Thai Journal of Mathematics
Special Issue (Annual Meeting in Mathematics, 2006) : 19–23

Regularity of Full Order-Preserving Transformation Semigroups on Some Dictionary Posets

W. Mora and Y. Kemprasit

Abstract : For a poset X, let $OT(X)$ be the full order-preserving transformation semigroup on X. The following results are known. If X is any nonempty subset of $\mathbb{Z}$, then $OT(X)$ is a regular semigroup, that is, for every $\alpha \in OT(X)$, $\alpha = \alpha\beta\alpha$ for some $\beta \in OT(X)$. If X is an interval in $\mathbb{R}$, then $OT(X)$ is regular if and only if X is closed and bounded. We deal with the regularity of $OT(A \times A, \leq_d)$ where $\phi \neq A \subseteq \mathbb{Z}$ and $\leq_d$ is the dictionary partial order on $A \times A$. We have that if A is infinite, then the chain $(A \times A, \leq_d)$ is neither order-isomorphic to a subset of $\mathbb{Z}$ nor order-isomorphic to an interval in $\mathbb{R}$. Our purpose is to show that $OT(A \times A, \leq_d)$ is regular if and only if A is finite.

Keywords : Full order-preserving transformation semigroup, regular semigroup
2000 Mathematics Subject Classification : 20M17, 20M20

1 Introduction

An element a of a semigroup S is called *regular* if $a = aba$ for some $b \in S$, and S is called a *regular semigroup* if every element of S is regular.

For a nonempty set X, let $T(X)$ be the full transformation semigroup on X, that is, $T(X)$ is the semigroup, under composition, of all mappings $\alpha : X \to X$. The image of x under $\alpha \in T(X)$ is written by $x\alpha$, and the range (image) of $\alpha \in T(X)$ is denoted by ran α. It is well-known that $T(X)$ is a regular semigroup ([1], page 4 or [2], page 63).

A mapping φ from a poset X into a poset Y is said to be *order-preserving* if

$$\forall x, x^{'} \in X, \quad x \leq x^{'} \text{ in } X \;\Rightarrow\; x\varphi \leq x^{'}\varphi \text{ in } Y.$$

The posets X and Y are said to be *order-isomorphic* if there is an order-preserving bijection φ from X onto Y such that $\varphi^{-1} : Y \to X$ is order-preserving.

If X is a poset, let $OT(X)$ be the subsemigroup of $T(X)$ consisting of all order-preserving mappings, that is,

$$OT(X) = \{ \ \alpha \in T(X) \mid \ \alpha \text{ is order-preserving } \}.$$

It is known from [1, page 203] that $OT(X)$ is regular if X is a finite chain. In 2000, Y. Kemprasit and T. Changphas [3] extended this result to any chain which is order-isomorphic to a subset of $\mathbb{Z}$, the set of integers with their natural order. It was also proved in [3] that if X is an interval in $\mathbb{R}$, the set of real numbers with usual order, then $OT(X)$ is regular if and only if X is closed and bounded. In [4], the authors generalized full order-preserving transformation semigroups by using sandwich multiplication and investigated their regularity and also provided some isomorphism theorems.

The results in [3] motivate us to consider the regularity of $OT(A \times A, \leq_d)$ where $\phi \neq A \subseteq \mathbb{Z}$ and $\leq_d$ is the dictionary partial order, that is ,

$$(a,b) \leq_d (c,d) \text{ if and only if } \begin{array}{ll} \text{(i)} & a < c \text{ or} \\ \text{(ii)} & a = c \text{ and } b \leq d. \end{array}$$

Then $(A \times A, \leq_d)$ is a chain. If A is finite, then $(A \times A, \leq_d)$ is a finite chain, and hence $OT(A \times A, \leq_d)$ is a regular semigroup. If A is infinite, then $A \times A$ is countably infinite, so $(A \times A, \leq_d)$ is not isomorphic to any interval in $\mathbb{R}$. It will be shown that for an infinite subset A of $\mathbb{Z}$, $(A \times A, \leq_d)$ is not order-isomorphic to any subset of $\mathbb{Z}$. Our main purpose is to show that for any $\phi \neq A \subseteq \mathbb{Z}$, $OT(A \times A, \leq_d)$ is regular if and only if A is finite.

