Thai Journal of Mathematics
Volume 14 (2016) Number 3 : 741–755
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Generalized Vector Equilibrium Problems with Relatively Monotone Mappings

Suhel Ahmad Khan

Department of Mathematics, BITS-Pilani, Dubai Campus
P.O. Box 345055, Dubai, UAE
e-mail : khan.math@gmail.com

Abstract : In this paper, we consider a generalized vector equilibrium problem over product sets in topological vector spaces. We establish that generalized vector equilibrium problems and a system of generalized vector equilibrium problems have same solution set. Further we define the concepts of relative pseudo monotonicity and relatively generalized B-pseudomontonicity for the set-valued bifunctions. Using these concepts and fixed point theorems, we establish some existence results for generalized vector equilibrium problems and system of generalized vector equilibrium problems. The concepts and results presented in this paper extend and unify a number of known concepts and results in the literature.

Keywords : product sets, system of generalized vector equilibrium problems; relatively pseudo-monotone mapping; relatively generalized B-pseudomonotone mapping; u-hemicontinuous mapping.
2010 Mathematics Subject Classification : 49J40; 47N10; 90C47.

1 Introduction

Equilibrium problems theory has emerged as an interesting and fascinating branch of applicable mathematics. This theory has become a rich source of inspiration and motivation for the study of a large number of problem arising in economics, optimization, operation research in a general and unified way. Equilibrium problems include varitional inequalities as well as complementarity problems, convex optimization, saddle-point problems, and Nash equilibrium as special cases, see for example ([1]-[4]). Equilibrium problems have been generalized in various directions. Vector equilibrium porblem is one of the important generalizations of equilibrium problem. In recent past a number of researchers extensively studied various classes of vector equilibrium problems, see for example ([5]-[9],[10],[12]-[23],[14]).

Very recently, Konnov [15] and Allevi et al. [16] discussed the existence of solution of (vector) variational inequalities over product sets. Further, Allevi et al. [17] and Ansari et al. [6] extended the results of ([16],[15]) for vector (quasi) variational inequalities over (countable) product sets.

Motivated and inspired by recent work going in this direction, we consider a generalized vector equilibrium problem over product sets (for short, GVEP) in topological vector spaces. We establish that GVEP and a system of generalized vector equilibrium problems (for short, SGVEP) both have same solution set. Further we define the concept of relative pseudo monotonicity and relatvely generalized B-pseudomontonicity for the set-valued bifunction, which extend the concepts of relatively pseudomonotonicity and B-pseudomontonicity given in ([16],[5],[18],[10],[15]). Using these concepts and fixed point theorems, we establish some existence results for GVEP and SGVEP. The concepts and results presented in this paper extend and unify a number of known concepts and results in the literature, see for example ([17],[16],[15],[14]).

2 Preliminaries

Throughout the paper unless otherwise stated, let $I = \{1, ..., m\}$ be an index set. For each $s \in I$, let X_s be real linear topological space and K_s be a nonempty convex subset of X_s. Set
$$K = \prod_{s\in I} K_s. \tag{1}$$

Let Y be a linear topological space with a partial order induced by a convex, closed and solid cone C with $0 \notin \operatorname{int} C$. Set $\mathbb{R}^m_+ = \{\mu \in \mathbb{R}^m : \mu_s > 0, 1 \le s \le m\}$.

For each $s \in I$, let $G_s : K \to 2^{L(X_s,Y)}$ be a mapping so that if we set
$$G = (G_s : s \in I), \tag{2}$$
then $G : K \to 2^{L(X,Y)}$ where $X = \prod_{s\in I} X_s$.

We consider the following *generalized vector equilibrium problem* over product sets (GVEP): Find $u = (u_s)_{s\in I} \in K$ such that
$$\sum_{s\in I} G_s(u, v_s) \not\subseteq -\operatorname{int} C, \quad \forall v_s \in K_s,\ s \in I. \tag{3}$$

The *dual problem* of GVEP (3) is to find an element $u \in K$ such that
$$\sum_{s\in I} G_s(v, u_s) \not\subseteq \operatorname{int} C, \quad \forall v_s \in K_s,\ s \in I, \tag{4}$$

where $v = (v_s)_{s\in I}$.

