 H^T(x)KH(y)H^T(y)F dy \\ &= H^T(x)G + H^T(x)K \left(\int_0^1 H(y)H^T(y) dy \right) F. \end{aligned}$$

Using Eq. (2.4), yields

$$H^T(x)F = H^T(x)G + H^T(x)KPF, \Rightarrow F = G + KPF, \Rightarrow (I - KP)F = G.$$

After solving the above linear system, we can find F and then

$$f(x) \simeq H^T(x)F.$$

4 Convergence Analysis

In this sections, we show that the MHFs method in the previous sections, is convergent and its order of convergence is $O(h^4)$.

Assume that \mathcal{X}_m is the set of all continuous function that are quadratic polynomials when restricted to each of the subintervals $[x_0, x_2], \dots, [x_{m-2}, x_m]$. Obviously, the dimension of \mathcal{X}_m is $d_m = m + 1$. Moreover, let \mathcal{P}_m be the interpolatory projection operator from \mathcal{X} to \mathcal{X}_m . Also, assume that $x_i = ih$, $i = 0, \dots, m$. We can write $\mathcal{P}_m f$ in its Lagrange form:

$$\mathcal{P}_m f(x) = \sum_{i=0}^m f_i h_i(x). \quad (4.1)$$

If i is even, the interpolation error have two formulas on $[x_{i-2}, x_i]$

$$f(x) - \mathcal{P}_m f(x) = (x - x_{i-2})(x - x_{i-1})(x - x_i) F[x_{i-2}, x_{i-1}, x_i, x], \quad (4.2)$$

and

$$f(x) - \mathcal{P}_m f(x) = \frac{(x - x_{i-2})(x - x_{i-1})(x - x_i)}{6} f'''(\xi_x), \quad x_{i-2} \leq x \leq x_i, \quad (4.3)$$

for some $\xi_x \in [x_{i-2}, x_i]$. The quantity $F[x_{i-2}, x_{i-1}, x_i]$ is a *Newton divided difference* of order three for the function $f(x)$. From the above formulas [13],

$$\|f - \mathcal{P}_m f\|_\infty \leq \frac{\sqrt{3}}{27} h^3 \|f'''\|_\infty, \quad f \in \mathcal{X}. \quad (4.4)$$

According to (4.4), if $f \in \mathcal{X}$, then

$$\lim_{m \rightarrow \infty} \mathcal{P}_m f = f,$$

and if $f \in \mathcal{X}_m$, from (4.3) it is clear that

$$\mathcal{P}_m f = f.$$

Now, we approximate the solution of Eq. (1.1) by attempting to solve the problem

$$\mathcal{P}_m(I - \mathcal{K})f_m = \mathcal{P}_m g, \quad f_m \in \mathcal{X}_m, \quad (4.5)$$

or

$$(I - \mathcal{P}_m \mathcal{K})f_m = \mathcal{P}_m g. \quad (4.6)$$

Define the iterated projection solution by

$$\hat{f}_m = g + \mathcal{K}f_m. \quad (4.7)$$

This new approximation is often an improvement on f_m . Moreover, it is used to better understand the behavior of the original projection solution. Applying \mathcal{P}_m to both sides of Eq. (4.7), we have

$$\mathcal{P}_m \hat{f}_m = \mathcal{P}_m g + \mathcal{P}_m \mathcal{K}f_m = f_m, \Rightarrow \mathcal{P}_m \hat{f}_m = f_m. \quad (4.8)$$

Thus, f_m is the projection of \hat{f}_m into \mathcal{X}_m .

Substituting the relation (4.8) into (4.7) and rearranging terms yields the equation

$$(I - \mathcal{K}\mathcal{P}_m)\hat{f}_m = g, \quad (4.9)$$

then we have

$$f - \hat{f}_m = g + \mathcal{K}f - g + \mathcal{K}f_m = \mathcal{K}(f - f_m), \Rightarrow \|f - \hat{f}_m\|_\infty \leq \|\mathcal{K}\|_\infty \|f - f_m\|_\infty.$$

This proves the convergence of \hat{f}_m to f is at least as rapid as that of f_m to f . Often it will be more rapid, because operating on $f - f_m$ with \mathcal{K} sometimes causes cancellation due to the smoothing behavior of integration.

