Thai Journal of Mathematics
Volume 14 (2016) Number 3 : 797–804
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

On the Positive Colored Noise of the Price of Stock

Amnuay Kananthai

Department of Mathematics, Faculty of Science, Chiang Mai University
Chiang Mai 50200, Thailand
e-mail : malamnka@gmail.com

Abstract : In studying the fluctuation of the price of stock, there are three parameters affecting such fluctuation. The first is well known parameter which is known as the volatility of stock. The second is not really well know which is white noise and has not been computed. Fortunately, in this paper we can find the formula of such white noise and can be computed. The third parameter is the positive colored noise which is the aim of this paper. Such positive colored noise is derived from the white noise and it is interesting parameter particularly, in involving the infinitely fluctuation of the price of stock. Moreover, The relationships between three parameter has been also studied and obtaining the interesting results.

Keywords : Black-Scholes equation.
2010 Mathematics Subject Classification : 35K05; 91G20.

1 Introduction

In Stochastic Differential Equation(SDE), there is an interesting parameter which is call the white noise. In 1975, T. Hida introduced the Theory of White noise, see[1]. The other interesting parameter is the positive colored noise. It is believed that such colored noise causes the infinite fluctuations of the price of stock, see([2], pp 4-6).

We know that the white noise is the derivative of the Brownian motion, that is $\xi(t) = \dfrac{dB(t)}{dt}$ where $B(t)$ denotes the Brownian motion. Since $\xi(t) = \dfrac{dB(t)}{dt}$ does not exist in the classical or Newtonian sense but it exists or has meaning in the white noise space which is the generalized of the space of tempered distribution, see([2], pp 6-8). Now, the positive colored noise is defined by

$$C(t) = \frac{e^{\xi(t)}}{\mathbb{E}(e^{\xi(t)})}$$

where $\xi(t)$ is the white noise and $\mathbb{E}(e^{\xi(t)})$ is the expectation of $e^{\xi(t)}$. Actually $C(t)$ is the normalization of $e^{\xi(t)}$.

In this paper, we studied the white noise ξ and the positive colored noise $C(t)$ involving the stock model. Such stock model is given by

$$ds = \mu s dt + \sigma s dB \tag{1.1}$$

where $s(t)$ is the price of stock at time t, μ is the drift of stock, σ is volatility of stock and $B(t)$ is Brownian motion. By the concept of white noise, (1.1) can be written by

$$ds = \mu s dt + \sigma s \xi dt. \tag{1.2}$$

By Ito's formula, (1.1) has a solution

$$s = s_0 \exp\left[(\mu - \frac{\sigma^2}{2})t + \sigma B\right] \tag{1.3}$$

where s_0 is the price of stock at $t = 0$. By using the tempered distribution and Ito's formula then (1.2) becomes the formula

$$\xi = \frac{1}{t\sigma} ln \frac{s}{s_0} - \frac{\mu}{\sigma} + \frac{\sigma}{2} \text{ for } 0 < t. \tag{1.4}$$

Now,

$$\begin{aligned} e^{\xi} &= \exp\left[\frac{1}{t\sigma} ln \frac{s}{s_0} - \frac{\mu}{\sigma} + \frac{\sigma}{2}\right] \\ &= \exp\left[-\frac{\mu}{\sigma} + \frac{\sigma}{2}\right] \left(\frac{s}{s_0}\right)^{\frac{1}{t\sigma}} = \exp\left[\frac{\sigma^2 - 2\mu}{2\sigma}\right] \left(\frac{s}{s_0}\right)^{\frac{1}{t\sigma}} \end{aligned}$$

and the expectation of e^{ξ} , that is

$$\mathbb{E}(e^{\xi}) = \exp\left[\frac{\sigma^2 - 2\mu}{2\sigma}\right] \mathbb{E}\left[\left(\frac{s}{s_0}\right)^{\frac{1}{t\sigma}}\right].$$

Thus the positive colored noise

$$C(t) = \frac{e^{\xi}}{\mathbb{E}(e^{\xi})} = \frac{\exp\left[\frac{\sigma^2 - 2\mu}{2\sigma}\right] \left(\frac{s}{s_0}\right)^{\frac{1}{t\sigma}}}{\exp\left[\frac{\sigma^2 - 2\mu}{2\sigma}\right] \mathbb{E}\left[\left(\frac{s}{s_0}\right)^{\frac{1}{t\sigma}}\right]}.$$

We obtain

$$C(t) = \frac{s^{\frac{1}{t\sigma}}}{\mathbb{E}((s)^{\frac{1}{t\sigma}})}. \tag{1.5}$$

From (1.3) and by direct computation

$$C(t) = \exp\left[\frac{\sigma}{2} - \frac{\mu}{\sigma} - \frac{1}{2t}\right]\left(\frac{s}{s_0}\right)^{\frac{1}{\sigma t}},\ t > 0. \tag{1.6}$$

2 Preliminaries

The following Definitions and Lemmas are needed.

