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The Applications of Modified Generalized Mixed Equilibrium Problems to Nonlinear Problems

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Abstract : From the concept of generalized mixed equilibrium problems, we introduce a new problem to prove a strong convergence theorem for finding a common element of the set of fixed point of an infinite family of nonexpansive mappings and the set of a finite family of generalized mixed equilibrium problems in Hilbert space. We also apply our main result for generalized equilibrium problems and variational inequality problems.

Keywords : nonexpansive mapping; generalized mixed equilibrium problem; K-mapping, fixed point problem.
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1 Introduction

The fixed theory plays an important role in nonlinear functional analysis and becomes a very useful tool in various fields. In applications to neural networks, fixed point theorems can be used to design a dynamic neural network in order to solve steady state solutions (see [1]) and consider the stability of impulsive cellular neural networks with time-varying delays (see [2]). Some methods have been proposed to solve the fixed point theorem; see, for example, [3, 4] and the references therein. Let H be a real Hilbert space and C be a nonempty closed

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convex subset of H. A point $x \in C$ is called a *fixed point* of T if $Tx = x$. The set of fixed points of T is denoted by set $Fix(T) := \{x \in C : Tx = x\}$.
A mapping T of C into itself is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C.$$

Definition 1.1. Let $A : C \to H$ be a mapping. Then A

(i) is called *monotone* if

$$\langle Ax - Ay, x - y\rangle \geq 0, \forall x, y \in C.$$

(ii) is called *ρ-strongly monotone* if there exists a positive constant ρ such that

$$\langle Ax - Ay, x - y\rangle \geq \rho \|x - y\|^2, \forall x, y \in C.$$

(iii) is called *μ-Lipschitzian* if there exists a positive constant μ such that

$$\|Ax - Ay\| \leq \mu\|x - y\|, \forall x, y \in C.$$

(iv) is called *α-inverse strongly monotone* if there exists a positive real number $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2, \forall x, y \in C.$$

Let $A : C \to H$ be a mapping. The *variational inequality problem* is to find a point $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0 \tag{1.1}$$

for all $v \in C$. The set of solutions of the variational inequality is denoted by $VI(C, A)$. The applications of the variational inequality problem has been expanded to problems from economics, finance, optimization and game theory, see [5] and the references therein.

Let $F : C \times C \to \mathbb{R}$ be a bifunction and $A : C \to H$ be a nonlinear mapping and $\varphi : C \to \mathbb{R}$ be a real-valued function. The *generalized mixed equilibrium problem* is to find $x \in C$ such that

$$F(x, y) + \varphi(y) - \varphi(x) + \langle Ax, y - x\rangle \geq 0, \tag{1.2}$$

for all $y \in C$, see [6]. The set of solutions of (1.2) is denoted by $GMEP(F, \varphi, A)$, that is

$$GMEP(F, \varphi, A) = \{x \in C : F(x, y) + \varphi(y) - \varphi(x) + \langle Ax, y - x\rangle \geq 0, \forall y \in C\}.$$

If $\varphi = 0$, then (1.2) reduces to *the generalized equilibrium problem.* The set of solution of generalized equilibrium problem is denoted by $EP(F, A)$; see, for example, [7] and [8]. If $F = 0, \varphi = 0$, then problem (1.2) reduces to (1.1). If $A = 0$, then (1.2) reduces to *the mixed equilibrium problem.* The set of solutions of mixed equilibrium problem is denote by $MEP(F,\varphi)$; see, for example [9] and [10]. If $\varphi = 0, A = 0$, then problem (1.2) reduces to *the equilibrium problem.* The set of solutions of the equilibrium point is denoted by $EP(F)$. Finding a solution of equilibrium problem can be applied to many problems in physics, optimization and economics. Several people have proposed some useful methods for solving the generalized mixed equilibrium problem, generalized equilibrium problem, mixed equilibrium problem and equilibrium problem; see, for example, [7, 11, 12, 13, 14] and the references therein. In the past few years, many authors studied the systems of equilibrium problems and systems of generalized equilibrium problems. Several iterative methods have been proposed to solve the solution sets of such problems and the solution sets of various nonlinear operator problems in Hilbert spaces; see, for example, [15, 16, 17, 18, 19] and the references therein.

In 2008, Jian-Wen Peng and Jen-Chih Yao [6] defined a mapping $T_r^{(F,\varphi)} : H \to C$ as follows: For $r > 0$ and $x \in H$,

$$T_r^{(F,\varphi)}(x) = \big\{z \in C : F(z,y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\big\}.$$

They showed that $T_r^{(F,\varphi)}$ is single-valued and firmly nonexpansive and satisfies

$$Fix(T_r^{(F,\varphi)}) = MEP(F,\varphi).$$

In 2011, Gang Cai and Shangquan Bu [20] introduced a new iterative algorithm by hybrid method for finding a common element of the set of solutions of finite general mixed equilibrium problems and the set of solutions of a general variational inequality problem for a finite family of inverse strongly monotone mappings and the set of common fixed points of infinite family of strictly pseudocontractive mappings as follows:

$$\begin{cases} u_n = T_{r_{M,n}}^{(F_M,\varphi_M)}(I - r_{M,n}B_M)T_{r_{M-1,n}}^{(F_{M-1},\varphi_{M-1})}(I - r_{M-1,n}B_{M-1})\cdots T_{r_{1,n}}^{(F_1,\varphi_1)}(I - r_{1,n}B_1)x_n, \\ y_n = P_C(I - \lambda_N A_N)P_C(I - \lambda_{N-1}A_{N-1})\cdots P_C(I - \lambda_2 A_2)P_C(I - \lambda_1 A_1)u_n, \\ z_n = \alpha_n y_n + (1 - \alpha_n)S_n y_n, \\ C_{n+1} = \{z \in C_n : \|z_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \forall n \geq 1, \end{cases} \tag{1.3}$$

and proved a strong convergence theorem of the sequence $\{x_n\}$ under suitable conditions.

Recently, Gang Cai and Shangquan Bu [21] studied a new general iterative scheme for finding a common element of the set of solutions of finite general mixed equilibrium problems, the set of solutions of finite variational inequalities for cocoercive mappings, the set of solutions of common fixed points of an infinite family of nonexpansive mappings and the set of solutions of fixed points of a nonexpansive semigroup in Hilbert space as follows:

$$\begin{cases} x_1 = x \in C, \\ z_n = T_{r_{M,n}}^{(F_M,\varphi_M)}(I - r_{M,n}B_M)T_{r_{M-1,n}}^{(F_{M-1},\varphi_{M-1})}(I - r_{M-1,n}B_{M-1})\cdots T_{r_{1,n}}^{(F_1,\varphi_1)}(I - r_{1,n}B_1)x_n, \\ u_n = P_C(I - \lambda_{N,n}A_N)P_C(I - \lambda_{N-1,n}A_{N-1})\cdots P_C(I - \lambda_{2,n}A_2)P_C(I - \lambda_{1,n}A_1)z_n, \\ x_{n+1} = \alpha_n f(S_n x_n) + \beta_n x_n + (1 - \beta_n - \alpha_n)W(\tau_n)S_n u_n, \forall n \geq 1, \end{cases} \tag{1.4}$$

and proved a strong convergence theorem of the sequence $\{x_n\}$ under appropriate conditions of parameter $\{\alpha_n\}$ and $\{\beta_n\}$.

Very recently, Gang Cai and Shangquan Bu [22] introduced two iterative algorithms for finding a common element of the set of solutions of finite general mixed equilibrium problems and the set of solutions of finite variational inequalities for inverse strongly monotone mappings and the set of common fixed points of an asymptotically κ-strictly pseudocontractive mapping in the intermediate sense in a real Hilbert space as follows:

$$\begin{cases} u_n = T_{r_{M,n}}^{(F_M,\varphi_M)}(I - r_{M,n}B_M)T_{r_{M-1,n}}^{(F_{M-1},\varphi_{M-1})}(I - r_{M-1,n}B_{M-1})\cdots T_{r_{1,n}}^{(F_1,\varphi_1)}(I - r_{1,n}B_1)x_n, \\ z_n = P_C(I - \lambda_{N,n}A_N)P_C(I - \lambda_{N-1,n}A_{N-1})\cdots P_C(I - \lambda_{2,n}A_2)P_C(I - \lambda_{1,n}A_1)z_n, \\ k_n = \delta_n z_n + (1 - \delta_n)T^n z_n, \\ y_n = (1 - \alpha_n)x_n + \alpha_n k_n, \\ C_{n+1} = \left\{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\right\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \forall n \geq 1. \end{cases} \tag{1.5}$$

Under suitable conditions, they proved the sequence $\{x_n\}$ converges strongly to an element of a set $\bigcap_{i=k}^{M} GMEP\,(F_k, \varphi_k, A_k) \bigcap \bigcap_{i=1}^{N} VI\,(C, A_i) \bigcap F(T)$ where B_k and A_i is μ_k -inverse strongly monotone and η_i -inverse-strongly monotone, respectively, for every $k \in \{1, 2, \ldots, M\}$, $i \in \{1, 2, \ldots, N\}$.

After we have considered these research, we have the following questions.

1. Can we prove a strong convergence theorem for finding a common solution of the set of a finite family of generalized mixed equilibrium problems by not using the composite form of mappings $T_r^{(F,\varphi)}$ in (1.3), (1.4) and (1.5)?
2. Can we use the different method from [20], [21] and [22] to prove a strong convergence theorem for finding a common solution of the set of a finite family of generalized mixed equilibrium problems?