We denote by U^g and U^d the solution sets of GVEP (3) and its dual problem (4), respectively.

Next, we consider the *system of generalized vector equilibrium problem* (SGVEP): Find $u = (u_s)_{s\in I} \in K$ such that

$$G_s(u, v_s) \not\subseteq -\operatorname{int} C, \quad \forall v_s \in K_s,\ s \in I. \tag{5}$$

We denote by U^s the solution set of SGVEP (5).

Now, we recall the following fixed point theorems which are important in establishing the results of the paper.

Theorem 2.1 ([19])**.** *Let K be nonempty convex subset of a topological vector space (not necessarily Hausdorff) E and let $S, T : K \to 2^K$ be set-valued mappings. Assume that the following conditions hold:*

(i) For all $x \in K,\ S(x) \subseteq T(x)$;

(ii) For all $x \in K$, $T(x)$ is convex and $S(x)$ is nonempty;

(iii) For all $y \in K$, $S^{-1}(y) := \{x \in K :\ y \in S(x)\}$ is compactly open;

(iv) There exists a nonempty compact (not necessarily convex) subset D of K and $\tilde{y} \in D$ such that $K\backslash D \subset T^{-1}(\tilde{y})$.

Then, there exists $\hat{x} \in K$ such that $\hat{x} \in T(\hat{x})$.

For every nonempty set A, we denote by 2^A (respectively $\mathcal{F}(A)$) the family of all subsets (respectively finite subsets) of A.

Theorem 2.2 ([20])**.** *Let K be a nonempty convex subset of a topological vector space (not necessarily Hausdorff) E and let $T : K \to 2^K$ be a set-valued mapping. Assume that the following conditions hold:*

(i) For all $x \in K,\ T(x)$ is convex;

(ii) For each $A \in \mathcal{F}(K)$ and for all $y \in CoA,\ T^{-1}(y) \bigcap CoA$ is open in CoA, where CoA denotes the convex hull of set A;

(iii) For each $A \in \mathcal{F}(K)$ and all $x, y \in CoA$ and every net $\{x_\alpha\}$ in K converging to x such that $ty + (1-t)x \notin T(x_\alpha)$, for all α and for all $t \in [0,1]$, we have $y \notin T(x)$;

(iv) There exists a nonempty compact subset D of K and an element $\tilde{y} \in D$ such that $\tilde{y} \in T(x)$ for all $x \in K\backslash D$;

(v) For all $x \in D$, $T(x)$ is nonempty.

Then, there exists $\hat{x} \in K$ such that $\hat{x} \in T(\hat{x})$.

Theorem 2.3 ([21]). *Let K be a nonempty compact and convex set in Hausdorff topological vector space E. Let B be a subset of $K \times K$ having the following properties:*

(i) For each $u \in K$, $(u,u) \in B$;

(ii) For each $u \in K$, the set $B_u = \{v \in K : (u,v) \in B\}$ is closed;

(iii) For each $v \in K$, the set $B_v = \{u \in K : (u,v) \notin B\}$ is convex.

Then there exists a point $v_0 \in K$ such that $K \times \{v_0\} \subset B$.

Theorem 2.4 ([11]). *Let A and B be nonempty sets of a topological vector space E and let $F : A \to 2^B$ be such that:*

(i) For each $x \in A$, $F(x)$ is closed in B;

(ii) For each finite subset $\{x^1, \cdots, x^n\}$ of A, we have $Co\{x^1, \cdots, x^n\} \subset \bigcup_{i=1}^{n} T(x^i)$;

(iii) There exists a point $x \in A$, such that $F(x)$ is compact.

Then

$$\bigcap_{x \in X} F(x) \neq \emptyset.$$

3 Relationship between GVEP (3) and SGVEP (5)

We define the following concepts.