Definition 2.1. Let $B(t)$ be the stochastic process then $B(t)$ is called the *Brownian motion* if it has the following properties.

(i) $B(0) = 0$ almost surely(a.s)

(ii) $B(t) - B(s)$ is normal distribution with mean 0 and variance $t - s$ for all $0 \le s \le t$

(iii) For all time $0 < t_1 < t_2 < \cdots < t_n$ the random variable $B(t_1), B(t_2) - B(t_1), \cdots, B(t_n) - B(t_{n-1})$ are independent(independent increment).

Notice in particular that

$$\mathbb{E}(B(t)) = 0,\ \mathbb{E}(B^2(t)) = t \text{ for each time } t \ge 0.$$

Definition 2.2. The white noise $\xi(t)$ and the positive colored noise $C(t)$ are defined by

$$\xi(t) = \frac{dB(t)}{dt} \text{ and } C(t) = \frac{e^{\xi(t)}}{\mathbb{E}(e^{\xi(t)})}$$

where $\mathbb{E}$ represent the expectation.

Actually, $\xi(t)$ and $C(t)$ does not exist in the classical sense or Newtonian sense, but they exist in the white noise space which the generalized of the space of tempered distribution, see[[2], pp. 6-8].

Lemma 2.3. *The stock model given by (1.1) has a solution given by (1.3) which is also called the Geometric Brownian Motion.*

Proof. see[[3],pp. 59-61]. □

Lemma 2.4. *The stock model in the form of white noise given by (1.2) has formula of ξ given by (1.4)*

Proof. We have from (1.2)

$$ds = \mu s dt + \sigma s \xi dt.$$

By Taylor expansion and Ito's formula, we obtain by

$$\int_0^t d\ lns(\tau)d\tau = (\mu - \frac{\sigma^2}{2})t + \sigma \int_0^t \frac{dBT(\tau)}{d\tau} d\tau$$
$$= (\mu - \frac{\sigma^2}{2}) + \sigma \int_0^t \xi(\tau)d\tau. \qquad (2.1)$$

Since $\xi \in S'(\mathbb{R})$ where $S'(\mathbb{R})$ is the space of tempered distribution, see[[2], pp. 6-8]. Thus for every $\varphi \in S(\mathbb{R})$ where $S(\mathbb{R})$ is the Schwartz Space. Define

$$\xi(\varphi) =< \xi(t), \varphi(t) >= \int_{-\infty}^{\infty} \xi(t)\varphi(t)dt$$

φ is called the test function. Now, consider $\int_0^t \xi(\tau)d\tau$. Write

$$\int_0^t \xi(\tau)d\tau = \int_0^t \xi(\tau)\frac{\varphi(\tau)}{\varphi(\tau)}d\tau \text{ for } \varphi(\tau) \in S(\mathbb{R}).$$

Since $\xi(\tau) \in S'(\mathbb{R})$, then $\frac{\xi(\tau)}{\varphi(\tau)} \in S'(\mathbb{R})$. It follows that $\frac{\xi(\tau)}{\varphi(\tau)}$ is also tempered distribution. Write

$$F(\tau) = \frac{\xi(\tau)}{\varphi(\tau)}.$$

We have $F(\tau)$ is the smooth function of τ. Thus

$$\int_0^t \xi(\tau)d\tau =< F(\tau), \varphi(\tau) >= \int_0^t F(\tau)\varphi(\tau)d\tau.$$

By mean value theorem, there exists τ^* for $0 < \tau^* < t$

$$\int_0^t F(\tau)\varphi(\tau)d\tau = F(\tau^*)\varphi(\tau^*)\int_0^t d\tau$$
$$= \frac{\xi(\tau^*}{\varphi(\tau^*}\varphi(\tau^*)t$$
$$= \xi(\tau^*)t$$

Thus $\int_0^t \xi(\tau)d\tau = \xi(\tau^*)dt$. Thus, By (2.1)

$$ln\frac{s(t)}{s_0} = (\mu - \frac{\sigma^2}{2})t + \sigma\xi t.$$

Thus, we obtain

$$\xi = \frac{1}{t\sigma} ln \frac{s}{s_0} - \frac{\mu}{\sigma} + \frac{\sigma}{2} \text{ for } t > 0.$$

Thus we obtain the formula of white noise ξ in (1.4) as required. □

Lemma 2.5. *The positive colored noise given by the definition 2.2 can be written in the from (1.5) and can be computed as the formula given by (1.6)*