2 Main Results

Let $\mathbb{Z}^+$ and $\mathbb{Z}^-$ denote the set of positive integers and the set of negative integers, respectively. If $A \subseteq \mathbb{Z}$ is infinite, then A has one of following properties:

(i) A is bounded below but not bounded above.
(ii) A is bounded above but not bounded below.
(iii) A is neither bounded below nor bounded above.

Then (i), (ii) and (iii) imply respectively that

(i)′ $A = \{ a_i \mid i \in \mathbb{Z}^+ \}$ where $a_1 < a_2 < a_3 < \ldots$,
(ii)′ $A = \{ a_i \mid i \in \mathbb{Z}^- \}$ where $a_{-1} > a_{-2} > a_{-3} > \ldots$ and
(iii)′ $A = \{ a_i \mid i \in \mathbb{Z} \}$ where $\ldots < a_{-2} < a_{-1} < a_0 < a_1 < a_2 < \ldots$.

We first show that for an infinite subset A of $\mathbb{Z}$, the chain $(A \times A, \leq_d)$ is not order-isomorphic to a subset of $\mathbb{Z}$.

For convenience, if S_1 and S_2 are subsets of a chain, we write $S_1 < S_2$ if $x < y$ for all $x \in S_1$ and $y \in S_2$.

Proposition 2.1. *For an infinite subset A of $\mathbb{Z}$, $(A \times A, \leq_d)$ is not order-isomorphic to a subset of $\mathbb{Z}$.*

Proof. First, we recall that for a sequence (x_n) in $S \subseteq \mathbb{Z}$, if $x_1 < x_2 < x_3 < \ldots$, then $\{ x_n \mid n \in \mathbb{Z}^+ \}$ has no upper bound in S. Also, if $x_1 > x_2 > x_3 > \ldots$, then $\{ x_n \mid n \in \mathbb{Z}^+ \}$ has no lower bound in S.

Case 1 : $A = \{ a_i \mid i \in \mathbb{Z}^+ \}$ where $a_1 < a_2 < \ldots$ or $A = \{ a_i \mid i \in \mathbb{Z} \}$

where $\ldots < a_{-1} < a_0 < a_1 < \ldots$. Then (a_2, a_1) is an upper bound of $\{\ (a_1, a_i) \mid i \in \mathbb{Z}^+\ \}$ and $(a_1, a_1) <_d (a_1, a_2) <_d (a_1, a_3) <_d \ldots$. Hence we have that $(A \times A, \leq_d)$ is not order-isomorphic to a subset of $\mathbb{Z}$.

Case 2 : $A = \{\ a_i \mid i \in \mathbb{Z}^-\ \}$ where $a_{-1} > a_{-2} > \ldots$. Then (a_{-2}, a_{-1}) is a lower bound of $\{\ (a_{-1}, a_i) \mid i \in \mathbb{Z}^-\ \}$ and $(a_{-1}, a_{-1}) >_d (a_{-1}, a_{-2}) >_d (a_{-1}, a_{-3}) >_d \ldots$. It follows that $(A \times A, \leq_d)$ is not order-isomorphic to a subset of $\mathbb{Z}$. □

To obtain the main result, the following fact is needed.

Lemma 2.2. *If α and β are elements of $T(X)$ such that $\alpha = \alpha\beta\alpha$, then ran$(\beta\alpha)$= ran α and $x\beta\alpha = x$ for all $x \in$ ran α.*

Proof. Since ran $\alpha = \text{ran}(\alpha\beta\alpha) \subseteq \text{ran}(\beta\alpha) \subseteq$ ran α, we have $\text{ran}(\beta\alpha) =$ ran α. If $x \in X$, then $x\alpha = x\alpha\beta\alpha = (x\alpha)\beta\alpha$. This implies that $x\beta\alpha = x$ for all $x \in$ ran α. □

Theorem 2.3. *Let $\phi \neq A \subseteq \mathbb{Z}$. Then $OT(A \times A, \leq_d)$ is a regular semigroup if and only if A is finite.*