Definition 3.1. For each $s \in I$, the mapping $G_s : K \times K_s \to 2^Y$ is said to be

(a) *u-hemicontinuous* in the first argument, if for any $u, v, z \in K$ and $\lambda \in [0,1]$, the mapping $\lambda \to G_s(u + \lambda(v-u), z_s)$ is upper semicontinuous at 0^+;

(b) *pseudo (w,C)-monotone*, if for all $u, v \in K$, we have

$$G_s(u, v_s) \nsubseteq -\operatorname{int} C \Rightarrow \sum_{s \in I} G_s(v, u_s) \nsubseteq \operatorname{int} C.$$

Lemma 3.2. *GVEP (3) implies SGVEP (5).*

Proof. The proof follows immediately from (1) and (2). □

Lemma 3.3. *If for each $s \in I$, the mapping $G_s : K \times K_s \to 2^Y$ is pseudo (w,C)-monotone, then $U^s \subseteq U^d$.*

Proof. The proof is directly followed by pseudo (*w,C*)-monotonicity of G_s. □

Lemma 3.4. *If for each $s \in I$, the mapping $G_s : K \times K_s \to 2^Y$ is u-hemicontinuous in the first argument with condition $\sum_{s\in I} G_s(u, u_s) = 0, \quad \forall u_s \in K_s,\ s \in I$ and C-convex in the second argument, then $U^d \subseteq U^g$.*

Proof. Let us consider $u \in U^d$, then

$$\sum_{s\in I} G_s(v, u_s) \not\subseteq \operatorname{int} C, \quad \forall v_s \in K_s,\ s \in I.$$

Since, for each $s \in I$, K_s is convex, $]u_s, v_s[:= \alpha v_s + (1-\alpha)u_s \in K_s$, $\quad \forall \alpha \in (0, 1]$, and hence, we have

$$\sum_{s\in I} G_s(z, u_s) \not\subseteq \operatorname{int} C, \quad \forall z_s \in]u_s, v_s[.$$

Again, since $G_s(z, .)$ is convex, we have

$$0 = \sum_{s\in I} G_s(z, z_s) \in \alpha \sum_{s\in I} G_s(z, v_s) + (1-\alpha) \sum_{s\in I} G_s(z, u_s).$$

From above inclusions, we have

$$\sum_{s\in I} G_s(z, v_s) \not\subseteq -\operatorname{int} C, \quad \forall v_s \in K_s,\ s \in I.$$

By u-hemicontinuity of G, the preceding inclusion implies that

$$\sum_{s\in I} G_s(u, v_s) \not\subseteq -\operatorname{int} C, \quad \forall v_s \in K_s,$$

i.e., $u \in U^g$ which implies that $U^d \subseteq U^g$. This completes the proof. □

Combining above three Lemmas, we have the following result.

Proposition 3.5. *If for each $s \in I$, the mapping $G_s : K \times K_s \to 2^Y$ is u-hemicontinuous in the first argument with condition $\sum_{s\in I} G_s(u, u_s) = 0, \quad \forall u_s \in K_s,\ s \in I$, C-convex in the second argument and pseudo (w,C)-monotone. Then GVEP (3) and SGVEP (5) both have same solution set.*

Now, we prove the following Minty's type Lemma which plays an important role in establishing existence result for GVEP (3).

Lemma 3.6. *If for each $s \in I$, the mapping $G_s : K \times K_s \to 2^Y$ is u-hemicontinuous in the first argument with condition $\sum_{s\in I} G_s(u, u_s) = 0,\ \forall u_s \in K_s,\ s \in I$, C-convex in the second argument and pseudo (w,C)-monotone, then the following two problems are equivalent.*

(I) Find $u \in K$ *such that*

$$\sum_{s\in I} G_s(u, v_s) \nsubseteq -intC, \quad \forall v_s \in K_s, \; s \in I. \tag{6}$$

(II) Find $u \in K$ *such that*

$$\sum_{s\in I} G_s(v, u_s) \nsubseteq intC, \quad \forall v_s \in K_s, \; s \in I. \tag{7}$$

Proof. (I)$\Longrightarrow$(II). It follows from Lemma 3.2 that GVEP (3) implies SGVEP (5), i.e.,

$$\sum_{s\in I} G_s(u, v_s) \nsubseteq -\operatorname{int} C \Longrightarrow G_s(u, v_s) \nsubseteq -\operatorname{int} C.$$

Further by pseudo (w,C)-monotonicity of G_s, we have

$$G_s(u, v_s) \nsubseteq -\operatorname{int} C \Longrightarrow \sum_{s\in I} G_s(v, u_s) \nsubseteq \operatorname{int} C.$$