Lemma 4.1. *suppose $\mathcal{P}_m : \mathcal{X} \rightarrow \mathcal{X}_m$, be defined by Eq. (4.1), where $m \geq 2$ and be even integer, then*

$$\|\mathcal{P}_m\|_\infty = 1.$$

Proof. Let us first consider

$$\|\mathcal{P}_m\|_\infty = \sup\{\|\mathcal{P}_m f\|_\infty : \|f\|_\infty = 1\}. \quad (4.10)$$

If $f \in \mathcal{X}$ and $\|f\|_\infty = 1$ then we have

$$\begin{aligned} \mathcal{P}_m f(x) &= \sum_{i=0}^m f_i h_i(x) \leq \|f\|_\infty \sum_{i=0}^m h_i(x) = \|f\|_\infty = 1, \\ &\Rightarrow \|\mathcal{P}_m\|_\infty \leq 1. \end{aligned} \quad (4.11)$$

Now, let $f(x) = 1$, $x \in D$, then $\|f\|_\infty = 1$ and

$$\mathcal{P}_m f(x) = \sum_{i=0}^m f_i h_i(x) = \sum_{i=0}^m h_i(x) = 1. \quad (4.12)$$

From (4.10) and (4.12), we have

$$\|\mathcal{P}_m\|_\infty \geq 1. \quad (4.13)$$

Combining both of the inequalities (4.11) and (4.13) leads to

$$\|\mathcal{P}_m\|_\infty = 1.$$

□

Theorem 4.2. *Assume \mathcal{X} be a Banach space, $\mathcal{K} : \mathcal{X} \rightarrow \mathcal{X}$ is bounded and $(I - \mathcal{K}) : \mathcal{X} \rightarrow \mathcal{X}$ is one-to-one and onto. Further if*

$$\lim_{n \rightarrow \infty} \|\mathcal{K} - \mathcal{P}_n \mathcal{K}\|_\infty = 0,$$

then for all sufficiently large even integer m , say $m \geq M$, the operator $(I - \mathcal{P}_m \mathcal{K})^{-1} : \mathcal{X} \rightarrow \mathcal{X}$ exists as a bounded operator. Also, it is uniformly bounded:

$$\sup_{m \geq M} \|(I - \mathcal{P}_m \mathcal{K})^{-1}\|_\infty < \infty. \quad (4.14)$$

For the solutions of Eq.s (1.1) and (4.6)

$$f - f_m = (I - \mathcal{P}_m \mathcal{K})^{-1}(f - \mathcal{P}_m f),$$

and

$$\frac{1}{\|(I - \mathcal{P}_m \mathcal{K})\|_\infty} \|f - \mathcal{P}_m f\|_\infty \leq \|f - f_m\|_\infty \leq \|(I - \mathcal{P}_m \mathcal{K})^{-1}\|_\infty \|f - \mathcal{P}_m f\|_\infty. \quad (4.15)$$

Proof. For the proof see [12].

□

Theorem 4.2 shows that $\{f_m\}$ converges to f if and only if $\{\mathcal{P}_m f\}$ converges to f . Further, if convergence does occur, then $\|f - f_m\|_\infty$ and $\|f - \mathcal{P}_m f\|_\infty$ tend to zero with exactly the same speed.

Lemma 4.3. Let $\mathcal{P}_m : \mathcal{X} \rightarrow \mathcal{Y}$, be a sequence of bounded linear operators, with $m \geq 2$ and be even integer, \mathcal{X} and \mathcal{Y} Banach space. If $\{\mathcal{P}_m f\}$ converges for all $f \in \mathcal{X}$, then the convergence is uniform on compact subsets of \mathcal{X} .

Proof. For the proof see [12].