Let $F : C \times C \to \mathbb{R}$ be a bifunction. For every $i = 1, 2, ..., N$, let $A_i : C \to H$ be mappings and $\varphi : C \to \mathbb{R}$ be a real-valued function. From (1.2), we introduce the new problem is to find $x \in C$ such that

$$F(x, y) + \varphi(y) - \varphi(x) + \langle \sum_{i=1}^{N} a_i A_i x, y - x\rangle \geq 0, \tag{1.6}$$

for all $y \in C$ and $\sum_{i=1}^{N} a_i = 1$. This problem is called *the modified generalized mixed equilibrium problem*. The set of solutions of (1.6) is denoted by $GMEP(F, \varphi, \sum_{i=1}^{N} a_i A_i)$, that is,

$$GMEP(F, \varphi, \sum_{i=1}^{N} a_i A_i) = \Big\{x \in C : F(x, y) + \varphi(y) - \varphi(x) + \langle \sum_{i=1}^{N} a_i A_i x, y - x\rangle \geq 0, \forall y \in C, \sum_{i=1}^{N} a_i = 1\Big\}.$$

If $A = A_i$ for every $i = 1, 2, ..., N$, then $GMEP(F, \varphi, \sum_{i=1}^{N} a_i A_i)$ reduces to $GMEP(F, \varphi, A)$.

In this paper, using (1.6), we prove a strong convergence theorem for finding a common element of the set of fixed point of an infinite family of nonexpansive mappings and the set of a finite family of generalized mixed equilibrium problems in Hilbert space. We also utilize our main result to prove a convergence theorem for a finite family of generalized equilibrium problems and a finite family of variational inequalities problems.

2 Preliminaries

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. We denote weak and strong convergence by notations " $\rightharpoonup$ " and " $\rightarrow$ ", respectively. In a real Hilbert space H, it is well known that

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha \|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2,$$

for all $x, y \in H$ and $\alpha \in [0,1]$. It is well known that H satisfies *Opial's condition*[23], i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\lim_{n\to\infty} \inf \|x_n - x\| < \lim_{n\to\infty} \inf \|x_n - y\|,$$

holds for every $y \in H$ with $y \neq x$.
Let P_C be the metric projection of H onto C i.e., for $x \in H, P_C x$ satisfies the property

$$\|x - P_C x\| = \min_{y\in C} \|x - y\|.$$

The following lemmas are needed to prove the main theorem.

Lemma 2.1 (See [24])**.** *Given $x \in H$ and $y \in C$. Then, $P_C x = y$ if and only if there holds the inequality*

$$\langle x - y, y - z\rangle \geq 0, \forall z \in C.$$

Lemma 2.2 (See [25])**.** *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1-\alpha_n)s_n + \delta_n, \forall n \geq 0,$$

where α_n is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} s_n = 0.$

Lemma 2.3. *Let H be a real Hilbert space. Then, the following inequality holds*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle,$$

for all $x, y \in H$.

Lemma 2.4 (See [24])**.** *Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let A be a mapping of C into H. Let $u \in C$. Then, for $\lambda > 0$,*

$$u = P_C(I - \lambda A)u \Leftrightarrow u \in VI(C, A),$$

where P_C is the metric projection of H onto C.

Definition 2.5 (See [26])**.** Let C be a nonempty convex subset of a real Banach space X. Let $\{T_i\}_{n=1}^{\infty}$ be an infinite family of nonexpensive mappings of C into itself and let $\lambda_1, \lambda_2, \dots,$ be real numbers in $[0,1]$. Define the mapping $K_n : C \to C$ as follows:

$$\begin{aligned}
U_{n,0} =& I,\\
U_{n,1} =& \lambda_1 T_1 U_{n,0} + (1-\lambda_1)U_{n,0},\\
U_{n,2} =& \lambda_2 T_2 U_{n,1} + (1-\lambda_2)U_{n,1},\\
&\vdots\\
U_{n,k} =& \lambda_k T_k U_{n,k-1} + (1-\lambda_k)U_{n,k-1},\\
U_{n,k+1} =& \lambda_{k+1} T_{k+1} U_{n,k} + (1-\lambda_{k+1})U_{n,k},\\
&\vdots\\
U_{n,n-1} =& \lambda_{n-1} T_{n-1} U_{n,n-2} + (1-\lambda_{n-1})U_{n,n-2},\\
K_n =& U_{n,n} = \lambda_n T_n U_{n,n-1} + (1-\lambda_n)U_{n,n-1}.
\end{aligned}$$

Such a mapping K_n is called the *K-mapping* generated by $T_1, T_2, ..., T_n$ and $\lambda_1, \lambda_2, ...\lambda_n$.

For solving the generalized mixed equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F, φ and C satisfy the following conditions:
$(A1)$ $F(x,x) = 0$ for all $x \in C$;
$(A2)$ F is monotone, i.e., $F(x,y) + F(y,x) \le 0$ for all $x, y \in C$;
$(A3)$ For each $x, y, z \in C$,

$$\lim_{t\downarrow 0} F\left(tz + (1-t)x, y\right) \le F(x,y);$$

$(A4)$ For each $x \in C, y \mapsto F(x,y)$ is convex and lower semicontinuous;
$(B1)$ For each $x \in H$ and $r > 0$ there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0;$$

$(B2)$ C is a bounded set.

Then, we have following lemma.

Lemma 2.6 (See [6]). *Assume that $F : C \times C \to \mathbb{R}$ satisfies $(A1)-(A4)$ and let $\varphi : C \to \mathbb{R}$ be a proper lower semicontinuous and convex function. Assume that either $(B1)$ or $(B2)$ holds. For $r > 0$ and $x \in H$, define a mapping $T_r^{(F,\varphi)} : H \to C$ as follows:*

$$T_r^{(F,\varphi)}(x) = \left\{z \in C : F(z,y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \forall y \in C\right\} \quad (2.1)$$

for all $z \in H$. Then, the following hold:

1. *for each $x \in H, T_r^{(F,\varphi)} \neq \emptyset$;*
2. *$T_r^{(F,\varphi)}$ is single-valued;*
3. *$T_r^{(F,\varphi)}$ is firmly nonexpansive, i.e., for any $x, y \in H$,*
$$\|T_r^{(F,\varphi)}(x) - T_r^{(F,\varphi)}(y)\|^2 \leq \langle T_r^{(F,\varphi)}(x) - T_r^{(F,\varphi)}(y), x-y\rangle;$$
4. *$Fix(T_r^{(F,\varphi)}) = MEP(F,\varphi)$;*
5. *$MEP(F,\varphi)$ is closed and convex.*

Lemma 2.7 (See [26]). *Let C be a nonempty closed convex subset of a strictly convex Banach space. Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpanxive mappings of C into itself with $\bigcap_{i=1}^{\infty} Fix(T_i) \neq \emptyset$ and let $\lambda_1, \lambda_2, \dots,$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, 2, \dots,$ with $\sum_{i=1}^{\infty} \lambda_i < \infty$. For every $n \in \mathbb{N}$, let K_n be the K-mapping generated by $T_1, T_2, \dots, T_n$ and $\lambda_1, \lambda_2, \dots, \lambda_n$. Then for every $x \in C$ and $k \in \mathbb{N}, \lim_{n\to\infty} K_n x$ exists.*

For every $k \in \mathbb{N}$ and $x \in C$. Kangtunyakarn[26] defined a mapping $K : C \to C$ as follows:

$$Kx = \lim_{n\to\infty} K_n x. \quad (2.2)$$

Such a mapping K is called the K-*mapping* generated by $T_1, T_2, \dots$ and $\lambda_1, \lambda_2, \dots$.

Remark 2.8 (See [26]). *For every $n \in \mathbb{N}$, K_n is a nonexpansive mapping and $\lim_{n\to\infty} sup_{x\in D} \|K_n x - Kx\| = 0$, for every bounded subset D of C.*

Lemma 2.9 (See [26]). *Let C be a nonempty closed convex subset of a strictly convex Banach space. Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpanxive mappings of C into itself with $\bigcap_{i=1}^{\infty} Fix(T_i) \neq \emptyset$ and let $\lambda_1, \lambda_2, \dots,$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, 2, \dots,$ with $\sum_{i=1}^{\infty} \lambda_i < \infty$. Let K_n be the K-mapping generated by $T_1, T_2, \dots, T_n$ and $\lambda_1, \lambda_2, \dots, \lambda_n$ and let K be the K-mapping generated by $T_1, T_2, \dots$ and $\lambda_1, \lambda_2, \dots$. Then $Fix(K) = \bigcap_{i=1}^{\infty} Fix(T_i)$.*

Lemma 2.10. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfy $(A1)-(A4)$ and $\varphi : C \to \mathbb{R} \bigcup \{+\infty\}$ be a real value function. For every $i = 1, 2, \ldots, N$, let A_i be α_i-strongly monotone with $\bar{\alpha} = min\{\alpha_i\}$ and $\bigcap_{i=1}^N GMEP(F, \varphi, A_i) \neq \emptyset$. Then*

$$GMEP(F, \varphi, \sum_{i=1}^N a_i A_i) = \bigcap_{i=1}^N GMEP(F, \varphi, A_i)$$

where $\sum_{i=1}^N a_i = 1, 0 < a_i < 1$ for every $i = 1, 2, .., N$.