(II)$\Longrightarrow$(I). Suppose that $u \in K$ satisfies

$$G_s(v, u_s) \nsubseteq \operatorname{int} C, \quad \forall v_s \in K_s, \; s \in I.$$

Since, for each $s \in I, K_s$ is convex, $]u_s, v_s[:= \alpha v_s + (1-\alpha)u_s \in K_s \quad \forall \alpha \in (0, 1]$, and hence, we have

$$\sum_{s\in I} G_s(z, u_s) \nsubseteq \operatorname{int} C, \quad \forall z_s \in]u_s, v_s[. \tag{8}$$

Again, since $G_s(z, .)$ is convex, we have

$$0 = \sum_{s\in I} G_s(z, z_s) \in \alpha \sum_{s\in I} G_s(z, v_s) + (1-\alpha) \sum_{s\in I} G_s(z, u_s) \tag{9}$$

From inclusions (8) and (9), we have

$$\sum_{s\in I} G_s(z, v_s) \nsubseteq -\operatorname{int} C, \quad \forall v_s \in K_s, \; s \in I.$$

By u-hemicontinuity of G_s, the preceding inclusion implies that

$$\sum_{s\in I} G_s(u, v_s) \nsubseteq -\operatorname{int} C, \quad \forall v_s \in K_s.$$

This completes the proof. □

Theorem 3.7. *For each $s \in I$, let the mapping $G_s : K \times K_s \to 2^Y$ be C-convex and upper-semicontinuous in the second argument with condition $\sum_{s\in I} G_s(u,u_s) = 0,\ \forall u_s \in K_s,\ s \in I$ and let G_s be u-hemicontinuous in first argument and pseudo (w,C)-monotone. Suppose that there exists a nonempty convex and compact subset D of K and a point $\tilde{v} \in D$, such that for all $u \in K \backslash D$, $\sum_{s\in I} G_s(u,\tilde{v}_s) \subseteq -int C$. Then GVEP (3) is solvable.*

Proof. Define set-valued mappings $S, T : K \to 2^K$ by

$$S(v) = \{u \in K : \sum_{s\in I} G_s(u, v_s) \subseteq -\operatorname{int} C\}$$

$$T(v) = \{u \in K : \sum_{s\in I} G_s(v, u_s) \subseteq \operatorname{int} C\}.$$

Now, for each $v \in K$, we claim that $T(v)$ is convex. Indeed, let $u^1, u^2 \in T(v),\ \ p, q \geq 0$ such that $p + q = 1$ as K is convex. Hence

$$\sum_{s\in I} G_s(v, pu_s^1 + qu_s^2) \in p\sum_{s\in I} G_s(v, u_s^1) + q\sum_{s\in I} G_s(v, u_s^2) - C \subset \operatorname{int} C.$$

Therefore $pu^1 + qu^2 \in T(v)$. Hence our claim is then verified.

Further, it follows from pseudo (*w,C*)-monotonicity of G_s that $S(v) \subseteq T(v)$ for each $v \in K$. Since

$$S^{-1}(u) = \{v \in K : u \in S(v)\}$$

$$S^{-1}(u) = \{v \in K : \sum_{s\in I} G_s(u, v_s) \subseteq -\operatorname{int} C\}$$

$$[S^{-1}(u)]^c = \{v \in K : \sum_{s\in I} G_s(u, v_s) \nsubseteq -\operatorname{int} C\}.$$

It is easy observed from upper-semicontinuity of G_s in the second argument that $[S^{-1}(u)]^c$ is closed for each $u \in K$ and hence $S^{-1}(u)$ is open in K. Therefore, $S^{-1}(u)$ is compactly open.

Assume that, for all $v \in K,\ S(v)$ is nonempty. Then all the conditions of Theorem 2.1 are satisfied and therefore there exists $\hat{u} \in K$ such that $\hat{u} \in T(\hat{u})$. Hence it follows that

$$0 = \sum_{s\in I} G_s(\hat{u}, \hat{u_s}) \subseteq \operatorname{int} C,$$

which is impossible.