□

Lemma 4.4. Let $\{\mathcal{P}_m\}$ be a family of bounded projections on \mathcal{X} , where $m \geq 1$ and be even integer, \mathcal{X} be a Banach space and

$$\lim_{m \rightarrow \infty} \mathcal{P}_m f = f, \quad f \in \mathcal{X}.$$

If $\mathcal{K} : \mathcal{X} \rightarrow \mathcal{X}$ be compact, then

$$\lim_{m \rightarrow \infty} \|\mathcal{K} - \mathcal{P}_m \mathcal{K}\|_\infty = 0.$$

Proof. For the proof see [12].

□

According to the above lemmas and theorem, we can conclude if Eq. (1.1) is uniquely solvable for all $g \in \mathcal{X}$, then the collocation Eq. (4.5) is uniquely solvable for all sufficiently large m , say $m \geq M$, and the inverses $(I - \mathcal{P}_m \mathcal{K})^{-1}$ are uniformly bounded, say by $L > 0$.

Lemma 4.5. *Let \mathcal{X} be a Banach space, $\mathcal{K} : \mathcal{X} \rightarrow \mathcal{X}$ be a bounded linear operator and $\mathcal{P}_m : \mathcal{X} \rightarrow \mathcal{X}_m$ defined by Eq. (4.1). Assume $(I - \mathcal{P}_m \mathcal{K})^{-1}$ exists from \mathcal{X} onto \mathcal{X} and Eq. (4.14) is satisfied. Then $(I - \mathcal{K} \mathcal{P}_m)^{-1}$ also exists, and*

$$(I - \mathcal{K} \mathcal{P}_m)^{-1} = I + \mathcal{K}(I - \mathcal{P}_m \mathcal{K})^{-1} \mathcal{P}_m, \quad (4.16)$$

so

$$\sup_{m \geq M} \|(I - \mathcal{K} \mathcal{P}_m)^{-1}\|_\infty < \infty. \quad (4.17)$$

Proof. We have

$$\begin{aligned} (I - \mathcal{K} \mathcal{P}_m)[I + \mathcal{K}(I - \mathcal{P}_m \mathcal{K})^{-1} \mathcal{P}_m] &= I - \mathcal{K} \mathcal{P}_m + (I - \mathcal{K} \mathcal{P}_m) \mathcal{K}(I - \mathcal{P}_m \mathcal{K})^{-1} \mathcal{P}_m = \\ &= I - \mathcal{K} \mathcal{P}_m + \mathcal{K}(I - \mathcal{P}_m \mathcal{K})(I - \mathcal{P}_m \mathcal{K})^{-1} \mathcal{P}_m = I - \mathcal{K} \mathcal{P}_m + \mathcal{K} \mathcal{P}_m = I. \end{aligned}$$

A similar proof work to show

$$[I + \mathcal{K}(I - \mathcal{P}_m \mathcal{K})^{-1} \mathcal{P}_m](I - \mathcal{K} \mathcal{P}_m) = I,$$

thus Eq. (4.16) is satisfied. Also, we have

$$\begin{aligned} \|(I - \mathcal{K} \mathcal{P}_m)^{-1}\|_\infty &= \|I + \mathcal{K}(I - \mathcal{P}_m \mathcal{K})^{-1} \mathcal{P}_m\|_\infty \leq \|I\|_\infty + \|\mathcal{K}\|_\infty \|(I - \mathcal{P}_m \mathcal{K})^{-1}\|_\infty \|\mathcal{P}_m\|_\infty \\ &= 1 + N \|(I - \mathcal{P}_m \mathcal{K})^{-1}\|_\infty, \end{aligned}$$

where $\|\mathcal{K}\|_\infty \leq N$. So

$$\sup_{m \geq M} \|(I - \mathcal{K} \mathcal{P}_m)^{-1}\|_\infty \leq \sup_{m \geq M} \{1 + N \|(I - \mathcal{P}_m \mathcal{K})^{-1}\|_\infty\} < \infty.$$