Proof. We have the positive colored noise $C(t)$ defined by $C(t) = \frac{e^{\xi(t)}}{\mathbb{E}(e^{\xi(t)})}$ where $\xi(t) = \frac{dB(t)}{dt}$ is the white noise, $B(t)$ is the Brownian Motion and $\mathbb{E}(e^{\xi(t)})$ is the expectation of $e^{\xi(t)}$. Now from (1.4)

$$\begin{aligned}
\xi(t) &= \frac{1}{t\sigma} ln \frac{s}{s_0} - \frac{\mu}{\sigma} + \frac{\sigma}{2}, \\
e^{\xi(t)} &= \exp\left[ln \left(\frac{s}{s_0}\right)^{\frac{1}{t\sigma}} - \frac{\mu}{\sigma} + \frac{\sigma}{2} \right] \\
&= \exp\left[ln \left(\frac{s}{s_0}\right)^{\frac{1}{t\sigma}} \right] \exp\left[\frac{\sigma^2 - 2\mu}{2\sigma}\right] \\
&= \exp\left[\frac{\sigma^2 - 2\mu}{2\sigma}\right] \left(\frac{s}{s_0}\right)^{\frac{1}{t\sigma}} \\
\mathbb{E}(e^{\xi(t)}) &= \exp\left[\frac{\sigma^2 - 2\mu}{2\sigma}\right] \mathbb{E}\left(\frac{s}{s_0}\right)^{\frac{1}{t\sigma}}.
\end{aligned}$$

Thus

$$C(t) = \frac{\exp\left[\frac{\sigma^2 - 2\mu}{2\sigma}\right] \left(\frac{s}{s_0}\right)^{\frac{1}{t\sigma}}}{\exp\left[\frac{\sigma^2 - 2\mu}{2\sigma}\right] \mathbb{E}\left(\frac{s}{s_0}\right)^{\frac{1}{t\sigma}}} = \frac{\left(\frac{s}{s_0}\right)^{\frac{1}{t\sigma}}}{\mathbb{E}\left(\frac{s}{s_0}\right)^{\frac{1}{t\sigma}}} = \frac{(s)^{\frac{1}{t\sigma}}}{\mathbb{E}((s)^{\frac{1}{t\sigma}})}.$$

Thus we obtain (1.5). Now, from (1.3)

$$\begin{aligned}
\left(\frac{s}{s_0}\right)^{\frac{1}{t\sigma}} &= \exp\left[((\mu - \frac{\sigma^2}{2})t + \sigma B)\frac{1}{t\sigma}\right] \\
&= \exp\left[\frac{B}{t} + (\frac{\mu}{\sigma} - \frac{\sigma}{2})\right].
\end{aligned}$$

Thus

$$\mathbb{E}\left(\frac{s}{s_0}\right)^{\dfrac{1}{t\sigma}} = \mathbb{E}\left(\exp\left[\frac{B}{t}+(\frac{\mu}{\sigma}-\frac{\sigma}{2})\right]\right) = \exp\left[\frac{\mu}{\sigma}-\frac{\sigma}{2}\right]\mathbb{E}(\exp[\frac{B}{t}]).$$

Since B is Gaussian or normal Distribution, with mean 0 and variance t. Now write $B=\sqrt{t}Z$ where z is standard normal distribution with mean 0 and variance 1. So we have

$$\begin{aligned}
\mathbb{E}(\exp[\frac{B}{t}] &= \mathbb{E}(\exp[\frac{\sqrt{t}z}{t}] \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp[\frac{\sqrt{t}}{t}x]\exp[-\frac{x^2}{2}]dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp[-\frac{x^2}{2}+\frac{sqrtt}{t}x]dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp[-\frac{1}{2}(\frac{x^2}{2}-2\frac{\sqrt{t}}{t}x)]dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp[-\frac{1}{2}(x-\frac{\sqrt{t}}{t})^2+\frac{t}{2t^2}]dx.
\end{aligned}$$

Put $u=\dfrac{1}{\sqrt{2}}(x-\dfrac{\sqrt{t}}{t}),\ dx=\sqrt{2}du$. Thus

$$\mathbb{E}(\exp[\frac{B}{t}]) = \frac{1}{\sqrt{2\pi}}\exp[\frac{1}{2t}]\int_{-\infty}^{\infty}e^{-u^2}\sqrt{2}du = \frac{\sqrt{2}\sqrt{\pi}}{\sqrt{2\pi}}\exp[\frac{1}{2t}] = \exp[\frac{1}{2t}].$$

Thus

$$\begin{aligned}
\mathbb{E}\left(\frac{s}{s_0}\right)^{\dfrac{1}{t\sigma}} &= \exp[\frac{\mu}{\sigma}-\frac{\sigma}{2}]\mathbb{E}(\exp[\frac{B}{t}]) \\
&= \exp[\frac{\mu}{\sigma}-\frac{\sigma}{2}]\exp[\frac{1}{2t}].
\end{aligned}$$