Hence, there exists $\bar{v} \in K$ such that $S(\bar{v}) = \emptyset$. This implies that, for all $u \in K$, $\sum_{s\in I} G_s(u, \bar{v}_s) \nsubseteq \operatorname{int} C$ that is, there exists $u \in K$ such that

$$\sum_{s\in I} G_s(v, u_s) \nsubseteq \operatorname{int} C, \quad \forall v_s \in K_s,\ s \in I.$$

By Lemma 3.6, above inclusion implies that there exists $u \in K$ such that

$$\sum_{s\in I} G_s(u, v_s) \nsubseteq -\operatorname{int} C, \quad \forall v_s \in K_s, s \in I.$$

This completes the proof. □

4 Parametric Generalized Vector Equilibrium Problem with Relatively Monotone Mapping

Now, we extend the notion of pseudo (w,C)-monotone to the set-valued vector case.

Definition 4.1. For each $s \in I$, the mapping $G_s : K \times K_s \to 2^Y$ is said to be

(a) *relatively monotone*, if there exists $\alpha, \beta \in \mathbb{R}^m_+$ such that $\forall u, v \in K$, we have

$$\sum_{s\in I} \alpha_s G_s(u, v_s) - \sum_{s\in I} \beta_s G_s(v, u_s) \subseteq C;$$

(b) *relatively w-pseudomonotone*, if there exists $\alpha, \beta \in \mathbb{R}^m_+$ such that $\forall u, v \in K$, we have

$$\sum_{s\in I} \beta_s G_s(u, v_s) \nsubseteq -\operatorname{int} C \Longrightarrow \sum_{s\in I} \alpha_s G_s(v, u_s) \nsubseteq \operatorname{int} C.$$

Remark 4.2. *(i) If for each $s \in I$, $G_s(u, v_s) = T_s(u)(v_s - u_s)$, where $T_s : K \to 2^{L(X_s,Y)}$, then Definition 4.1 (a)-(b) reduce to the concepts of relatively monotonicity and relatively w-pseudomonotone monotonicity of T_s given in Allevi et al. [15].*

(ii) It what follows, we reserve the symbol α and β for parameters associated to relative (pseudo) monotonocity of G . It is clear that relative monotonocity implies relative w-pseudo monotonocity, but the reverse assertions are not true in general.

We now consider a parametric form of generalized vector equlibrium problem. Fix an element $\gamma \in \mathbb{R}^m_+$ and for each $s \in I$, consider the mapping $G_s^{(\gamma_s)}$:

$K \times K_s \to 2^Y$. Then, we consider the following *parametric generalized vector equlibrium problem* (for short, PGVEP): Find $u \in K$ such that

$$\sum_{s \in I} \gamma_s G_s(u, v_s) \nsubseteq -\operatorname{int} C, \quad \forall v_s \in K_s,\ s \in I. \tag{10}$$

and its dual problem: Find $u \in K$ such that

$$\sum_{s \in I} G_s^{(\gamma_s)}(v, u_s) \nsubseteq \operatorname{int} C, \quad \forall v_s \in K_s,\ s \in I. \tag{11}$$

We denote by U_γ^g and U_γ^d the solution sets of problem PGVEP (10) and (11), respectively.

Lemma 4.3. *PGVEP (10) implies SGVEP (5).*

Proof. The proof follows immediately from (1) and (2). □

Lemma 4.4. *If for each $s \in I$, the set K_s is convex and the mapping $G_s : K \times K_s \to 2^Y$ is u-hemicontinuous in the first argument, then $U_\gamma^d \subseteq U_\gamma^g$.*

Proof. For each $s \in I$, $G_s^{(\gamma_s)}$ will also be u-hemicontinuous in the first argument and the result follows from Lemma 3.4. □

Now, we establish existence result for SGVEP (5).

Theorem 4.5. *For each $s \in I$, let K_s be convex; G_s be relatively w-pseudomonotone with nonempty compact values and G_s be u-hemicontinuous in first argument and upper semicontinuous in second argument with condition $\sum_{s \in I} G_s(u, u_s) = 0, \ \forall u_s \in K_s,\ s \in I$. Suppose that there exists a nonempty, convex and compact subset D of K and a point $\tilde{v} \in D$ such that for all $u \in K \backslash D$, $\sum_{s \in I} \alpha_s G_s(u, \tilde{v}_s) \subseteq -intC$. Then SGVEP (5) is solvable.*

Proof. Define set-valued mappings $A, B : K \to 2^K$ by

$$B(v) = \{u \in K : \sum_{s \in I} \beta_s G_s(u, v_s) \nsubseteq -\operatorname{int} C\}$$

$$A(v) = \{u \in K : \sum_{s \in I} \alpha_s G_s(v, u_s) \nsubseteq \operatorname{int} C\}$$

We divide the proof into the following three steps.