□

Theorem 4.6. *Assume that the integral Eq. (1.1) is uniquely solvable for all $g \in \mathcal{X}$. Further assume that the solution $f \in \mathcal{X}$ and that the kernel function $k(x, y)$ is at least once continuously differentiable with respect to y . Let \mathcal{P}_m be the interpolatory projection (4.1) defined by piecewise quadratic interpolation. Then*

$$\|f - f_m\|_\infty \leq L \|f - \mathcal{P}_m f\|_\infty \leq \frac{\sqrt{3}L}{27} h^3 \|f'''\|_\infty, \quad m \geq M,$$

where $\|(I - \mathcal{P}_m \mathcal{K})^{-1}\|_\infty \leq L$. Then for the iterated collocation method, we have

$$\|f - \hat{f}_m\|_\infty \leq ch^4, \quad (4.18)$$

where $c > 0$ is a suitable constant. Consequently,

$$\max_{i=0,1,\dots,m} |f(x_i) - f_m(x_i)| = O(h^4). \quad (4.19)$$

Proof. From the relations (4.4) and (4.15), we have

$$\|f - f_m\|_\infty \leq \|(I - \mathcal{P}_m \mathcal{K})^{-1}\|_\infty \|f - \mathcal{P}_m f\|_\infty \leq L \|f - \mathcal{P}_m f\|_\infty \leq \frac{\sqrt{3}L}{27} h^3 \|f'''\|_\infty,$$

where $m \geq M$. For the error in \hat{f}_m , first rewrite (1.1) as

$$(I - \mathcal{K} \mathcal{P}_m)f = g + \mathcal{K}f - \mathcal{K} \mathcal{P}_m f.$$

Subtract Eq. (4.9) to obtain

$$(I - \mathcal{K} \mathcal{P}_m)(f - \hat{f}_m) = \mathcal{K}(I - \mathcal{P}_m)f,$$

or

$$f - \hat{f}_m = (I - \mathcal{K} \mathcal{P}_m)^{-1} \mathcal{K}(I - \mathcal{P}_m)f.$$

To examine the error in \hat{f}_m , we make a detailed examination of $\mathcal{K}(I - \mathcal{P}_m)f$. Using (4.1) yields to

$$\mathcal{K}(I - \mathcal{P}_m)f(x) = \int_0^1 k(x, y) \left\{ f(y) - \sum_{i=0}^m f_i h_i(y) \right\} dy.$$

On the other hand from (4.2), we have

$$\begin{aligned} \mathcal{K}(I - \mathcal{P}_m)f(x) = & \\ & \sum_{k=1}^{\frac{m}{2}} \int_{x_{2k-2}}^{x_{2k}} k(x, y)(y - x_{2k-2})(y - x_{2k-1})(y - x_{2k}) F[x_{2k-2}, x_{2k-1}, x_{2k}, y] dy. \end{aligned}$$

Examining the integral in more detail, write it as

$$\int_{x_{2k-2}}^{x_{2k}} u(x, y)w(y)dy, \quad (4.20)$$

with

$$u(x, y) = k(x, y)F[x_{2k-2}, x_{2k-1}, x_{2k}, y],$$

and

$$w(y) = (y - x_{2k-2})(y - x_{2k-1})(y - x_{2k}).$$

Assume that

$$v(y) = \int_{x_{2k-2}}^y (z - x_{2k-2})(z - x_{2k-1})(z - x_{2k})dz, \quad x_{2k-2} \leq y \leq x_{2k}.$$

