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Transformation Semigroups Admitting Nearring Structure

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Abstract : By *a right* [*left*] *nearring* we mean a triple $(N, +, \cdot)$ where $(N, +)$ is an ablien group, $(N, \cdot)$ is a semigroup and $(x + y) \cdot z = x \cdot z + y \cdot z$ $[z \cdot (x + y) = z \cdot x + z \cdot y]$ for all $x, y, z \in N$. A semigroup is said to *admit a right* [*left*] *nearring structure* if there is an operation $+$ on S such that $(S, +, \cdot)$ is a right [left] nearring or there is an operation $+$ on S^0 such that $(S^0, +, \cdot)$ is a right [left] nearring where $\cdot$ is the operation on S and S^0, respectively. For a nonempty set X, let $G(X)$, $T(X)$, $P(X)$ and $I(X)$ denote respectively the symmetric group on X, the full transformation semigroup on X, the partial transformation semigroup on X and the 1-1 partial transformation semigroup on X. Our purpose is to characterize when $G(X)$, $T(X)$, $P(X)$ and $I(X)$ admit a right nearring structure and a left nearring structure. The remarkable results are as follows : $T(X)$ and $P(X)$ admit a right nearring structure for every set X while they admit a left nearring structure only the case $|X| = 1$.

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1 Introduction

The cardinality of a set X will be denoted by $|X|$.

If S is a semigroup which does not process a zero or $|S| = 1$, let S^0 denote the semigroup S with an adjoined zero 0 ; otherwise $S^0 = S$. An *idempotent* of a semigroup S is an element $a \in S$ with $a^2 = a$.

A *right* [*left*] *nearring* is a triple $(N, +, \cdot)$ where
(i) $(N, +)$ is a group,
(ii) $(N, \cdot)$ is a semigroup and
(iii) for all $x, y, z \in N, (x + y) \cdot z = x \cdot z + y \cdot z$ $[z \cdot (x + y) = z \cdot x + z \cdot y]$.
Subnearrings of a right [left] nearring are defind naturally. Nearrings have the following basic properties :

Proposition 1.1. ([2], page 19) *If* $(N, +, \cdot)$ *is a right* [*left*] *nearring, then*
(i) $0 \cdot x = 0$ $[x \cdot 0 = 0]$ *for all* $x \in N$ *where* 0 *is the identity of* $(N, +)$ *and*
(ii) $(-x) \cdot y = -(x \cdot y)$ $[x \cdot (-y) = -(x \cdot y)]$ *for all* $x, y \in N$.

Hence if $(N,+,\cdot)$ is a right [left] nearring, then $(N,\cdot)$ has a left [right] zero.

Some standard examples of right nearrings are $(M(\mathbb{R}),+,\circ)$, $(C(\mathbb{R}),+,\circ)$ and $(D(\mathbb{R}),+,\circ)$ where

$$\begin{aligned} M(\mathbb{R}) &= \text{the set of all mappings } f:\mathbb{R}\to\mathbb{R}, \\ C(\mathbb{R}) &= \{f\in M(\mathbb{R}) \mid f \text{ is continuous}\}, \\ D(\mathbb{R}) &= \{f\in M(\mathbb{R}) \mid f \text{ is differentiable}\} \end{aligned}$$

and $+$ and $\circ$ are respectively the usual addition and composition of functions. We see that all the right nearrings $(M(\mathbb{R}),+,\circ)$, $(C(\mathbb{R}),+,\circ)$ and $(D(\mathbb{R}),+,\circ)$ are additively commutative. Throughout, our right nearrings and our left nearrings are assumed to be additively commutative. Observe that the right nearrings given above are not left nearrings. Then they are not rings.

