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On The Kernel of Black-Scholes Equation Related to the Risk Neutrality for Cash-or-Nothing Options

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Abstract : In this paper, we studied the Kernel or the elementary solution of the Black-Scholes Equation and we can relate such Kernel to the risk neutrality for cash-or-nothing option. We obtained interesting new results.

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1 Introduction

It is well known that the Black-Scholes equation plays an important role in finding the option price of the stock market. This equation is given by

$$\frac{\partial}{\partial t}u(s,t) + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2}{\partial s^2}u(s,t) + rs\frac{\partial}{\partial s}u(s,t) - ru(s,t) = 0 \tag{1.1}$$

with the terminal condition

$$u(s,t) = (s_T - p)^+ \text{ for } 0 \leq t \leq T \tag{1.2}$$

where $u(s,t)$ is the option price at time t, σ is the volatility of stock, s_T is the price of stock at the expiration time T, r is the interest rate and p is the strike price. They obtain the solution $u(s,t)$ of (1.1) satisfies (1.2) of the form

$$u(s,t) = sN(d_1) - pe^{-r(T-t)}N(d_2) \tag{1.3}$$

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where $d_1 = \dfrac{ln(\frac{s}{p}) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$ and $d_2 = \dfrac{ln(\frac{s}{p}) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$ and denote by $N(x)$ the normal distribution $N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{\frac{y^2}{2}} dy$. see ([1], p91). The equation (1.3) is called the Black-Scholes Formula. In this paper, we transformed (1.1) and (1.2) to equation

$$\frac{\partial}{\partial\tau}V(R,\tau) - \frac{1}{2}\sigma^2\frac{\partial^2}{\partial R^2}V(R,\tau) - (r - \frac{\sigma^2}{2})\frac{\partial}{\partial R}V(R,\tau) + rV(R,\tau) = 0 \qquad (1.4)$$

and $V(R,0) = (e^R - p)^+$ by $R = lns$, $\tau = T - t$ and $u(s,t) = V(R,\tau)$. Since $(e^R - p)^+$ is continuous function of R, write the initial condition

$$V(R,0) = (e^R - p)^+ = f(R). \qquad (1.5)$$

We obtain the Kernel of (1.4) satisfies (1.5) in the form

$$K(R,\tau) = \frac{e^{-r\tau}}{\sqrt{2\pi\sigma^2\tau}}\exp\left[-\frac{(R - (r - \frac{1}{2}\sigma^2)\tau)^2}{2\sigma^2\tau}\right]. \qquad (1.6)$$

Next, we study the cash-or-nothing option with the risk neutrality by starting with the stock model

$$ds = \mu s dt + \sigma s dB \qquad (1.7)$$

where s is the price of stock at time t, μ is the drift of stock, B is Brownian motion and σ is the volatility of stock.

Equation (1.7) can be dirived to geometric Brownian motion of the form

$$s(t) = s_0\exp\left[\sigma B(t) + (\mu - \frac{\sigma^2}{2})t\right]$$

or

$$ln\frac{s(t)}{s_0} = \sigma B(t) + (\mu - \frac{\sigma^2}{2})t. \qquad (1.8)$$

Note that, the right hand side of (1.8) is a normal random variable whose mean is $(\mu - \frac{\sigma^2}{2})t$ and whose variance is $\sigma^2 t$.

Since $s(t)$ is a random variable for each time t, and the density function for $s(t)$ is given by

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi t}}]\exp\left[-\frac{(ln\frac{x}{s_0} - (\mu - \frac{\sigma^2}{2})t)^2}{2\sigma^2 t}\right] \qquad (1.9)$$

for $x > 0$ with $f(x) = 0$, for $x \leq 0$ see ([2],p56).
Note that, the payoff of the call option is given by payoff= $(s_T - p)^+$ where s_T is the price of stock at the expiration time T and p is the strike price. We also have the discounted expectation of the payoff $e^{-r(T-t)}E(s_T - p)^+$ where r is the interest rate and $0 \leq t \leq T$. Let

$$W(s,t) = e^{-r(T-t)}E(s_T - p)^+.$$

By the definition of expectation and the density function given by (1.9). We obtain

$$w(s,t) = e^{-r(T-t)} \int_0^\infty \frac{(x-p)^+}{x\sigma\sqrt{2\pi(T-t)}}]\exp\left[-\frac{(ln\frac{x}{s} - (\mu - \frac{\sigma^2}{2})(T-t))^2}{2\sigma^2(T-t)}\right]. \tag{1.10}$$