Then $v'(y) = w(y)$, $v(y) \geq 0$ on $[x_{2k-2}, x_{2k}]$, and $v(x_{2k-2}) = v(x_{2k}) = 0$. The integral Eq. (4.20) becomes

$$\begin{aligned} \int_{x_{2k-2}}^{x_{2k}} u(x, y)v'(y)dy &= u(x, y)v(y)\Big|_{x_{2k-2}}^{x_{2k}} - \int_{x_{2k-2}}^{x_{2k}} u_y(x, y)v(y)dy \\ &= - \int_{x_{2k-2}}^{x_{2k}} u_y(x, y)v(y)dy, \end{aligned}$$

and

$$\left| \int_{x_{2k-2}}^{x_{2k}} u_y(x, y)v(y)dy \right| \leq \|u_y\|_\infty \int_{x_{2k-2}}^{x_{2k}} v(y)dy = \frac{4h^5}{15} \|u_y\|_\infty,$$

where

$$\begin{aligned} u_y(x, y) &= \frac{\partial}{\partial y} \{k(x, y)F[x_{2k-2}, x_{2k-1}, x_{2k}, y]\} = \frac{\partial k(x, y)}{\partial y} F[x_{2k-2}, x_{2k-1}, x_{2k}, y] \\ &+ k(x, y)F[x_{2k-2}, x_{2k-1}, x_{2k}, y]. \end{aligned}$$

The last formula uses a standard result for the differentiation of Newton divided differences (see [13]). To have this derivation be valid, we must have $u \in C(D)$, and this is true if $f \in \mathcal{X}$ and $k_x \in C(D)$. Combining these results, we have

$$\mathcal{K}(I - \mathcal{P}_m)f(x) = O(h^4). \quad (4.21)$$

By substituting (4.17) and (4.21), we have

$$\|f - \hat{f}_m\|_\infty \leq ch^4.$$

Eq. (4.19) comes from noting first that the property $\mathcal{P}_m \hat{f}_m = f_m$ implies

$$f_m(x_i) = \hat{f}_m(x_i), \quad i = 0, 1, \dots, m,$$

and second from applying Eq. (4.18). \square

5 Numerical Examples

In this section, some examples will be given to demonstrate the efficiency of our method. The errors have been calculated by using

$$e_m = \|f(x) - f_m(x)\| = \max \left\{ |f(x) - f_m(x)|, x \in D \right\}.$$

All computations have been performed on a Intel CPU PC. using a Matlab code.

Example 5.1. Consider the following linear Fredholm integral equation [14]:

$$f(x) = (1-x)e^x + x + \int_0^1 x^2 e^{y(x-1)} f(y) dy; \quad x \in D, \quad (5.1)$$

with the exact solution $f(x) = e^x$.

Table 1 and Fig. 2 illustrate the error results for this example. Also, we compare the maximum absolute error computed by the present method and rationalized hat functions method [10] in Table 2. This fact is obvious from Table 2 that the results obtained by the present method is better than that obtained in [10].

Table 1:
Absolute error for $m = 8, 16, 32$ of $f(x)$ of Eq. (5.1)

Nodes x	Present method		
	m=8	m=16	m=32
x = 0.0	0.000000e-0	0.000000e-0	0.000000e-0
x = 0.1	7.051122e-5	1.678878e-5	1.074000e-6
x = 0.2	1.442252e-4	9.440411e-6	2.420387e-6
x = 0.3	1.790884e-4	1.067389e-5	2.598107e-6
x = 0.4	9.099707e-5	2.401660e-5	1.466063e-6
x = 0.5	6.567563e-7	4.100712e-8	2.562325e-9
x = 0.6	1.167226e-4	2.765041e-5	1.768881e-6
x = 0.7	2.378355e-4	1.556802e-5	3.990328e-6
x = 0.8	2.943947e-4	1.754360e-5	4.286971e-6
x = 0.9	1.519933e-4	3.947407e-5	2.409461e-6
x = 1.0	2.253516e-6	1.410860e-7	8.821773e-9

Table 2:
Approximate infinity-norm of absolute error for Example 5.1

Methods	ϵ_m
Method of [10]	
m = 8	7.9e-3
m = 16	1.9e-3
m = 32	5.0e-4
Present method	
m = 8	3.1e-4
m = 16	4.0e-5
m = 32	4.6e-6