Since the multiplicative structure of any ring is by definition a semigroup with zero, it is valid to ask whether a given semigroup S has S^0 isomorphic to the multiplicative structure of some ring. If it does, S is said to *admit a ring structure*. Equivalently, S admits a ring structure if and only if there is an operation $+$ on S^0 such that $(S^0,+,\cdot)$ is a ring where $\cdot$ is the operation on S^0. Semigroups admitting ring structure have long been studied. For examples, see [4], [6], [1], [7] and [5]. Right [left] nearrings are a generalization of rings and by definition and Proposition 1.1(i), their multiplicative structures are semigroups with left [right] zero. Hence we have the following fact.

$$\begin{array}{l}\text{If } (S,\cdot) \text{ is a semigroup without left [right] zero, then there is}\\ \text{no operation } + \text{ on } S \text{ such that } (S,+,\cdot) \text{ is a right [left] nearring.}\end{array} \tag{1}$$

A right [left] nearring $(N,+,\cdot)$ is called *zero-symmetric* if $0\cdot x = x\cdot 0 = 0$ for all $x\in N$. The following question is reasonable.

$$\begin{array}{l}\text{For a given semigroup } S, \text{ is there an operation } + \text{ on } S^0\\ \text{such that } (S^0,+,\cdot) \text{ is a zero-symmetric right [left] nearring}\\ \text{where } \cdot \text{ is the operation on } S^0?\end{array} \tag{2}$$

Note that a right [left] zero of $(N,\cdot)$ is an idempotent. Because of (1) and (2), the definition of a semigroup admitting right [left] nearring structure is reasonably given as follows : A semigroup S is said to *admit a right [left] nearring structure* if there is an operation $+$ on S such that $(S,+,\cdot)$ is a right [left] nearring where $\cdot$ is the operation on S or there is an operation $+$ on S^0 such that $(S^0,+,\cdot)$ is a right [left] nearring where $\cdot$ is the operation on S^0. Equivalently, a semigroup S admits a right [left] nearring structure if and only if S or S^0 is isomorphic to the multiplicative structure of some right [left] nearring. We note here that if $(S^0,+,\cdot)$ is a right [left] nearring, then by Proposition 1.1(i), 0 is the identity of $(S^0,+,\cdot)$.

The following fact will be quoted.

Proposition 1.2. ([2], page 7) *If $(G,+)$ is a group and $M(G)$ is the set of all mappings $f : G \to G$, then $(M(G),+,\circ)$ is a right nearring where $+$ and $\circ$ are the usual addition and composition of functions, respectively.*

For a nonempty set X, let

$$\begin{aligned} G(X) &= \text{the symmetric semigroup on } X,\\ T(X) &= \text{the full transformation semigroup on } X,\\ P(X) &= \text{the partial transformation semigroup on } X,\\ I(X) &= \text{the 1-1 partial transformation semigroup on } X. \end{aligned}$$

Then $G(X) \subseteq T(X) \subseteq P(X)$ and $G(X) \subseteq I(X) \subseteq P(X)$. Also, $P(X)$ and $I(X)$ have a zero 0 (the empty transformation) but $G(X)$ and $T(X)$ have no zero if $|X| > 1$. The domain and the range of $f \in P(X)$ are denoted by $\mathrm{dom} f$ and $\mathrm{ran} f$, respectively. For convenience, for $\emptyset \neq A \subseteq X$ and $x \in X$, let A_x be the constant mapping whose domain and range are A and $\{x\}$, respectively and the identity map on A may be denoted by 1_A. For distinct $a, b \in X$, let $(a\ b) \in G(X)$ be defined by $(a\ b)(a) = b$, $(a\ b)(b) = a$ and $(a\ b)(x) = x$ for all $x \in X \smallsetminus \{a, b\}$. Elements of $P(X)$ may be written by bracket notation. For examples,

$$\begin{aligned} \begin{pmatrix} a \\ b \end{pmatrix} &= \text{ the mapping in } I(X) \text{ whose domain and}\\ &\quad\ \text{range are } \{a\} \text{ and } \{b\}\text{, respectively,}\\ \begin{pmatrix} a & x \\ b & x \end{pmatrix}_{x \in X \smallsetminus \{a\}} &= \text{ the mapping } f \in T(X) \text{ defined by}\\ &\quad\ f(x) = \begin{cases} b & \text{if } x = a,\\ x & \text{ortherwise.} \end{cases} \end{aligned}$$

Observe that

$$A_x = \begin{pmatrix} a \\ x \end{pmatrix}_{a \in A} \text{ and } (a\ b) = \begin{pmatrix} a & b & x \\ b & a & x \end{pmatrix}_{x \in X \smallsetminus \{a,b\}}.$$

The following facts are known.