We see that (1.10) contains the drift μ. If we put $\mu = r$, then (1.10) become the option value of the risk neutrality. Let us study the cash-or-nothing options,

$$\text{the payoff} = \begin{cases} A, & \text{if } s_T > p \\ 0, & \text{if } s_T < p \end{cases} \tag{1.11}$$

where A is positive real number, see([2], p163) with the condition (1.11). Then (1.10) become.

$$w(s,t) = Ae^{-r(T-t)} \int_0^\infty \frac{1}{x\sigma\sqrt{2\pi(T-t)}}]\exp\left[-\frac{(ln\frac{x}{s} - (r - \frac{\sigma^2}{2})(T-t))^2}{2\sigma^2(T-t)}\right] dx \tag{1.12}$$

with $\mu = r$. By changing the variable, say $y = -\left[\frac{(ln\frac{x}{s} - (r - \frac{\sigma^2}{2})(T-t))^2}{2\sigma^2(T-t)}\right]$ together with the integral $\int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} du = \sqrt{2\pi}$, then (1.12) reduces to

$$w(s,t) = Ae^{-r(T-t)}N(d_2) \tag{1.13}$$

where d_2 is given by the Black-Scholes formula (1.3). Put $R = ln\frac{x}{s}$ in (1.12), we obtain

$$w(s,t) = Ae^{-r(T-t)} \int_{ln\frac{p}{s}}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2(T-t)}}]\exp\left[-\frac{(R - (r - \frac{\sigma^2}{2})(T-t))^2}{2\sigma^2(T-t)}\right] dR \tag{1.14}$$

$$= A\int_{ln\frac{p}{s}}^{\infty} K(R,\tau)dR \tag{1.15}$$

where

$$K(R,\tau)=\frac{e^{-r\tau}}{\sqrt{2\pi\sigma^2\tau}}\exp\left[-\frac{(R-(r-\frac{1}{2}\sigma^2)\tau)^2}{2\sigma^2\tau}\right]$$

is the Kernel of (1.14) with $\tau = T - t$. We see that (1.12) and (1.15) is the relationship between the Kernel and risk neutrality for cash-or-nothing option.

Let $u=\dfrac{R-(r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{2\tau}}$ then $dR=\sigma\sqrt{2\tau}du$. Now for $ln\dfrac{p}{s}\leq R<\infty$

$$\frac{ln\frac{p}{s}-(r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{2\tau}}\leq\frac{R-(r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{2\tau}}<\infty$$

$$\frac{ln\frac{p}{s}-(r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{2\tau}}\leq u<\infty.$$

Thus (1.14) becomes

$$w(s,t)=Ae^{-r\tau}\int_d^\infty e^{-u^2}du\cdot\frac{\sigma\sqrt{2\tau}}{\sqrt{2\pi\sigma^2\tau}}$$

where

$$d=\frac{ln\frac{p}{s}-(r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{2\tau}} \tag{1.16}$$

since $\int_{-\infty}^{\infty}=\int_{-\infty}^{d}+\int_d^\infty$. Thus

$$\begin{aligned}w(s,t)&=Ae^{-r\tau}\frac{\sigma\sqrt{2\tau}}{\sqrt{2\pi\sigma^2\tau}}\left[\int_{-\infty}^{\infty}e^{-u^2}du-\int_{-\infty}^{d}e^{-u^2}du\right]\\&=Ae^{-r\tau}\left[\frac{\sigma\sqrt{2\tau}\cdot\sqrt{\pi}}{\sqrt{2\pi\sigma^2\tau}}-\frac{1}{\sqrt{\pi}}\int_{-\infty}^{d}e^{-u^2}du\right]\\&=Ae^{-r\tau}\left[1-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d}e^{-\frac{y^2}{2}}dy\right]\text{ where } u=\frac{y}{\sqrt{2}}.\end{aligned}$$

Thus

$$w(s,t)=Ae^{-r\tau}[1-N(d)]=Ae^{-r\tau}N(-d) \tag{1.17}$$

(Note that $1-N(d)=N(-d)$). For $t=T$ where T is the expiration time and we have $\tau=0$. It follows that $d=ln\frac{p}{s_0}$ in (1.16) and since $s>p$ for call option. We have $d=-\infty$ with $ln\dfrac{p}{s}<0$. It follows that $N(d)=0$ for $d=-\infty$. Thus $w(s,T)=A(1-0)=A$ which is the payoff in(1.11). We can also compare (1.17) with (1.13) which is similar form.

2 Preliminaries

The following definition and basic concepts are needed.