Proof. It is easy to see that $\bigcap_{i=1}^N GMEP(F, \varphi, A_i) \subseteq GMEP(F, \varphi, \sum_{i=1}^N a_i A_i)$. Next, we will show that $GMEP(F, \varphi, \sum_{i=1}^N a_i A_i) \subseteq \bigcap_{i=1}^N GMEP(F, \varphi, A_i)$.
Let $x_0 \in GMEP(F, \varphi, \sum_{i=1}^N a_i A_i)$ and $x^* \in \bigcap_{i=1}^N GMEP(F, \varphi, A_i)$. Then we have

$$F(x_0, y) + \varphi(y) - \varphi(x_0) + \langle \sum_{i=1}^N a_i A_i x_0, y - x_0 \rangle \geq 0, \forall y \in C. \tag{2.3}$$

Since $\bigcap_{i=1}^N GMEP(F, \varphi, A_i) \subseteq GMEP(F, \varphi, \sum_{i=1}^N a_i A_i)$, we have

$$F(x^*, y) + \varphi(y) - \varphi(x^*) + \langle \sum_{i=1}^N a_i A_i x^*, y - x^* \rangle \geq 0, \forall y \in C. \tag{2.4}$$

Since $x_0, x^* \in C$, (2.3) and (2.4), we have

$$F(x_0, x^*) + \varphi(x^*) - \varphi(x_0) + \langle \sum_{i=1}^N a_i A_i x_0, x^* - x_0 \rangle \geq 0 \tag{2.5}$$

and

$$F(x^*, x_0) + \varphi(x_0) - \varphi(x^*) + \langle \sum_{i=1}^N a_i A_i x^*, x_0 - x^* \rangle \geq 0. \tag{2.6}$$

Summing up (2.5), (2.6) and (A_2), we have

$$\begin{aligned} 0 \leq & \langle \sum_{i=1}^N a_i A_i x^*, x_0 - x^* \rangle + \langle \sum_{i=1}^N a_i A_i x_0, x^* - x_0 \rangle \\ = & \langle \sum_{i=1}^N a_i A_i x^*, x_0 - x^* \rangle - \langle \sum_{i=1}^N a_i A_i x_0, x_0 - x^* \rangle \\ = & \sum_{i=1}^N a_i \langle A_i x^* - A_i x_0, x_0 - x^* \rangle \end{aligned}$$

$$
\begin{aligned}
&= -\sum_{i=1}^{N} a_i\langle A_ix^* - A_ix_0, x^* - x_0\rangle\\
&\le -\sum_{i=1}^{N} a_i\alpha_i\|x^* - x_0\|^2\\
&= -\bar{\alpha}\|x_0 - x^*\|^2.
\end{aligned}
$$

It implies that

$$
x_0 = x^*. \tag{2.7}
$$

By (2.7), then we have

$$
x_0 \in \bigcap_{i=1}^{N} GMEP(F,\varphi,A_i).
$$

Hence

$$
GMEP(F,\varphi,\sum_{i=1}^{N} a_iA_i) \subseteq \bigcap_{i=1}^{N} GMEP(F,\varphi,A_i).
$$

□

Remark 2.11. *For every $i = 1, 2, \ldots, N$,*

1. *$\sum_{i=1}^{N} a_iA_i$ is $\bar{\alpha}$-strongly monotone .*
2. *If A_i is α_i-strongly monotone and L_i- Lipschitzian with $\bar{\alpha} = min\{\alpha_i\}$ and $\bar{L} = max\{L_i\}$, respectively, then $\sum_{i=1}^{N} a_iA_i$ is $\frac{\bar{\alpha}}{L^2}$-inverse strongly monotone mapping.*

Proof. To prove (1), since A_i be α_i-strongly monotone mappings for every $i = 1, 2, ..., N$ and $\bar{\alpha} = min_{i=1,2,...,N}\{\alpha_i\}$. Let $x, y \in C$, then we have

$$
\begin{aligned}
\langle\sum_{i=1}^{N} a_iA_ix - \sum_{i=1}^{N} a_iA_iy, x - y\rangle &= \sum_{i=1}^{N} a_i\langle A_ix - A_iy, x - y\rangle\\
&\ge \sum_{i=1}^{N} a_i\alpha_i\|x - y\|^2\\
&\ge \bar{\alpha}\|x - y\|^2.
\end{aligned}
$$

Hence $\sum_{i=1}^{N} a_iA_i$ is a $\bar{\alpha}$-strongly monotone mapping.

To prove (2), since A_i is a L_i-Lipschitzian mapping for every $i = 1, 2, ..., N$ and $\bar{L} = max_{i=1,2,..,N}\{L_i\}$, then

$$
\begin{aligned}
\left\|\sum_{i=1}^{N} a_iA_ix - \sum_{i=1}^{N} a_iA_iy\right\| &= \sum_{i=1}^{N} a_i\left\|A_ix - A_iy\right\|\\
&\le \sum_{i=1}^{N} a_iL_i\left\|x - y\right\|\\
&\le \bar{L}\left\|x - y\right\|. \qquad (2.8)
\end{aligned}
$$

From (1) and (2.8), we have

$$\begin{aligned}
\langle\sum_{i=1}^{N} a_iA_ix - \sum_{i=1}^{N} a_iA_iy, x-y\rangle &= \sum_{i=1}^{N} a_i\langle A_ix - A_iy, x-y\rangle \\
&= \sum_{i=1}^{N} a_i\langle A_ix - A_iy, x-y\rangle \\
&\geq \sum_{i=1}^{N} a_i\alpha_i\|x-y\|^2 \\
&\geq \bar{\alpha}\|x-y\|^2 \\
&\geq \frac{\bar{\alpha}}{\bar{L}^2}\|\sum_{i=1}^{N} a_iA_ix - \sum_{i=1}^{N} a_iA_iy\|^2.
\end{aligned}$$

Then $\sum_{i=1}^{N} a_iA_i$ is $\frac{\bar{\alpha}}{\bar{L}^2}$-inverse strongly monotone. □

3 Main Results

In this section, we introduce the following iterative algorithm and prove a strong convergence for solving a common element of the set of fixed point of an infinite family of nonexpansive mappings and the set of a finite family of generalized mixed equilibrium problems in Hilbert space.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i = 1, 2, \ldots, N$, let $F : C \times C \to \mathbb{R}$ be a bifunction satisfing $(A1) - (A4)$ and $\varphi : C \to \mathbb{R}\bigcup\{+\infty\}$ be a proper lower semicontinuous function and convex function. Let A_i be α_i-strongly monotone and L_i-Lipschitzian mappings from C into H where $\bar{\alpha} = min_{i=1,2,\ldots,N}\{\alpha_i\}$ and $\bar{L} = max_{i=1,2,\ldots,N}\{L_i\}$. Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive mapping of C into itself with $\bigcap_{i=1}^{\infty} Fix(T_i) \neq \emptyset$ and $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, 2, \ldots$ with $\sum_{i=1}^{\infty} \lambda_i < \infty$. For every $n \in \mathbb{N}$, let K_n be the K-mapping generated by $T_1, T_2, \ldots, T_n$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ and let K be the K-mapping generated by $T_1, T_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$, i.e., $Kx = \lim_{n\to\infty} K_nx$ for every $x \in C$. Assume $\mathcal{F} := \bigcap_{i=1}^{\infty} Fix(T_i) \bigcap \bigcap_{i=1}^{N} GMEP(F, \varphi, A_i) \neq \emptyset$. For every $n \in \mathbb{N}$, assume the either $(B1)$ or $(B2)$ holds and let the sequences $\{x_n\}$ and $\{u_n\}$ be generated by $x_1, u \in C$ and*

$$\begin{aligned}
&F(u_n, y) + \varphi(y) - \varphi(u_n) + \left\langle \sum_{i=1}^{N} a_n^i A_ix_n, y - u_n \right\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \\
&x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n \left(b_n u_n + (1 - b_n)K_n u_n\right), \forall n \geq 1, \qquad (3.1)
\end{aligned}$$

where the sequence $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{b_n\} \subset [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$, for all $n \geq 1$. Suppose the following conditions hold:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0$*;*

(ii) $0 < a \leq \beta_n, b_n \leq b < 1$*, for some* $a, b \in \mathbb{R}$ *and for all* $n \geq 1$*;*

(iii) $0 < c \leq r_n \leq d < \frac{2\bar{\alpha}}{L^2}$*, for some* $c, d \in \mathbb{R}$ *and for all* $n \geq 1$*;*

(iv) $\sum_{i=1}^{N} a_n^i = 1$*, for all* $n \geq 1$*;*

(v) $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty,$
$\sum_{n=1}^{\infty} \left|a_{n+1}^j - a_n^j\right| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty.$

Then the sequence $\{x_n\}$ *converges strongly to* $z_0 = P_{\mathcal{F}}u$*.*

Proof. First, we show that $I - r_n \sum_{i=1}^{N} a_n^i A_i$ is a nonexpansive mapping. Since $\sum_{i=1}^{N} a_n^i A_i$ is $\frac{\bar{\alpha}}{L^2}$-inverse strongly monotone mapping. Put $S_n^N = \sum_{i=1}^{N} a_n^i A_i$ for all $n \in \mathbb{N}$. For any $x, y \in C$, we have

$$\begin{aligned}
\left\|(I - r_n S_n^N)x - (I - r_n S_n^N)y\right\|^2 &= \left\|(x - y) - r_n(S_n^N x - S_n^N y)\right\|^2 \\
&= \|x - y\|^2 + r_n^2 \left\|S_n^N x - S_n^N y\right\|^2 - 2r_n\langle x - y, S_n^N x - S_n^N y\rangle \\
&\leq \|x - y\|^2 + r_n^2 \left\|S_n^N x - S_n^N y\right\|^2 - 2r_n \frac{\bar{\alpha}}{L^2} \left\|S_n^N x - S_n^N y\right\|^2 \\
&= \|x - y\|^2 + r_n \left(r_n - \frac{2\bar{\alpha}}{L^2}\right) \left\|S_n^N x - S_n^N y\right\|^2 \\
&\leq \|x - y\|^2 .
\end{aligned}$$