Proposition 1.3. ([7]) *Let X be a nonempty set. Then the following statements hold.*

(i) *$G(X)$ admits a ring structure if and only if $|X| \leq 2$*

(ii) *If $S(X)$ is $T(X)$, $P(X)$ or $I(X)$, then $S(X)$ admits a ring structure if and only if $|X| = 1$.*

Two-sided distribution was used for the proof of Proposition 1.3. These results motivate us to characterize when these standard transformation semigroups admit a right nearring structure and admit a left nearring structure. The following important facts are helpful for our work.

Proposition 1.4. ([7]) *For any nonempty set X, there is an operation $+$ on X such that $(X, +)$ is an abelian group.*

It can be seen from the proof of Proposition 1.4 in [7] that the identity of the group $(X, +)$ can be specified. Hence we have

Proposition 1.5. *If X is a nonempty set and $a \in X$, then there is an operation $+$ on X such that $(X, +)$ is an abelian group with identity a.*

Proposition 1.6. ([3], page 41) *Let X be a nonempty set and θ a symbol not representing any element of X. For $f \in P(X)$, define $f^* \in T(X \cup \{\theta\})$ by*

$$f^*(x) = \begin{cases} f(x) & \text{if } x \in \mathrm{dom} f, \\ \theta & \text{otherwise.} \end{cases}$$

Then $f \mapsto f^*$ is an isomorphism from $P(X)$ onto the subsemigroup of $T(X \cup \{\theta\})$ consisting of all $g \in T(X \cup \{\theta\})$ with $g(\theta) = \theta$.

2 Transformation Semigroups

For convenience, let $\mathcal{R}, \mathcal{RNR}$ and $\mathcal{LNR}$ denote respectively the class of all semigroups admitting a ring structure, the class of all semigroups admitting a right nearring structure and the class of all semigroups admitting a left nearring structure. Then we have $\mathcal{R} \subseteq \mathcal{RNR} \cap \mathcal{LNR}$.

Theorem 2.1. *For a nonempty set X,*
(i) *$G(X) \in \mathcal{RNR}$ if and only if $|X| \leq 2$ and*
(ii) *$G(X) \in \mathcal{LNR}$ if and only if $|X| \leq 2$.*

Proof. If $|X| \leq 2$, then by Proposition 1.3(i), $G(X) \in \mathcal{R}$. Hence the converses of (i) and (ii) hold since $\mathcal{R} \subseteq \mathcal{RNR}$ and $\mathcal{R} \subseteq \mathcal{LNR}$.

Assume that $|X| > 2$. Let a, b, c be distinct elements of X. Since $G(X)$ is a group and 1_X is the only idempotent of $G(X)$, it follows that $G(X)$ has neither a right zero nor a left zero. First, suppose that $G(X) \in \mathcal{RNR}$. Then there is an operation $+$ on $G^0(X)$ such that $(G^0(X), +, \circ)$ is a right nearring. Thus $1_X + (a\ b) = f$ for some $f \in G^0(X)$, so

$$f = 1_X + (a\ b) = ((a\ b) + 1_X)(a\ b) = (1_X + (a\ b))(a\ b) = f(a\ b)$$

Case 1: $f \neq 0$. Then $(a\ b) = 1_X$, a contradiction.

Case 2: $f = 0$. Then $1_X + (a\ b) = 0$. Since $(a\ b) \neq (a\ c)$, $1_X + (a\ c) \neq 0$. Let $1_X + (a\ c) = g \in G(X)$. Then

$$g = 1_X + (a\ c) = ((a\ c) + 1_X)(a\ c) = g(a\ c),$$

so $(a\ c) = 1_X$, a contradiction.