Definition 2.1. Let f be an integrable function, the *Fourier transform* of f is defined by

$$\mathfrak{F}f(x) = \widehat{f}(w) = \int_{-\infty}^{\infty} e^{-i\omega x} f(x)dx \tag{2.1}$$

and the inverse

$$\mathfrak{F}^{-1}\widehat{f}(\omega) = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\widehat{f}(\omega)d\omega. \tag{2.2}$$

Lemma 2.2. *The Black-Scholes Equation given by (1.1) and the terminal condition (1.2) can be transformed to (1.4) with the initial condition (1.5) by changing the variable $R = lns$ and write $u(s,t) = V(R,\tau)$.*

Proof. We have

$$\frac{\partial}{\partial t}u(s,t) = \frac{\partial}{\partial t}V(R,\tau) = \frac{\partial}{\partial \tau}V(R,\tau)\cdot\frac{\partial \tau}{\partial t} = -\frac{\partial}{\partial \tau}V(R,\tau),$$

since $\tau = T - t$ and

$$\frac{\partial}{\partial s}u(s,t) = \frac{\partial}{\partial R}V(R,\tau)\cdot\frac{\partial R}{\partial s} = \frac{1}{s}\frac{\partial}{\partial R}V(R,\tau)$$
$$\frac{\partial^2}{\partial s^2}u(s,t) = \frac{\partial}{\partial s}\left(\frac{1}{s}\frac{\partial}{\partial R}V(R,\tau)\right) = -\frac{1}{s^2}\frac{\partial V}{\partial R} + \frac{1}{s^2}\frac{\partial^2 V}{\partial R^2}.$$

Substitute into (1.1) and (1.2) we have obtain (1.4) and (1.5) □

Lemma 2.3. *Given the equation in (1.4) with the initial condition in (1.5). Then we obtain the solution of (1.4) in the convolution form*

$$V(R,\tau) = K(R,\tau) * f(R)$$

where $K(R,\tau)$ is the kernel given by (1.6) and $f(R)$ is the continuous function given by (1.5).

Proof. Take the Fourier transform defined by (2.1) to the equation (1.4). We obtain

$$\frac{\partial}{\partial \tau}\widehat{V}(\omega,\tau) + \frac{1}{2}\sigma^2\omega^2\widehat{V}(\omega,\tau) + i\omega(r - \frac{1}{2}\sigma^2)\widehat{V}(\omega,\tau) + r\widehat{V}(\omega,\tau) = 0. \tag{2.3}$$

Now, for any fixed ω, (2.3) is the ordinary differential equation of variable τ and we obtain

$$\widehat{V}(\omega,\tau) = C(\omega)\exp\left[-(\frac{1}{2}\sigma^2\omega^2 + i\omega(r - \frac{1}{2}\sigma^2) + r)\tau\right]. \tag{2.4}$$

Since in (1.5), we also have

$$\widehat{V}(\omega,0)=\widehat{f}(\omega). \tag{2.5}$$

By the inverse Fourier transform in (2.2).

$$\begin{aligned}V(R,\tau)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega R}\widehat{V}(\omega,\tau)d\omega\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega R}\widehat{f}(\omega)\exp\left[-(\frac{1}{2}\sigma^2\omega^2+i\omega(r-\frac{1}{2}\sigma^2)+r)\tau\right]d\omega \text{ by (2.4).}\end{aligned}$$

Now, let $\widehat{f}(\omega)=\int_{-\infty}^{\infty}e^{-i\omega y}f(y)dy$. Thus

$$\begin{aligned}V(R,\tau)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{i\omega R}e^{-i\omega y}\exp\left[-(\frac{1}{2}\sigma^2\omega^2+i\omega(r-\frac{1}{2}\sigma^2)+r)\tau\right]f(y)dyd\omega\\&=\frac{e^{-r\tau}}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left[-(\frac{1}{2}\sigma^2\omega^2+i\omega((r-\frac{1}{2}\sigma^2)+r)\tau-(R-y)))\right]f(y)dyd\omega\\&=\frac{e^{-r\tau}}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left[-\left(\frac{(r-\frac{1}{2}\sigma^2)\tau-(R-y)}{2\sigma^2\tau}\right)^2\right]\times\\&\quad\exp\left[-\frac{1}{2}\sigma^2\tau\left(\omega-i\frac{(r-\frac{1}{2}\sigma^2)\tau-(R-y)}{\sigma^2\tau}\right)^2\right]f(y)dyd\omega.\end{aligned}$$