Step I. We prove that $\bigcap\limits_{v\in K} \overline{B(v)} \neq \emptyset$. Let z be in the convex hull of any finite subset $\{u^1, \cdots, u^n\}$ of K. Then $z = \sum\limits_{j=1}^{n} \mu_j u^j$ for some $\mu_j > 0,\ j = 1, ..., n;\ \sum\limits_{j=1}^{n} \mu_j = 1$. If $z \notin \bigcup\limits_{j=1}^{n} B(v^j)$, then

$$\sum_{s\in I} \beta_s G_s(z, v_s^j) \nsubseteq -\operatorname{int} C \Longrightarrow \sum_{j=1}^{n} \mu_j \big(\sum_{s\in I} \beta_s G_s(z, v_s^j)\big) \subseteq -\operatorname{int} C; \quad \forall j = 1, ..., n.$$

$$0 = \sum_{s\in I} \beta_s G_s(z, z_s) = \sum_{s\in I} \beta_s G_s\big(z, \sum_{j=1}^{n} \mu_j {v_s}^j\big)$$
$$\sum_{j=1}^{n} \mu_j \big(\sum_{s\in I} \beta_s G_s(z, {v_s}^j)\big) \subseteq -\operatorname{int} C,$$

which is a contradiction to the assumption $0 \in -\operatorname{int} C$. Therefore, the mapping $\overline{B} : K \to 2^K$ defined by $\overline{B}(v) = \overline{B(v)}$, the closure of $B(v)$, is also a KKM mapping. By assumption there exists a nonempty convex compact subset D of K and point $\tilde{v} \in D$ such that for all $u \in K\backslash D$, $\sum\limits_{s\in I} \alpha_s G_s(u, v_s) \subseteq -\operatorname{int} C$. This implies that $\overline{B(v)} \subset D$. Hence $\overline{B(\tilde{v})}$ is compact. Therefore by Theorem 2.4., we get

$$\bigcap_{v\in K} \overline{B(v)} \neq \emptyset.$$

Step II. We prove that $\bigcap\limits_{v\in K} A(v) \neq \emptyset$. From relative w-pseudomonotonocity of G_s, it follows that $B(v) \subseteq A(v)$. Next, we claim that for each $v \in K$, $A(v)$ is closed. Indeed for any $v \in K$, there exists a net $\{u^\theta\}$ in $A(v)$ such that $\{u^\theta\}$ converges to $u \in K$. Then we have $\sum\limits_{s\in I} \alpha_s G_s(v, {u_s}^\theta) \nsubseteq \operatorname{int} C$ for each θ and for each $v \in K$. That is, for each θ, there exists $p_s^\theta \in G_s(v, {u_s}^\theta),\ s \in I$ such that $\sum\limits_{s\in I} \alpha_s p_s^\theta \notin \operatorname{int} C$. Since for each $s \in I$, the set $M_s := \{{u_s}^\theta\} \cup \{u_s\}$ is compact and hence $p_s^\theta \in G_s(v, M_s),\ s \in I$. Since $G_s(v, M_s)$ is compact, $\{p_s^\theta\}$ has a convergent subnet with limit, say p_s for each $s \in I$. Without loss of generality, we can assume that p_s^θ converges to p_s for each $s \in I$. Since $Y\backslash\{\operatorname{int} C\}$ is closed, then by upper semicontinuity of $G_s(v, .)$, $p_s \in G_s(v, u_s)$, and hence $\sum\limits_{s\in I} \alpha_s G_s(v, u_s^\theta) \nsubseteq \operatorname{int} C$ implies that $\sum\limits_{s\in I} \alpha_s G_s(v, u_s) \nsubseteq \operatorname{int} C$. Hence $u \in A(v)$, for each $v \in K$.