This proves that $G(X) \notin \mathcal{RNR}$.

Next, to show that $G(X) \notin \mathcal{LNR}$, suppose on the contrary that $G(X) \in \mathcal{LNR}$. Then there is an operation $+$ on $G^0(X)$ such that $(G^0(X), +, \circ)$ is a left nearring. Then $1_X + (a\ b) = f$ for some $f \in G^0(X)$. Then

$$f = 1_X + (a\ b) = (a\ b)((a\ b) + 1_X) = (a\ b)f$$

Case 1: $f \neq 0$. Then $(a\ b) = 1_X$, a contradiction.

Case 2: $f = 0$. Then $1_X + (a\ c) = g$ for some $g \in G(X)$. Since

$$g = 1_X + (a\ c) = (a\ c)((a\ c) + 1_X) = (a\ c)g,$$

we have $(a\ c) = 1_X$, a contradiction.

Hence the proof is complete. □

Theorem 2.2. (i) *For any nonempty set X, $T(X) \in \mathcal{RNR}$.*
(ii) *For any nonempty set X, $T(X) \in \mathcal{LNR}$ if and only if $|X| = 1$.*

Proof. (i) Let X be a nonempty set. By Proposition 1.4, there is an operation $+$ on X such that $(X, +)$ is an abelian group. For $f, g \in T(X)$, define

$$(f + g)(x) = f(x) + g(x) \text{ for all } x \in X.$$

By Proposition 1.2, $(T(X), +, \circ)$ is a right nearring, so $T(X) \in \mathcal{RNR}$.

(ii) Suppose that $|X| > 1$. Let $a, b \in X$ be distinct. Since $X_a f = X_a$ and $X_b f = X_b$ for all $f \in T(X)$, it follows that $T(X)$ has no right zero. Suppose that there is an operation $+$ on $T^0(X)$ such that $(T^0(X), +, \circ)$ is a left nearring. Then $X_a + X_b = f$ for some $f \in T^0(X)$.

Case 1: $f \neq 0$. Then

$$X_a + X_a = X_a\,(X_a + X_b) = X_a f = X_a$$

which implies that $X_a = 0$, a contradiction.

Case 2: $f = 0$. Then $X_a + X_b = 0$ and

$$X_a + X_a = X_a(X_a + X_b) = X_a 0 = 0.$$

It follows that $X_b = X_a$, a contradiction.

The converse is obtained from Proposition 1.3(ii) since $\mathcal{R} \subseteq \mathcal{LNR}$. □

Theorem 2.3. (i) *For any nonempty set X, $P(X) \in \mathcal{RNR}$.*
(ii) *For a nonempty set X, $P(X) \in \mathcal{LNR}$ if and only if $|X| = 1$.*

Proof. (i) Let X be a nonempty set. Let θ be a symbol not representing any element of X. For $f \in P(X)$, define $f^* \in T(X \cup \{\theta\})$ as in Proposition 1.6. By Proposition 1.6, the mapping $f \mapsto f^*$ is an isomorphism from $P(X)$ onto the subsemigroup $\{g \in T(X \cup \{\theta\}) \mid g(\theta) = \theta\}$ of $T(X \cup \{\theta\})$. From Proposition 1.5, there is an operation $+$ on $X \cup \{\theta\}$ such that $(X \cup \{\theta\}, +)$ is an abelian group with identity θ. Then $(T(X \cup \{\theta\}), +, \circ)$ is a right nearring by Proposition 1.2 where

$$(f + g)(x) = f(x) + g(x) \text{ for all } f, g \in T(X \cup \{\theta\}) \text{ and } x \in X.$$

If $g, h \in T(X \cup \{\theta\})$ are such that $g(\theta) = \theta = h(\theta)$, then

$$(g + h)(\theta) = g(\theta)+h(\theta) = \theta + \theta = \theta,\ (gh)(\theta) = g(h(\theta)) = g(\theta) = \theta,$$
$$(-g)(\theta) = -g(\theta) = -\theta = \theta.$$

Therefore $\{g \in T(X \cup \{\theta\}) \mid g(\theta) = \theta\}$ is a subnearring of the right nearring $(T(X \cup \{\theta\}), +, \circ)$. But $P(X)$ is isomorphic to $(\{g \in T(X \cup \{\theta\}) \mid g(\theta) = \theta\}, \circ)$, so $P(X) \in \mathcal{RNR}$.