Then, $I - r_n S_n^N$ is a nonexpansive mapping for all $n \geq 1$.
The proof can be divided into 5 steps.
Step 1. We will show that $\{x_n\}$ is bounded. Since

$$F(u_n, y) + \varphi(y) - \varphi(u_n) + \left\langle S_n^N x_n, y - u_n\right\rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C,$$

by Lemma 2.6, we have $u_n = T_{r_n}^{(F,\varphi)}(I - r_n S_n^N)x_n$ and

$$Fix\left(T_{r_n}^{(F,\varphi)}(I - r_n S_n^N)\right) = GMEP(F, \varphi, S_n^N). \tag{3.2}$$

Let $z \in \mathcal{F}$. From Lemma 2.10 and (3.2), we have

$$z \in \bigcap_{i=1}^{N} GMEP(F, \varphi, A_i) = GMEP(F, \varphi, S_n^N) = Fix(T_{r_n}^{(F,\varphi)}(I - r_n S_n^N).$$

By nonexpansiveness of $T_{r_n}^{(F,\varphi)}$, we have

$$\begin{aligned}
\|x_{n+1}-z\| &= \|\alpha_n u + \beta_n x_n + \gamma_n \left(b_n u_n + (1-b_n)K_n u_n\right) - z\| \\
&= \|\alpha_n (u-z) + \beta_n (x_n - z) + \gamma_n \left(b_n (u_n - z) + (1-b_n)(K_n u_n - z)\right)\| \\
&\leq \alpha_n \|u-z\| + \beta_n \|x_n - z\| + \gamma_n \left(b_n \|u_n - z\| + (1-b_n)\|K_n u_n - z\|\right) \\
&= \alpha_n \|u-z\| + \beta_n \|x_n - z\| + \gamma_n \|u_n - z\| \\
&= \alpha_n \|u-z\| + \beta_n \|x_n - z\| + \gamma_n \|T_{r_n}^{(F,\varphi)}(I - r_n S_n^N)x_n - z\| \\
&\leq \alpha_n \|u-z\| + \beta_n \|x_n - z\| + \gamma_n \|x_n - z\| \\
&= \alpha_n \|u-z\| + (1-\alpha_n)\|x_n - z\|. \qquad (3.3)
\end{aligned}$$

Put $M_1 = max\{\|u-z\|, \|x_1 - z\|\}$. From (3.3) and mathematical induction, we have $\|x_n - z\| \leq M_1$, for all $n \geq 1$. It implies that, $\{x_n\}$ is bounded and so is $\{u_n\}$.

Step 2. We will show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|(\alpha_n - \alpha_{n-1})u + \beta_n (x_n - x_{n-1}) + (\beta_n - \beta_{n-1})x_{n-1} + \gamma_n \Big(b_n (u_n - u_{n-1}) \\
&\quad + (b_n - b_{n-1})u_{n-1} + (1-b_n)(K_n u_n - K_{n-1}u_{n-1}) + (b_{n-1} - b_n)K_{n-1}u_{n-1}\Big) \\
&\quad + (\gamma_n - \gamma_{n-1})\left(b_{n-1}u_{n-1} + (1-b_{n-1})K_{n-1}u_{n-1}\right)\| \\
&\leq |\alpha_n - \alpha_{n-1}|\|u\| + \beta_n \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|x_{n-1}\| + \gamma_n \Big(b_n \|u_n - u_{n-1}\| \\
&\quad + |b_n - b_{n-1}|\|u_{n-1}\| + (1-b_n)\|K_n u_n - K_{n-1}u_{n-1}\| + |b_n - b_{n-1}|\|K_{n-1}u_{n-1}\|\Big) \\
&\quad + |\gamma_n - \gamma_{n-1}|\left(b_{n-1}\|u_{n-1}\| + (1-b_{n-1})\|K_{n-1}u_{n-1}\|\right) \\
&\leq |\alpha_n - \alpha_{n-1}|\|u\| + \beta_n \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|x_{n-1}\| + \gamma_n \Big(b_n \|u_n - u_{n-1}\| \\
&\quad + |b_n - b_{n-1}|\|u_{n-1}\| + (1-b_n)\|K_n u_n - K_n u_{n-1}\| + (1-b_n)\|K_n u_{n-1} - K_{n-1}u_{n-1}\| \\
&\quad + |b_n - b_{n-1}|\|K_{n-1}u_{n-1}\|\Big) + |\gamma_n - \gamma_{n-1}|\left(b_{n-1}\|u_{n-1}\| + (1-b_{n-1})\|K_{n-1}u_{n-1}\|\right) \\
&\leq |\alpha_n - \alpha_{n-1}|\|u\| + \beta_n \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|x_{n-1}\| + \gamma_n \Big(b_n \|u_n - u_{n-1}\| \\
&\quad + |b_n - b_{n-1}|\|u_{n-1}| + (1-b_n)\|u_n - u_{n-1}\| + (1-b_n)\|K_n u_{n-1} - K_{n-1}u_{n-1}\| \\
&\quad + |b_n - b_{n-1}|\|K_{n-1}u_{n-1}\|\Big) + |\gamma_n - \gamma_{n-1}|\left(b_{n-1}\|u_{n-1}\| + (1-b_{n-1})\|K_{n-1}u_{n-1}\|\right) \\
&\leq |\alpha_n - \alpha_{n-1}|\|u\| + \beta_n \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|x_{n-1}\| + \gamma_n \Big(\|u_n - u_{n-1}\| \\
&\quad + |b_n - b_{n-1}|\|u_{n-1}\| + \|K_n u_{n-1} - K_{n-1}u_{n-1}\| + |b_n - b_{n-1}|\|K_{n-1}u_{n-1}\|\Big) \\
&\quad + |\gamma_n - \gamma_{n-1}|\left(\|u_{n-1}\| + \|K_{n-1}u_{n-1}\|\right). \qquad (3.4)
\end{aligned}$$

Applying the method of [26], Lemma 2.11, we have

$$K_n u_{n-1} - K_{n-1}u_{n-1} = \lambda_n (T_n K_{n-1}u_{n-1} - K_{n-1}u_{n-1}).$$

It follows that

$$\|K_n u_{n-1} - K_{n-1}u_{n-1}\| = \lambda_n \|T_n K_{n-1}u_{n-1} - K_{n-1}u_{n-1}\|. \tag{3.5}$$

Since $u_n = T_{r_n}^{(F,\varphi)}(I - r_n S_n^N)x_n$ where $S_n^N = \sum_{i=1}^N a_n^i A_i$. By the definition of $T_{r_n}^{(F,\varphi)}$, we have

$$F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - (I - r_n S_n^N)x_n\rangle \geq 0, \forall y \in C. \tag{3.6}$$

Similarly

$$F(u_{n+1}, y)+\varphi(y)-\varphi(u_{n+1})+\frac{1}{r_{n+1}}\langle y-u_{n+1}, u_{n+1}-(I-r_{n+1}S_{n+1}^N)x_{n+1}\rangle \geq 0, \forall y \in C. \tag{3.7}$$

From (3.6) and (3.7), we obtain

$$F(u_n, u_{n+1}) + \varphi(u_{n+1}) - \varphi(u_n) + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - (I - r_n S_n^N)x_n\rangle \geq 0 \tag{3.8}$$

and

$$F(u_{n+1}, u_n)+\varphi(u_n)-\varphi(u_{n+1}) + \frac{1}{r_{n+1}}\langle u_n-u_{n+1}, u_{n+1}-(I-r_{n+1}S_{n+1}^N)x_{n+1}\rangle \geq 0. \tag{3.9}$$

Summing up (3.8), (3.9) and $A2$, we have

$$\tfrac{1}{r_n}\langle u_{n+1}-u_n, u_n-(I-r_n S_n^N)x_n\rangle+\tfrac{1}{r_{n+1}}\langle u_n-u_{n+1}, u_{n+1}-(I-r_{n+1}S_{n+1}^N)x_{n+1}\rangle \geq 0.$$

It follows that

$$\langle u_{n+1} - u_n, \frac{u_n - (I - r_n S_n^N)x_n}{r_n} - \frac{u_{n+1} - (I - r_{n+1}S_{n+1}^N)x_{n+1}}{r_{n+1}}\rangle \geq 0.$$

Since $r_n > 0$, we have

$$\begin{aligned}
0 &\leq \langle u_{n+1} - u_n, u_n - (I - r_n S_n^N)x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - (I - r_{n+1}S_{n+1}^N)x_{n+1})\rangle \\
&= \langle u_{n+1} - u_n, u_n - u_{n+1}\rangle + \langle u_{n+1} - u_n, u_{n+1} - (I - r_n S_n^N)x_n \\
&\quad - \frac{r_n}{r_{n+1}}(u_{n+1} - (I - r_{n+1}S_{n+1}^N)x_{n+1})\rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|u_{n+1}-u_n\|^2 &\leq\langle u_{n+1}-u_n, u_{n+1}-(I - r_n S_n^N)x_n - \tfrac{r_n}{r_{n+1}}(u_{n+1}-(I-r_{n+1}S_{n+1}^N)x_{n+1})\rangle \\
&=\langle u_{n+1} - u_n, (I - r_{n+1}S_{n+1}^N)x_{n+1} - (I - r_n S_n^N)x_n \\
&\quad + \left(1 - \tfrac{r_n}{r_{n+1}}\right)(u_{n+1} - (I - r_{n+1}S_{n+1}^N)x_{n+1})\rangle \\
&\leq\|u_{n+1} - u_n\|\Bigg(\|(I - r_{n+1}S_{n+1}^N)x_{n+1} - (I - r_n S_n^N)x_n\| \\
&\quad + \left|1 - \tfrac{r_n}{r_{n+1}}\right| \|u_{n+1} - (I - r_{n+1}S_{n+1}^N)x_{n+1}\|\Bigg).
\end{aligned}$$