Proof. Suppose that A is infinite. Let $c \in A$ be a fixed element and define $\alpha : A \times A \to A \times A$ by

$$(\{x\} \times A)\alpha = \{(c, x)\} \text{ for all } x \in A,$$

that is,

$$(x, y)\alpha = (c, x) \text{ for all } x, y \in A.$$

Then we have

$$\text{ran}\alpha = \{c\} \times A. \tag{2.1}$$

Since $\{x\} \times A < \{y\} \times A$ and $(c, x) <_d (c, y)$ for all $x, y \in A$ with $x < y$, we deduce that $\alpha \in OT(A \times A, \leq_d)$. To show that α is not regular in $OT(A \times A, \leq_d)$, suppose on the contrary that $\alpha = \alpha\beta\alpha$ for some $\beta \in OT(A \times A, \leq_d)$. By (1) and Lemma 2.2,

$$(c, x)\beta\alpha = (c, x) \text{ for all } x \in A. \tag{2.2}$$

Case 1 : $A = \{\ a_i \mid i \in \mathbb{Z}^+\ \}$ where $a_1 < a_2 < \ldots$ or $A = \{\ a_i \mid i \in \mathbb{Z}\ \}$ where $\ldots < a_{-1} < a_0 < a_1 < \ldots$. Then $c < e$ for some $e \in A$. Hence $(c, x) <_d (e, e)$ for all $x \in A$. This implies that $(c, x)\beta\alpha \leq_d (e, e)\beta\alpha$ for all $x \in A$. By (2),

$$(c, x) \leq_d (e, e)\beta\alpha \quad \text{for all } x \in A.$$

Since $(e, e)\beta\alpha \in$ ran α, by (1), $(e, e)\beta\alpha = (c, f)$ for some $f \in A$. Consequently,

$$(c, x) \leq_d (c, f) \quad \text{for all } x \in A,$$

so $x \leq f$ for all $x \in A$. This is a contradiction since A has no maximum.

Case 2 : $A = \{\ a_i \mid i \in \mathbb{Z}^-\ \}$ where $a_{-1} > a_{-2} > \dots$. Then $r < c$ for some $r \in A$. Thus $(r, r) <_d (c, x)$ for all $x \in A$ which implies that $(r, r)\beta\alpha \leq_d (c, x)\beta\alpha$ for all $x \in A$. Therefore we have from (2) that

$$(r, r)\beta\alpha \leq_d (c, x) \quad \text{for all } x \in A.$$

But $(r, r)\beta\alpha \in \operatorname{ran} \alpha$, so by (1), $(r, r)\beta\alpha = (c, s)$ for some $s \in A$. Hence

$$(c, s) \leq_d (c, x) \quad \text{for all } x \in A.$$

This implies that $s \leq x$ for all $x \in A$ which is a contradiction since A has no minimum.

This proves that if $OT(A \times A, \leq_d)$ is regular, then A is finite. The converse holds because $(A \times A, \leq_d)$ is a finite chain if A is finite. □

Remark 2.4. (1) The given proof of Proposition 2.1 uses the basic fact of $\mathbb{Z}$. As mentioned previously, if a chain X is order-isomorphic to a subset of $\mathbb{Z}$, then $OT(X)$ is a regular semigroup. Then Proposition 2.1 can be referred as a corollary of Theorem 2.3.
(2) From the proof of Theorem 2.3, we define $\alpha \in OT(A \times A, \leq_d)$ depending on a given $c \in A$ where A is an infinite subset of $\mathbb{Z}$. Then α can be written as α_c. Observe that $\alpha_{c_1} \neq \alpha_{c_2}$ if $c_1 \neq c_2$ in A. Since each α_c is a nonregular element of $OT(A \times A, \leq_d)$, we deduce that $OT(A \times A, \leq_d)$ has an infinite number of nonregular elements. Since every constant mapping from $A \times A$ into $A \times A$ is a regular element of $OT(A \times A, \leq_d)$, it follows that $OT(A \times A, \leq_d)$ also contains an infinite number of regular elements.

References

[1] P.M. Higgins, *Techniques of semigroup theory,* Oxford University Press, New York, 1992.

[2] J.M. Howie, *Fundamentals of semigroup theory,* Clarendon Press. Oxford, 1995.

[3] Y. Kemprasit and T. Changphas, Regular order-preserving transformation semigroups, *Bull. Austral. Math. Soc.* **62** (2000), 511-524.

[4] Y. Kemprasit and S. Jaidee, Regularity and isomorphism theorems of generalized order-preserving transformation semigroups, *Vietnam J. Math.* **33** (2000),253-260.

(Received 25 May 2006)

W. Mora and Y. Kemprasit
Department of Mathematics
Faculty of Science
Chulalongkorn University
Bangkok 10330, THAILAND.
e-mail: `Yupaporn.K@chula.ac.th`