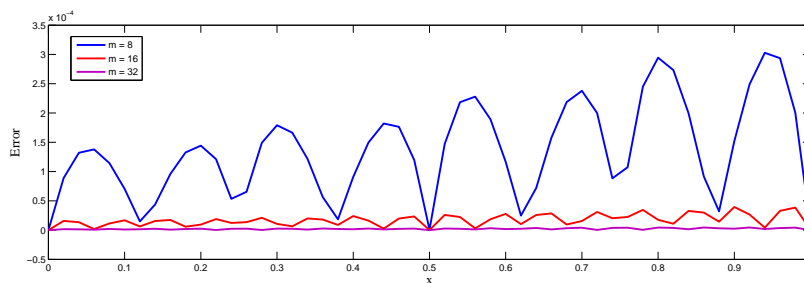


Figure 2. Absolute value of error, Example 5.1 with $m = 8, 16, 32$.

Example 5.2. Consider the following linear Fredholm integral equation [12]:

$$f(x) = e^{-x} \cos(x) - \frac{0.2e^{x-1}(1-x+(x-1)\cos(1)+\sin(1))}{x^2-2x+2} + \int_0^1 0.2e^{xy} f(y) dy ; \quad (5.2)$$

where $x \in D$ and the exact solution is $f(x) = e^{-x} \cos(x)$.

Table 3 and Fig. 3 illustrate the error results for this example. Also, we compare the maximum absolute error computed by the present method and rationalized hat functions method [10] in Table 2. This fact is obvious from Table 4 that the results obtained by the present method is better than that obtained in [10].

Table 3:
Absolute error for $m = 8, 16, 32$ of $f(x)$ of Eq. (5.2)

Nodes x	Present method		
	m=8	m=16	m=32
x = 0.0	7.759476e-7	4.853733e-8	3.034217e-9
x = 0.1	9.800906e-5	2.688648e-5	1.595818e-6
x = 0.2	1.829946e-4	1.019395e-5	2.414831e-6
x = 0.3	1.041159e-4	7.480676e-6	1.985549e-6
x = 0.4	4.695818e-5	1.012831e-5	6.876549e-7
x = 0.5	7.225526e-7	4.508725e-8	2.816836e-9
x = 0.6	1.672490e-5	5.248148e-6	2.848585e-7
x = 0.7	2.565871e-5	9.971313e-7	1.965583e-7
x = 0.8	9.320964e-6	1.664778e-7	5.168838e-9
x = 0.9	8.169541e-6	2.397558e-6	1.279250e-7
x = 1.0	1.586006e-6	9.919542e-8	6.200819e-9

Table 4:
Approximate infinity-norm of absolute error for Example 5.1

Methods	ϵ_m
Method of [10]	
m = 8	1.8e-3
m = 16	4.4e-4
m = 32	1.1e-4
Present method	
m = 8	2.0e-4
m = 16	2.7e-5
m = 32	3.0e-6

6 Conclusion

In this paper we have proposed method for solving linear Fredholm integral equations based on MHFs. This method converts the linear Fredholm integral

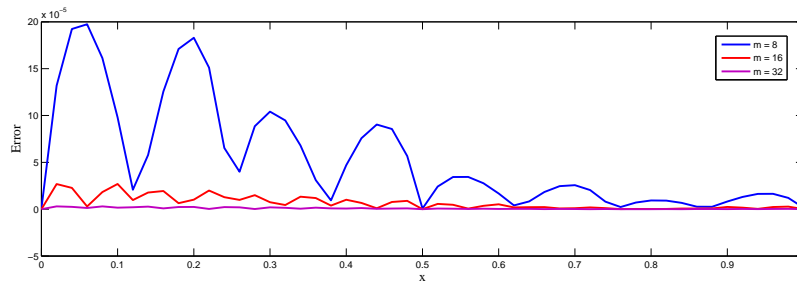


Figure 3. Absolute value of error, Example 5.2 with $m = 8, 16, 32$.

equations into a linear system of algebraic equations. Furthermore, it is proved that **MHFs** method is convergence and the order of convergence of this method is $O(h^4)$. A comparison is made between the numerical (and exact) solutions of [10]. In addition, the method can also be extended to the system of linear Fredholm integral equations, but some modifications are required.

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