Then

$$\begin{aligned}
\|u_{n+1}-u_n\| \leq& \|(I-r_{n+1}S_{n+1}^N)x_{n+1}-(I-r_nS_n^N)x_n\| \\
&+\frac{1}{r_{n+1}}|r_{n+1}-r_n|\,\|u_{n+1}-(I-r_{n+1}S_{n+1}^N)x_{n+1}\| \\
\leq& \|(I-r_{n+1}S_{n+1}^N)x_{n+1}-(I-r_{n+1}S_{n+1}^N)x_n\|+\|(I-r_{n+1}S_{n+1}^N)x_n \\
&-(I-r_nS_n^N)x_n\|+\frac{1}{r_{n+1}}|r_{n+1}-r_n|\,\|u_{n+1}-(I-r_{n+1}S_{n+1}^N)x_{n+1}\| \\
\leq& \|x_{n+1}-x_n\|+\|r_{n+1}S_{n+1}^Nx_n-r_nS_n^Nx_n\| \\
&+\frac{1}{r_{n+1}}|r_{n+1}-r_n|\,\|u_{n+1}-(I-r_{n+1}S_{n+1}^N)x_{n+1}\| \\
\leq& \|x_{n+1}-x_n\|+r_{n+1}\|S_{n+1}^Nx_n-S_n^Nx_n\|+|r_{n+1}-r_n|\|S_n^Nx_n\| \\
&+\frac{1}{r_{n+1}}|r_{n+1}-r_n|\,\|u_{n+1}-(I-r_{n+1}S_{n+1}^N)x_{n+1}\| \\
=& \|x_{n+1}-x_n\|+r_{n+1}\|\sum_{i=1}^N(a_{n+1}^i-a_n^i)A_ix_n\|+|r_{n+1}-r_n|\|S_n^Nx_n\| \\
&+\frac{1}{r_{n+1}}|r_{n+1}-r_n|\,\|u_{n+1}-(I-r_{n+1}S_{n+1}^N)x_{n+1}\| \\
\leq& \|x_{n+1}-x_n\|+r_{n+1}\sum_{i=1}^N|a_{n+1}^i-a_n^i|\|A_ix_n\|+|r_{n+1}-r_n|\|S_n^Nx_n\| \\
&+\frac{1}{r_{n+1}}|r_{n+1}-r_n|\,\|u_{n+1}-(I-r_{n+1}S_{n+1}^N)x_{n+1}\| \\
\leq& \|x_{n+1}-x_n\|+d\sum_{i=1}^N|a_{n+1}^i-a_n^i|\|A_ix_n\|+|r_{n+1}-r_n|\|S_n^Nx_n\| \\
&+\frac{1}{c}|r_{n+1}-r_n|\,\|u_{n+1}-(I-r_{n+1}S_{n+1}^N)x_{n+1}\|. \qquad (3.10)
\end{aligned}$$

Substitute (3.5) and (3.10) into (3.4), we have

$$\begin{aligned}
\|x_{n+1}-x_n\| \leq& |\alpha_n-\alpha_{n-1}|\|u\|+\beta_n\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\|x_{n-1}\| \\
&+\gamma_n\Big(\Big(\|x_n-x_{n-1}\|+d\sum_{i=1}^N|a_n^i-a_{n-1}^i|\|A_ix_{n-1}\| \\
&+|r_n-r_{n-1}|\|S_{n-1}^Nx_{n-1}\|+\frac{1}{c}|r_n-r_{n-1}|\,\|u_n-(I-r_{n+1}S_n^N)x_n\|\Big) \\
&+|b_n-b_{n-1}|\|u_{n-1}\|+\lambda_n\|T_nK_{n-1}u_{n-1}-K_{n-1}u_{n-1}\| \\
&+|b_{n-1}-b_n|\|K_{n-1}u_{n-1}\|\Big)+|\gamma_n-\gamma_{n-1}|\left(\|u_{n-1}\|+\|K_{n-1}u_{n-1}\|\right)
\end{aligned}$$

$$
\begin{aligned}
\leq&(1-\alpha_n)\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|u\|+|\beta_n-\beta_{n-1}|\|x_{n-1}\| \\
&+d\sum_{i=1}^{N}|a_n^i-a_{n-1}^i|\|A_ix_{n-1}\|+|r_n-r_{n-1}|\|S_{n-1}^Nx_{n-1}\| \\
&+\frac{1}{c}\,|r_n-r_{n-1}|\,\|u_n-(I-r_{n+1}S_n^N)x_n\| \\
&+|b_n-b_{n-1}|\|u_{n-1}\|+\lambda_n\|T_nK_{n-1}u_{n-1}-K_{n-1}u_{n-1}\|+|b_{n-1}-b_n|\|K_{n-1}u_{n-1}\| \\
&+|\gamma_n-\gamma_{n-1}|\,(\|u_{n-1}\|+\|K_{n-1}u_{n-1}\|) \\
\leq&(1-\alpha_n)\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|M_2+|\beta_n-\beta_{n-1}|M_2+d\sum_{i=1}^{N}|a_n^i-a_{n-1}^i|M_2 \\
&+|r_n-r_{n-1}|M_2+\frac{1}{c}\,|r_n-r_{n-1}|\,M_2+|b_n-b_{n-1}|M_2+\lambda_nM_2 \\
&+|b_{n-1}-b_n|M_2+|\gamma_n-\gamma_{n-1}|M_2,
\end{aligned}
$$

where $M_2 := max_{n\in\mathbb{N}}\{\|u\|,\|x_n\|,\|u_n\|,\|A_ix_{n-1}\|,\|S_n^Nx_n\|,\|u_n-(I-r_{n+1}S_n^N)x_n\|,$ $\|K_nu_n\|,(\|u_n\|+\|K_nu_n\|),\|T_nK_{n-1}u_{n-1}-K_{n-1}u_{n-1}\|\}$. From Lemma 2.2, the conditions (ii) and (v), we have

$$
\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.11}
$$

Step 3. We show that $\lim_{n\to\infty}\|u_n-x_n\|=\lim_{n\to\infty}\|K_nu_n-u_n\|=0$. To show this, let $z\in\mathcal{F}$. Since $u_n=T_{r_n}^{(F,\varphi)}(I-r_nS_n^N)x_n$ and $T_{r_n}^{(F,\varphi)}$ is a firmly nonexpensive mapping, we have

$$
\begin{aligned}
\|T_{r_n}^{(F,\varphi)}(I-r_nS_n^N)x_n-z\|^2 &\leq \langle (I-r_nS_n^N)x_n-(I-r_nS_n^N)z,u_n-z\rangle \\
&=\frac{1}{2}\big(\|(I-r_nS_n^N)x_n-(I-r_nS_n^N)z\|^2+\|u_n-z\|^2 \\
&\quad-\|(I-r_nS_n^N)x_n-(I-r_nS_n^N)z-u_n+z\|^2\big) \\
&\leq\frac{1}{2}\big(\|x_n-z\|^2+\|u_n-z\|^2-\|(x_n-u_n)-r_n(S_n^Nx_n-S_n^Nz)\|^2\big) \\
&=\frac{1}{2}\big(\|x_n-z\|^2+\|u_n-z\|^2-\|(x_n-u_n)\|^2-(r_n)^2\|S_n^Nx_n-S_n^Nz\|^2 \\
&\quad+2r_n\langle x_n-T_{r_n}^{(F,\varphi)}(I-r_nS_n^N)x_n,S_n^Nx_n-S_n^Nz\rangle\big) \\
&\leq\frac{1}{2}\big(\|x_n-z\|^2+\|u_n-z\|^2-\|x_n-u_n\|^2-(r_n)^2\|S_n^Nx_n-S_n^Nz\|^2 \\
&\quad+2r_n\|x_n-T_{r_n}^{(F,\varphi)}(I-r_nS_n^N)x_n\|\|S_n^Nx_n-S_n^Nz\|\big),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|u_n-z\|^2 \leq&\|x_n-z\|^2-\|x_n-u_n\|^2 \\
&+2r_n\|x_n-T_{r_n}^{(F,\varphi)}(I-r_nS_n^N)x_n\|\|S_n^Nx_n-S_n^Nz\|.
\end{aligned} \tag{3.12}
$$

From the definition of u_n and Remark (2.11), we have

$$\begin{aligned}
\|u_n - z\|^2 =&\|T_{r_n}^{(F,\varphi)}(I - r_n S_n^N)x_n - T_{r_n}^{(F,\varphi)}(I - r_n S_n^N)z\|^2 \\
\leq&\|(I - r_n S_n^N)x_n - (I - r_n S_n^N)z\|^2 \\
=&\|(x_n - z) - r_n(S_n^N x_n - S_n^N z)\|^2 \\
=&\|x_n - z\|^2 - 2r_n\langle x_n - z, S_n^N x_n - S_n^N z\rangle + (r_n)^2\|S_n^N x_n - S_n^N z\|^2 \\
\leq&\|x_n - z\|^2 - 2r_n\left(\frac{\bar{\alpha}}{\bar{L}^2}\right)\|S_n^N x_n - S_n^N z\|^2 + (r_n)^2\|(S_n^N x_n - S_n^N z)\|^2 \\
=&\|x_n - z\|^2 - r_n\left(2\frac{\bar{\alpha}}{\bar{L}^2} - r_n\right)\|S_n^N x_n - S_n^N z\|^2. \qquad (3.13)
\end{aligned}$$