(ii) Assume that $|X| > 1$. Let a and b be distinct elements of X. Suppose that there is an operation $+$ on $P(X)$ such that $(P(X), +, \circ)$ is a left nearring. Then $\begin{pmatrix} a \\ a \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} = f$ for some $f \in P(X)$. Then

$$\begin{pmatrix} a \\ a \end{pmatrix} f = \begin{pmatrix} a \\ a \end{pmatrix} \left(\begin{pmatrix} a \\ a \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} \right) = \begin{pmatrix} a \\ a \end{pmatrix} + 0 = \begin{pmatrix} a \\ a \end{pmatrix} \text{ and}$$
$$\begin{pmatrix} b \\ b \end{pmatrix} f = \begin{pmatrix} b \\ b \end{pmatrix} \left(\begin{pmatrix} a \\ a \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} \right) = 0 + \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

It follows that $a = \begin{pmatrix} a \\ a \end{pmatrix}(a) = \left(\begin{pmatrix} a \\ a \end{pmatrix} f \right)(a) = \begin{pmatrix} a \\ a \end{pmatrix} f(a)$ and $b = \begin{pmatrix} a \\ b \end{pmatrix}(a) = \left(\begin{pmatrix} b \\ b \end{pmatrix} f \right)(a) = \begin{pmatrix} b \\ b \end{pmatrix} f(a)$ which imply respectively that $f(a) = a$ and $f(a) = b$. This is a contradiction.

The converse holds by Proposition 1.3(ii). □

Theorem 2.4. *Let X be a nonempty set.*
(i) *$I(X) \in \mathcal{RNR}$ if and only if $|X| = 1$.*
(ii) *$I(X) \in \mathcal{LNR}$ if and only if $|X| = 1$.*

Proof. (i) Assume that $|X| > 1$ and let $a, b \in X$ be distinct. Suppose that there is an operation $+$ on $I(X)$ such that $(I(X), +, \circ)$ is a right nearring. Then $\begin{pmatrix} a \\ a \end{pmatrix} + \begin{pmatrix} b \\ a \end{pmatrix} = f$ for some $f \in I(X)$, so

$$f \begin{pmatrix} a \\ a \end{pmatrix} = \left(\begin{pmatrix} a \\ a \end{pmatrix} + \begin{pmatrix} b \\ a \end{pmatrix} \right) \begin{pmatrix} a \\ a \end{pmatrix} = \begin{pmatrix} a \\ a \end{pmatrix} + 0 = \begin{pmatrix} a \\ a \end{pmatrix},$$
$$f \begin{pmatrix} b \\ b \end{pmatrix} = \left(\begin{pmatrix} a \\ a \end{pmatrix} + \begin{pmatrix} b \\ a \end{pmatrix} \right) \begin{pmatrix} b \\ b \end{pmatrix} = 0 + \begin{pmatrix} b \\ a \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix}.$$

These implies that $f(a) = \left(f\begin{pmatrix} a \\ a \end{pmatrix}\right)(a) = \begin{pmatrix} a \\ a \end{pmatrix}(a) = a$ and $f(b) = \left(f\begin{pmatrix} b \\ b \end{pmatrix}\right)(b) = \begin{pmatrix} b \\ a \end{pmatrix}(b) = a$. This is a contradiction since f is 1-1.

The converse is obtained from Proposition 1.3(ii).

(ii) The proof can be given the same as that of Theorem 2.3(ii). □

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