It follow that

$$C(t) = \frac{\left(\frac{s}{s_0}\right)^{\dfrac{1}{r\sigma}}}{\mathbb{E}\left(\frac{s}{s_0}\right)^{\dfrac{1}{t\sigma}}} = \frac{\left(\frac{s}{s_0}\right)^{\dfrac{1}{t\sigma}}}{\exp[\frac{\mu}{\sigma}-\frac{\sigma}{2}+\frac{1}{2t}]} = \exp[\frac{\sigma}{2}-\frac{\mu}{\sigma}-\frac{1}{2t}]\left(\frac{s}{s_0}\right)^{\dfrac{1}{t\sigma}}.$$

We obtain (1.6) as required. □

3 Main Result

From the Introduction part, the preliminaries part Lemma 2 .1, Lemma 2.2 and Lemma 2.3 leading to the main theorem.

Theorem 3.1. *The positive colored noise* $C(t)$ *given by* $C(t) = \dfrac{e^{\xi(t)}}{\mathbb{E}(e^{\xi(t)})}$ *where* $\xi(t) = \dfrac{bB(t)}{dt}$ *is the white noise,* $B(t)$ *is Brownian Motion and* $\mathbb{E}(e^{\xi(t)})$ *is the expectation of* $e^{\xi(t)}$*. Then*

$$C(t) = exp[\frac{\sigma}{2} - \frac{\mu}{\sigma} - \frac{1}{2t}]\left(\frac{s}{s_0}\right)^{\frac{1}{t\sigma}}$$

where $s(t)$ *is the price of stock at any time* t*,* s_0 *is the price of stock at* $t = 0$*,* σ *is the volatility and* μ *is the drift of stock.*

Proof. Follows from Lemma 2.1-Lemma 2.3 □

By simulation $s(t)$ from geometric Brownian motion choose $\mu = .05$, $\sigma = 0.4$ and $s_0 = 40$, T=1 and dt = .001. the graph of stock price $s(t)$, white noise $\xi(t)$ and positive colored noise $C(t)$ as show below

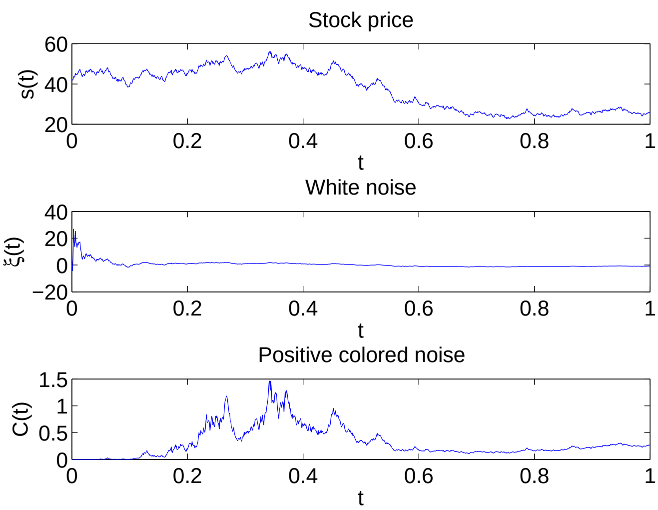

Figure 1: Simulation of the stock price and computation of white noise and positive colored noise.

4 Conclusion

We know that the price s of stock is random variable or the stochastic process and it fluctuation like Brownian motion. So the price of stock given by (1.3) has the graph of fluctuation. Compare with the white noise ξ given by (1.4) and the positive colored noise $C(t)$ given by (1.6). The graph of such white noise ξ has a high fluctuation at the beginning of the short period of time and low fluctuation of the the long period of time and slow down smoothly to zero as $t \to \infty$. That means if we let time t going so long The white noise ξ will not appear in the stock model. Thus we only have $ds = \mu s dt$. It follows that $s = s_0 e^{\mu t}$. In this case $\mu = r$, That mean we deposit money s_0 in the bank with interest μ at the time t

For positive colored noise $C(t)$, we obtain the graph is much more fluctuation than the price given by (1.3) and the white noise given by (1.4). That gives us more knowledge of the fluctuation and can be distinguished between the types of stock.

Acknowledgements : The authors would like to thank the Research Administration Center, Chiangmai University for financial supports.

References

[1] T. Hida, White Noise Analysis. An overview and some future direction, IIAS Reports 1995-001, 1995.

[2] H.M. Kuo, White Noise Distribution Theory, CRC Press, Boca Raton 1996.

[3] M. Baxter, R. Rennie, Financial Calculus, Cambridge University Press 1996.

(Received 2 October 2016)
(Accepted 16 December 2016)