From the definition of x_n and (3.13), we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 \leq&\alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \gamma_n\left(b_n\|u_n - z\| + (1 - b_n)\|K_n u_n - z\|\right)^2 \\
\leq&\alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \gamma_n\|u_n - z\|^2 \\
\leq&\alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \gamma_n\Big(\|x_n - z\|^2 \\
&-r_n\left(2\frac{\bar{\alpha}}{\bar{L}^2} - r_n\right)\|S_n^N x_n - S_n^N z\|^2\Big) \\
\leq&\alpha_n\|u - z\|^2 + \|x_n - z\|^2 - \gamma_n r_n\left(2\frac{\bar{\alpha}}{\bar{L}^2} - r_n\right)\|S_n^N x_n - S_n^N z\|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
\gamma_n r_n\left(2\frac{\bar{\alpha}}{\bar{L}^2} - r_n\right)\|S_n^N x_n - S_n^N z\|^2 \leq& \alpha_n\|u - z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\
\leq&\alpha_n\|u - z\|^2 + (\|x_n - z\| + \|x_{n+1} - z\|)(\|x_{n+1} - x_n\|).
\end{aligned}$$

From the condition (i) and (3.11), we have

$$\lim_{n\to\infty}\|S_n^N x_n - S_n^N z\| = 0. \qquad (3.14)$$

From the definition of x_n and (3.12), we obtain

$$\begin{aligned}
\|x_{n+1} - z\|^2 \leq&\alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \gamma_n\|u_n - z\|^2 \\
\leq&\alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \gamma_n\Big(\|x_n - z\|^2 - \|x_n - u_n\|^2 \\
&+ 2r_n\|x_n - T_{r_n}^{(F,\varphi)}(I - r_n S_n^N)x_n\|\|S_n^N x_n - S_n^N z\|\Big) \\
\leq&\alpha_n\|u - z\|^2 + \|x_n - z\|^2 - \gamma_n\|x_n - u_n\|^2 \\
&+ 2\gamma_n r_n\|x_n - T_{r_n}^{(F,\varphi)}(I - r_n S_n^N)x_n\|\|S_n^N x_n - S_n^N z\|.
\end{aligned}$$

It implies that

$$\begin{aligned}
\gamma_n\|x_n - u_n\|^2 \leq&\alpha_n\|u - z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\
&+ 2\gamma_n r_n\|x_n - T_{r_n}^{(F,\varphi)}(I - r_n S_n^N)x_n\|\|S_n^N x_n - S_n^N z\| \\
\leq&\alpha_n\|u - z\|^2 + (\|x_n - z\| - \|x_{n+1} - z\|)\|x_{n+1} - x_n\| \\
&+ 2\gamma_n r_n\|x_n - T_{r_n}^{(F,\varphi)}(I - r_n S_n^N)x_n\|\|S_n^N x_n - S_n^N z\|.
\end{aligned}$$

From the condition (i), (3.11) and (3.14), we have

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.15}$$

From the definition of x_n, we have

$$\begin{aligned} x_{n+1} - x_n =& \alpha_n(u - x_n) + \gamma_n \left(b_n(u_n - x_n) + (1 - b_n)(K_n u_n - x_n)\right) \\ =& \alpha_n(u - x_n) + \gamma_n b_n(u_n - x_n) + \gamma_n(1 - b_n)(K_n u_n - u_n). \end{aligned}$$

From the conditions (i), (ii), (3.11) and (3.15), we have

$$\lim_{n\to\infty} \|K_n u_n - u_n\| = 0. \tag{3.16}$$

Step 4. We show that $\lim_{n\to\infty} \sup \langle u - z_0, x_n - z_0\rangle \leq 0$, where $z_0 = P_{\mathcal{F}} u$. To show this, take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_{n\to\infty} \sup \langle u - z_0, x_n - z_0\rangle = \lim_{k\to\infty} \langle u - z_0, x_{n_k} - z_0\rangle . \tag{3.17}$$

Without loss of generality, we may assume that $x_{n_k} \rightharpoonup \omega$ as $k \to \infty$ where $\omega \in C$. From (3.15), we obtain $u_{n_k} \rightharpoonup \omega$ as $k \to \infty$.

Assume $\omega \notin \bigcap_{i=1}^{\infty} Fix(T_i)$. From Lemma 2.9, we have $Fix(K) = \bigcap_{i=1}^{\infty} Fix(T_i)$. From Opial's condition, (3.16) and Remark 2.8, we have

$$\begin{aligned} \lim_{k\to\infty} \inf \|u_{n_k} - \omega\| <& \lim_{k\to\infty} \inf \|u_{n_k} - K\omega\| \\ \leq& \lim_{k\to\infty} \inf(\|u_{n_k} - K_{n_k} u_{n_k}\| + \|K_{n_k} u_{n_k} - K_{n_k}\omega\| + \|K_{n_k}\omega - K\omega\|) \\ \leq& \lim_{k\to\infty} \inf \|u_{n_k} - \omega\| . \end{aligned}$$

This is a contradiction, we have

$$\omega \in \bigcap_{i=1}^{\infty} Fix(T_i). \tag{3.18}$$

From $u_n = T_{r_n}^{(F,\varphi)}(I - r_n S_n^N)x_n$, we have

$$F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle S_n^N x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C.$$

From (A2), we have

$$\varphi(y) - \varphi(u_n) + \langle S_n^N x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n).$$

In particular

$$\varphi(y) - \varphi(u_{n_j}) + \langle S_{n_j}^N x_{n_j}, y - u_{n_j}\rangle + \frac{1}{r_{n_j}}\langle y - u_{n_j}, u_{n_j} - x_{n_j}\rangle \geq F(y, u_{n_j}).$$

It follows that

$$\varphi(y) - \varphi(u_{n_j}) + \langle S_{n_j}^N x_{n_j}, y - u_{n_j}\rangle + \langle y - u_{n_j}, \frac{u_{n_j} - x_{n_j}}{r_{n_j}}\rangle \geq F(y, u_{n_j}). \quad (3.19)$$

Put $y_t = ty + (1-t)\omega$ where for all $t \in (0,1]$, we have $y_t \in C$. From (3.19), we have

$$\begin{aligned}
\varphi(y_t) - \varphi(u_{n_j}) + \langle y_t - u_{n_j}, S_{n_j}^N y_t\rangle &\geq \langle y_t - u_{n_j}, S_{n_j}^N y_t\rangle - \langle S_{n_j}^N x_{n_j}, y_t - u_{n_j}\rangle \\
&- \langle y_t - u_{n_j}, \frac{u_{n_j} - x_{n_j}}{r_{n_j}}\rangle + F(y_t, u_{n_j}) \\
&=\langle y_t - u_{n_j}, S_{n_j}^N y_t - S_{n_j}^N u_{n_j} + S_{n_j}^N u_{n_j}\rangle - \langle y_t - u_{n_j}, S_{n_j}^N x_{n_j}\rangle \\
&- \langle y_t - u_{n_j}, \frac{u_{n_j} - x_{n_j}}{r_{n_j}}\rangle + F(y_t, u_{n_j}) \\
&=\langle y_t - u_{n_j}, S_{n_j}^N y_t - S_{n_j}^N u_{n_j}\rangle + \langle y_t - u_{n_j}, S_{n_j}^N u_{n_j} - S_{n_j}^N x_{n_j}\rangle \\
&- \langle y_t - u_{n_j}, \frac{u_{n_j} - x_{n_j}}{r_{n_j}}\rangle + F(y_t, u_{n_j}).
\end{aligned}$$

From (3.15), we have $\|S_{n_j}^N u_{n_j} - S_{n_j}^N x_{n_j}\| \to 0$ and $\frac{u_{n_j} - x_{n_j}}{r_{n_j}} \to 0$. From monotonicity of $S_{n_j}^N$ and $(A4)$, we have

$$\varphi(y_t) - \varphi(\omega) + \langle y_t - \omega, S_{n_j}^N y_t\rangle \geq F(y_t, \omega). \quad (3.20)$$

Form $(A1)$ and (3.20), we have

$$\begin{aligned}
0 &= F(y_t, y_t) + \varphi(y_t) - \varphi(y_t) \\
&= F(y_t, ty + (1-t)\omega) + \varphi(ty + (1-t)\omega) - \varphi(y_t) \\
&\leq tF(y_t, y) + (1-t)F(y_t, \omega) + t\varphi(y) + (1-t)\varphi(\omega) - \varphi(y_t) \\
&\leq tF(y_t, y) + (1-t)\varphi(y_t) - (1-t)\varphi(\omega) + (1-t)\langle y_t - \omega, S_{n_j}^N y_t\rangle + t\varphi(y) \\
&\quad + (1-t)\varphi(\omega) - \varphi(y_t) \\
&= tF(y_t, y) + t\varphi(y) - t\varphi(y_t) + (1-t)\langle ty + (1-t)\omega - \omega, S_{n_j}^N y_t\rangle \\
&= tF(y_t, y) + t\varphi(y) - t\varphi(y_t) + (1-t)t\langle y - \omega, S_{n_j}^N y_t\rangle.
\end{aligned}$$