Put $u=\sigma\sqrt{\frac{\tau}{2}}(\omega-i\frac{(r-\frac{1}{2}\sigma^2)\tau-(R-y)}{\sigma^2\tau})$, $du=\sigma\sqrt{\frac{\tau}{2}}d\omega$, $d\omega=\frac{\sqrt{2}}{\sigma\sqrt{\tau}}du$. Thus

$$\begin{aligned}V(R,\tau)&=\frac{e^{-r\tau}}{2\pi}\int_{-\infty}^{\infty}\exp\left[-\left(\frac{(r-\frac{1}{2}\sigma^2)\tau-(R-y)}{2\sigma^2\tau}\right)^2\right]f(y)dy\int_{-\infty}^{\infty}e^{-u^2}du\frac{\sqrt{2}}{\sigma\sqrt{\tau}}\\&=\frac{e^{-r\tau}}{2\pi}\frac{\sqrt{2}\sqrt{\pi}}{\sigma\sqrt{\tau}}\int_{-\infty}^{\infty}\exp\left[-\left(\frac{(r-\frac{1}{2}\sigma^2)\tau-(R-y)}{2\sigma^2\tau}\right)^2\right]f(y)dy\\&=\frac{1}{\sqrt{2\pi\sigma^2\tau}}e^{-r\tau}\int_{-\infty}^{\infty}\exp\left[-\left(\frac{(r-\frac{1}{2}\sigma^2)\tau-(R-y)}{2\sigma^2\tau}\right)^2\right]f(y)dy\\&=K(R,\tau)*f(R).\quad \text{(Note that }\int_{-\infty}^{\infty}e^{-u^2}du=\sqrt{\pi})\end{aligned}$$

where

$$K(R,\tau)=\frac{1}{\sqrt{2\pi\sigma^2\tau}}e^{-r\tau}\int_{-\infty}^{\infty}\exp\left[-\left(\frac{(r-\frac{1}{2}\sigma^2)\tau-(R-y)}{2\sigma^2\tau}\right)^2\right] \tag{2.6}$$

is the kernel. Moreover, we have $lim_{\tau\to 0}K(R,\tau)=\delta(R)$ where is Dirac-delta distribution, see([3],p34). It follow that $V(R,0)=\delta(R)*f(R)=f(R)$. Thus (1.5) is satisfied. □

3 Main Results

Starting from (1.4) to (1.7) and Lemma 2.3 leading to the following theorems.

Theorem 3.1. *Given the equation*

$$\frac{\partial}{\partial\tau}V(R,\tau) - \frac{1}{2}\sigma^2\frac{\partial^2}{\partial R^2}V(R,\tau) - (r - \frac{\sigma^2}{2})\frac{\partial}{\partial R}V(R,\tau) + rV(R,\tau) = 0 \qquad (3.1)$$

and the initial condition $V(R,0) = (e^R - p)^+ = f(R)$ *then we obtain* $V(R,\tau) = K(R,\tau) * f(R)$ *as the solution of (3.1) where* $K(R,\tau)$ *is the kernel given by (2.6)*

Theorem 3.2. *Let* $w(s,t) = e^{-r(T-t)}E(s_T - p)^+$ *be the cash-or-nothing option for* $0 \leq t \leq T$ *and* $E(s_T - p)^+$ *is the expectation of the payoff* $(s_T - p)^+$, *Given the payoff*

$$(s_T - p)^+ = \begin{cases} A \text{ if } s_T > p \\ 0 \text{ if } s_T < p \end{cases}$$

where A is payoff real number. Then $w(s,t) = A\int_{ln\frac{p}{s}}^{\infty} K(R,\tau)dR$ *where* $K(R,\tau)$ *is given by (2.6). Moreover,* $w(s,t) = Ae^{-r(T-t)}N(-d)$ *where d is given by (1.16) and denote* $N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}dy$. *We have* $d = -\infty$ *at* $t+T$ *by (1.16). Thus* $N(-d) = 1$. *It follows that the payoff* $w(s,t) = A$ *which is the call payoff.*

4 Conclusion

The cash-or-nothing option with the risk neutrality with $\mu = r$ in (1.12) and obtain

$$w(s,t) = Ae^{-r(T-t)}N(d_2) \qquad (4.1)$$

where d_2 is given by the Black-Scholes formula (1.3). By approaching the Black-Scholes equation with the kernel $K(R,\tau)$ we obtain the cash-or-nothing option

$$w(s,t) = Ae^{-r(T-t)}N(-d) \qquad (4.2)$$

where d is given by (1.16). We see that (4.2) is related to (4.1) while (4.2) is more general and need more mathematical concepts which is helpful in advanced research in the area of Financial Mathematics.

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References

[1] J. Stemptli, V. Goodman, The Mathematics of Finance, Modeling and Hedging, The Brooks/cole series, series in Advance Mathematics, United State, United Kingdom, 2001.

[2] D.J. Itigham, An Introduction to Financial Option Valuation, Cambridge University Press, 2004.

[3] I.M. Gelfand, G.E. Shilov, Generalized Function, Vol. 1 Academic Press, 1972.

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