It implies that

$$0 \leq F(y_t, y) + \varphi(y) - \varphi(y_t) + (1-t)\langle y - \omega, S_{n_j}^N y_t\rangle.$$

Letting $t \to 0^+$ and (A3), we have

$$0 \leq F(\omega, y) + \varphi(y) - \varphi(\omega) + \langle y - \omega, S_{n_j}^N \omega\rangle, \forall y \in C.$$

Then $\omega \in GMEP(F, \varphi, \Sigma_{i=1}^N a_n^i A_i)$. From Lemma 2.10, we have

$$\omega \in \bigcap_{i=1}^N GMEP(F, \varphi, A_i). \quad (3.21)$$

From (3.18) and (3.21), we have $\omega \in \mathcal{F}$. Since $x_{n_k} \rightharpoonup \omega$ and $\omega \in \mathcal{F}$, hence we have

$$\lim_{n\to\infty} \sup \langle u - z_0, x_n - z_0\rangle = \lim_{k\to\infty} \langle u - z_0, x_{n_k} - z_0\rangle = \langle u - z_0, \omega - z_0\rangle \le 0. \tag{3.22}$$

Step 5. Finally, we will show that $\lim_{n\to\infty} x_n = z_0$, where $z_0 = P_{\mathcal{F}}u$. By nonexpansiveness of K_n, we have

$$\begin{aligned}\|x_{n+1} - z_0\|^2 =& \|\alpha_n u + \beta_n x_n + \gamma_n (b_n u_n + (1 - b_n)K_n u_n) - z_0\|^2 \\ \le& \|\beta_n(x_n - z_0) + \gamma_n (b_n(u_n - z_0) + (1 - b_n)(K_n u_n - z_0))\|^2 \\ &+ 2\alpha_n\langle u - z_0, x_{n+1} - z_0\rangle \\ \le& (1 - \alpha_n)\|x_n - z_0\|^2 + 2\alpha_n\langle u - z_0, x_{n+1} - z_0\rangle.\end{aligned}$$

Applying Lemma 2.2 and (3.22), we have the sequence $\{x_n\}$ converges strongly to $z_0 = P_{\mathcal{F}}u$. This completes the proof. □

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i = 1, 2, \ldots, N$, let $F : C \times C \to \mathbb{R}$ be a bifunction satisfing $(A1) - (A4)$ and $\varphi : C \to \mathbb{R}\bigcup\{+\infty\}$ be a proper lower semicontinuous function and convex function. Let A be α-strongly monotone and L -Lipschitzian mappings from C into H . Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive mapping of C into itself with $\bigcap_{i=1}^{\infty} Fix(T_i) \neq \emptyset$ and $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, 2, \ldots$ with $\sum_{i=1}^{\infty} \lambda_i < \infty$. For every $n \in \mathbb{N}$, let K_n be the K-mapping generated by $T_1, T_2, \ldots, T_n$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ and let K be the K-mapping generated by $T_1, T_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$, i.e., $Kx = \lim_{n\to\infty} K_n x$ for every $x \in C$. Assume $\mathcal{F} := \bigcap_{i=1}^{\infty} Fix(T_i) \bigcap GMEP(F, \varphi, A) \neq \emptyset$. For every $n \in \mathbb{N}$, assume the either $(B1)$ or $(B2)$ holds and let the sequences $\{x_n\}$ and $\{u_n\}$ be generated by $x_1, u \in C$ and*

$$\begin{aligned}&F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \\ &x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n (b_n u_n + (1 - b_n)K_n u_n), \forall n \ge 1,\end{aligned} \tag{3.23}$$

where the sequence $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{b_n\} \subset [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$, for all $n \ge 1$. Suppose the following conditions hold:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0$;

(ii) $0 < a \le \beta_n, b_n \le b < 1$, *for some* $a, b \in \mathbb{R}$ *and for all* $n \ge 1$;

(iii) $0 < c \le r_n \le d < \frac{2\bar{\alpha}}{L^2}$, *for some* $c, d \in \mathbb{R}$ *and for all* $n \ge 1$;

(iv) $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$
$\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty.$

Then the sequence $\{x_n\}$ *converges strongly to* $z_0 = P_{\mathcal{F}}u$.

Proof. Put $A \equiv A_i$ for every $1, 2, ..., N$ in Theorem 3.1. From Theorem 3.1, we obtain the desired result. □

Corollary 3.3. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $F : C \times C \to \mathbb{R}$ *be a bifunction satisfing* $(A1) - (A4)$. *Let* $\{T_i\}_{i=1}^{\infty}$ *be infinite family of nonexpansive mapping of* C *into itself with* $\bigcap_{i=1}^{\infty} Fix(T_i) \neq \emptyset$ *and* $\lambda_1, \lambda_2, \ldots$ *be real numbers such that* $0 < \lambda_i < 1$ *for every* $i = 1, 2, \ldots$ *with* $\sum_{i=1}^{\infty} \lambda_i < \infty$. *For every* $n \in \mathbb{N}$, *let* K_n *be the* K*-mapping generated by* $T_1, T_2, \ldots, T_n$ *and* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *and let* K *be the* K*-mapping generated by* $T_1, T_2, \ldots$ *and* $\lambda_1, \lambda_2, \ldots$, *i.e.,* $Kx = \lim_{n\to\infty} K_n x$ *for every* $x \in C$. *Assume* $\mathcal{F} := \bigcap_{i=1}^{\infty} Fix(T_i) \bigcap EP(F) \neq \emptyset$. *For every* $n \in \mathbb{N}$, *let the sequences* $\{x_n\}$ *and* $\{u_n\}$ *be generated by* $x_1, u \in C$ *and*

$$\begin{aligned} &F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \\ &x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n \left(b_n u_n + (1 - b_n) K_n u_n\right), \forall n \geq 1, \end{aligned} \tag{3.24}$$

where the sequence $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{b_n\} \subset [0, 1]$ *with* $\alpha_n + \beta_n + \gamma_n = 1$, *for all* $n \geq 1$. *Suppose the following conditions hold:*

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0$;

(ii) $0 < a \leq \beta_n, b_n \leq b < 1$, *for some* $a, b \in \mathbb{R}$ *and for all* $n \geq 1$;

(iii) $0 < c \leq r_n \leq d < \frac{2\bar{\alpha}}{L^2}$, *for some* $c, d \in \mathbb{R}$ *and for all* $n \geq 1$;

(iv) $\sum_{i=1}^{N} a_n^i = 1$, *for all* $n \geq 1$;

(v) $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$
$\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty.$

Then the sequence $\{x_n\}$ *converges strongly to* $z_0 = P_{\mathcal{F}}u$.

Proof. Put $\varphi \equiv 0$ and $A_i \equiv 0$ for every $1, 2, ..., N$ in Theorem 3.1. From Theorem 3.1, we obtain the desired result. □

4 Apply to Generalized Equilibrium Problem

In this section, we utilize our main results for the following result: From Lemma 2.10, the following result is related to generalized equilibrium problem:

Lemma 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfing $(A1) - (A4)$. For every $i = 1, 2, \ldots, N$, let A_i be α_i-strongly monotone from C into H with $\alpha_i > 0$, $\bar{\alpha} = min\{\alpha_i\}$ and $\bigcap_{i=1}^{N} EP(F, A_i) \neq \emptyset$. Then*

$$EP(F, \sum_{i=1}^{N} a_i A_i) = \bigcap_{i=1}^{N} EP(F, A_i)$$

where $0 < a_i < 1$, for every $i = 1, 2, ..., N$ and $\sum_{i=1}^{N} a_i = 1$.

Proof. Put $\varphi \equiv 0$. Then we obtain the desired result. □

Theorem 4.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i = 1, 2, \ldots, N$, let $F : C \times C \to \mathbb{R}$ be a bifunction satisfing $(A1)-(A4)$. Let A_i be α_i-strongly monotone and L_i-Lipschitzian mappings C into H where $\bar{\alpha} = min_{i=1,2,\ldots,N}\{\alpha_i\}$ and $\bar{L} = max_{i=1,2,\ldots,N}\{L_i\}$. Let $\{T_i\}_{i=1}^{\infty}$ be infinite family of nonexpansive mapping of C into itself with $\bigcap_{i=1}^{\infty} Fix(T_i) \neq \emptyset$ and $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, 2, \ldots$ with $\sum_{i=1}^{\infty} \lambda_i < \infty$. For every $n \in \mathbb{N}$, let K_n be the K-mapping generated by $T_1, T_2, \ldots, T_n$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ and let K be the K-mapping generated by $T_1, T_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$, i.e., $Kx = \lim_{n\to\infty} K_n x$ for every $x \in C$. Assume $\mathcal{F} := \bigcap_{i=1}^{\infty} Fix(T_i) \bigcap \bigcap_{i=1}^{N} EP(F, A_i) \neq \emptyset$. For every $n \in \mathbb{N}$, let the sequences $\{x_n\}$ and $\{u_n\}$ be generated by $x_1, u \in C$ and*

$$\begin{aligned} &F(u_n, y) + \left\langle \sum_{i=1}^{N} a_n^i A_i x_n, y - u_n \right\rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \\ &x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n \left(b_n u_n + (1 - b_n) K_n u_n \right), \forall n \geq 1, \end{aligned} \tag{4.1}$$

where the sequence $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{b_n\} \subset [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$, for all $n \geq 1$. Suppose the following conditions hold:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0$;

(ii) $0 < a \leq \beta_n, b_n \leq b < 1$, *for some* $a, b \in \mathbb{R}$ *and for all* $n \geq 1$;

(iii) $0 < c \leq r_n \leq d < \frac{2\bar{\alpha}}{\bar{L}^2}$, *for some* $c, d \in \mathbb{R}$ *and for all* $n \geq 1$;

(iv) $\sum_{i=1}^{N} a_n^i = 1$, *for all* $n \geq 1$;

(v) $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty,$
$\sum_{n=1}^{\infty} \left| a_{n+1}^j - a_n^j \right| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty.$

Then the sequence $\{x_n\}$ converges strongly to $z_0 = P_{\mathcal{F}} u$.

Proof. Put $\varphi \equiv 0$ in Theorem 3.1. By Lemma 4.1 and Theorem 3.1, we obtain the desired result. □

From Lemma 2.10, we have the result involving variational inequality problem as follows:

Lemma 4.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i = 1, 2, \ldots, N$, let A_i be α_i-strongly monotone from C into H with $\alpha_i > 0$, $\bar{\alpha} = min\{\alpha_i\}$ and $\bigcap_{i=1}^{N} VI(C, A_i) \neq \emptyset$. Then*

$$VI(C, \sum_{i=1}^{N} a_i A_i) = \bigcap_{i=1}^{N} VI(C, A_i)$$

where $0 < a_i < 1$ for every $i = 1, 2, .., N$ and $\sum_{i=1}^{N} a_i = 1$.

Proof. Put $F \equiv \varphi \equiv 0$ in Lemma 2.10. From Lemma 2.10, we obtain the desired result. □

Theorem 4.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i = 1, 2, \ldots, N$, let A_i be α_i-strongly monotone and L_i -Lipschitzian mappings from C into H where $\bar{\alpha} = min_{i=1,2,\ldots,N}\{\alpha_i\}$ and $\bar{L} = max_{i=1,2,\ldots,N}\{L_i\}$. Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive mapping of C into itself with $\bigcap_{i=1}^{\infty} Fix(T_i) \neq \emptyset$ and $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, 2, \ldots$ with $\sum_{i=1}^{\infty} \lambda_i < \infty$. For every $n \in \mathbb{N}$, let K_n be the K-mapping generated by $T_1, T_2, \ldots, T_n$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ and let K be the K-mapping generated by $T_1, T_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$, i.e, $Kx = \lim_{n\to\infty} K_n x$ for every $x \in C$. Assume $\mathcal{F} := \bigcap_{i=1}^{\infty} Fix(T_i) \bigcap \bigcap_{i=1}^{N} VI(C, A_i) \neq \emptyset$. For every $n \in \mathbb{N}$, let the sequences $\{x_n\}$ and $\{u_n\}$ be generated by $x_1, u \in C$ and*

$$\begin{aligned} x_{n+1} =& \alpha_n u + \beta_n x_n + \gamma_n \big(b_n P_C(I - r_n \sum_{i=1}^{N} a_n^i A_i) x_n \\ &+ (1 - b_n) K_n P_C(I - r_n \Sigma_{i=1}^{N} a_n^i A_i) x_n\big), \forall n \geq 1, \end{aligned} \tag{4.2}$$

where the sequence $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{b_n\} \subset [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$, for all $n \geq 1$. Suppose the following conditions hold:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0$;

(ii) $0 < a \leq \beta_n, b_n \leq b < 1$, *for some* $a, b \in \mathbb{R}$ *and for all* $n \geq 1$;

(iii) $0 < c \leq r_n \leq d < \frac{2\bar{\alpha}}{\bar{L}^2}$, *for some* $c, d \in \mathbb{R}$ *and for all* $n \geq 1$;

(iv) $\sum_{i=1}^{N} a_n^i = 1$, *for all* $n \geq 1$;

(v) $\sum_{n=1}^{\infty}|r_{n+1}-r_n|<\infty$, $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$, $\sum_{n=1}^{\infty}|\gamma_{n+1}-\gamma_n|<\infty$, $\sum_{n=1}^{\infty}\left|a_{n+1}^j-a_n^j\right|<\infty$, $\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n|<\infty$.

Then the sequence $\{x_n\}$ *converges strongly to* $z_0=P_{\mathcal{F}}u$.

Proof. Put $F\equiv\varphi\equiv 0$ in Theorem 3.1, we have

$$\langle y-u_n, x_n-r_n\Sigma_{i=1}^N a_n^i A_i x_n-u_n\rangle\geq 0, \forall y\in C.$$

It implies that

$$u_n=P_C(I-r_n\Sigma_{i=1}^N a_n^i A_i)x_n.$$

By Lemma 4.3 and Theorem 3.1, we obtain the desired result. □

Theorem 4.5. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *For every* $i=1,2,\ldots,N$, *let* $F:C\times C\to\mathbb{R}$ *be a bifunction satisfy* $(A1)-(A4)$ *and* $\varphi:C\to\mathbb{R}\bigcup\{+\infty\}$ *be a proper lower semicontinuous function and convex function. Let* A_i *be* α_i*-strongly monotone and* L_i*-Lipschitzian mappings from* C *into* H *where* $\bar{\alpha}=min_{i=1,2,\ldots,N}\{\alpha_i\}$ *and* $\bar{L}=max_{i=1,2,\ldots,N}\{L_i\}$. *Let* $\{D_i\}_{i=1}^{\infty}$ *be* d_i*-inverse strongly monotone mapping of* C *into* H *with* $\bar{d}=min_{i=1,2,\ldots,N}\{d_i\}$. *Define the mapping* $G_i:C\to C$ *by*

$$G_i x=P_C(I-\rho D_i)x, \forall x\in C, 0\leq\rho\leq 2\bar{d}$$

and $\lambda_1,\lambda_2,\ldots$ *be real numbers such that* $0<\lambda_i<1$ *for every* $i=1,2,\ldots$ *with* $\sum_{i=1}^{\infty}\lambda_i<\infty$. *For every* $n\in\mathbb{N}$, *let* K_n *be the* K*-mapping generated by* $G_1,G_2,\ldots,G_n$ *and* $\lambda_1,\lambda_2,\ldots,\lambda_n$ *and let* K *be the* K*-mapping generated by* $G_1,G_2,\ldots$ *and* $\lambda_1,\lambda_2,\ldots$. *i.e.,* $Kx=\lim_{n\to\infty}K_n x$ *for every* $x\in C$. *Assume* $\mathcal{F}:=\bigcap_{i=1}^{\infty}VI(C,D_i)\bigcap\bigcap_{i=1}^{N}GMEP(F,\varphi,A_i)\neq\emptyset$. *For every* $n\in\mathbb{N}$, *assume the either* $(B1)$ *or* $(B2)$ *holds and let the sequences* $\{x_n\}$ *and* $\{u_n\}$ *be generated by* $x_1,u\in C$ *and*

$$F(u_n,y)+\varphi(y)-\varphi(u_n)+\left\langle\sum_{i=1}^{N}a_n^i A_i x_n, y-u_n\right\rangle+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0,$$
$$x_{n+1}=\alpha_n u+\beta_n x_n+\gamma_n\left(b_n u_n+(1-b_n)K_n u_n\right), \forall n\geq 1, \tag{4.3}$$

where the sequence $\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{b_n\}\subset[0,1]$ *with* $\alpha_n+\beta_n+\gamma_n=1$, *for all* $n\geq 1$. *Suppose the following conditions hold:*

(i) $\sum_{n=1}^{\infty}\alpha_n=\infty, \lim_{n\to\infty}\alpha_n=0$;

(ii) $0<a\leq\beta_n,b_n\leq b<1$, *for some* $a,b\in\mathbb{R}$ *and for all* $n\geq 1$;

(iii) $0 < c \leq r_n \leq d < \frac{2\bar{\alpha}}{L^2}$, *for some* $c, d \in \mathbb{R}$ *and for all* $n \geq 1$*;*

(iv) $\sum_{i=1}^{N} a_n^i = 1$, *for all* $n \geq 1$*;*

(v) $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$,
$\sum_{n=1}^{\infty} \left|a_{n+1}^j - a_n^j\right| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then the sequence $\{x_n\}$ *converges strongly to* $z_0 = P_{\mathcal{F}}u$*.*

Proof. First, we show that $I - \rho D_i$ is a nonexpansive mapping for every $i = 1, 2, ..., N$. For any $x, y \in C$, we have

$$\begin{aligned}
\|(I - \rho D_i)x - (I - \rho D_i)y\|^2 &= \|(x - y) - \rho(D_i x - D_i y)\|^2 \\
&= \|x - y\|^2 + \rho^2 \|D_i x - D_i y\|^2 - 2\rho\langle x - y, D_i x - D_i y\rangle \\
&\leq \|x - y\|^2 + \rho^2 \|D_i x - D_i y\|^2 - 2\rho d_i \|D_i x - D_i y\|^2 \\
&\leq \|x - y\|^2 + \rho^2 \|D_i x - D_i y\|^2 - 2\rho\bar{d} \|D_i x - D_i y\|^2 \\
&\leq \|x - y\|^2 + \rho(\rho - 2\bar{d}) \|D_i x - D_i y\|^2 \\
&\leq \|x - y\|^2 .
\end{aligned}$$

Then, $I - \rho D_i$ is a nonexpansive mapping for every $i = 1, 2, ..., N$.
It implies that $P_C(I - \rho D_i)$ is a nonexpansive mapping for every $i \in \mathbb{N}$. By Lemma 2.4, we can conclude that

$$\bigcap_{i=1}^{\infty} VI(C, D_i) = \bigcap_{i=1}^{\infty} F(P_C(I - \rho D_i)).$$

From Theorem 3.1, we obtain the desired result. □

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