Thus, we conclude that $A(v)$ is closed. Therefore, $\overline{B(v)} \subseteq A(v)$ and hence from Step I, we have $\bigcap\limits_{v\in K} A(v) \neq \emptyset$.

Step III. We prove that $U^s \neq \emptyset$. From Step II, it follows that $U^d(\alpha) \neq \emptyset$. Now Lemma 4.3 and 4.4 yield $U^s \neq \emptyset$, as desired. $\square$

Next we have the following existence result without monotonicity.

Theorem 4.6. *For each $s \in I$, let K_s be compact and convex, and let the mapping $G_s : K \times K_s \to 2^Y$ be C-convex in first argument and upper semicontinuous in second argument with the condition $\sum_{s\in I} \alpha_s G_s(u, u_s) = 0, \quad \forall u_s \in K_s$, then there exists a point $u_0 \in K$ such that u_0 is a solution of SGVEP (5).*

Proof. Let us suppose that

$$H = \{(u, v) \in K \times K : \sum_{s\in I} \alpha_s G_s(u, v_s) \nsubseteq -\mathrm{int}\, C\}.$$

It is clear that $(u, v) \in H, \ \forall u \in K$. For each $u \in K$, the set

$$H_u = \{v \in K : \sum_{s\in I} \alpha_s G_s(u, v_s) \nsubseteq -\mathrm{int}\, C\}$$

is closed as can be easily seen from the upper semicontinuity of $G_s(u, .)$.

Now we claim that for $v \in K$, the set

$$H_v = \{u \in K : \sum_{s\in I} \alpha_s G_s(u, v_s) \subseteq -\mathrm{int}\, C\}$$

is convex. Indeed, let $u_1, u_2 \in H(v), \quad p, q \geq 0$ such that $p + q = 1$. Since K is convex,

$$\begin{aligned}\sum_{s\in I} \alpha_s G_s(pu_1 + qu_2, v_s) &\in p\sum_{s\in I} \alpha_s G_s(u_1, v_s) + q\sum_{s\in I} \alpha_s G_s(u_2, v_s) - C \\ &= -\mathrm{int}\, C - \mathrm{int}\, C - C \subset -\mathrm{int}\, C,\end{aligned}$$

which implies that $pu_1 + qu_2 \in H(v)$.

Thus our claim is then verified. All the assumptions of Theorem 2.3 are satisfied. Therefore by Theorem 2.3 , there exists a point $u_0 \in K$ such that $K \times \{u_0\} \subset H$, which implies that $u_0 \in K$ such that

$$\sum_{s\in I} \alpha_s G_s(u_0, v_s) \nsubseteq -\mathrm{int}\, C, \quad \forall v_s \in K_s, s \in I.$$

This completes the proof. □

We define the following concepts.

Definition 4.7. For each $s \in I$, let the mapping $G_s : K \times K_s \to 2^Y$ is said to be *relatively generalized B-pseudomonotone*, if for each net $\{u^\theta\}$ in K and $u, v \in K$ such that $u^\theta \to u$ and

$$\sum_{s\in I} \alpha_s G_s(u^\theta, tu_s + (1 - t)v_s) \nsubseteq -\mathrm{int}\, C, \quad \forall t \in [0, 1],$$

we have

$$\sum_{s\in I}\alpha_s G_s(u,v_s) \nsubseteq -\operatorname{int} C, \quad \forall v_s \in K_s, s\in I.$$

If for each $s\in I,\ G_s(u,v_s) = T_s(u)(v_s-u_s)$, where $T_s : K \to 2^{L(X_s,Y)}$, then we have the following definition reduced from Definition 4.7.

Definition 4.8. For each $s \in I$, the mapping $T_s : K \to 2^{L(X_s,Y)}$ be *relatively B-pseudomonotone*, if for each net $\{u^\theta\}$ in K and $u, v \in K$ such that $u^\theta \to u$ and

$$\sum_{s\in I}\alpha_s T_s(u^\theta)((tu_s-(1-t)v_s)-u^\theta_s) \nsubseteq -\operatorname{int} C, \quad \forall t\in[0,1],$$

we have

$$\sum_{s\in I}\alpha_s T_s(u)(v_s-u_s) \nsubseteq -\operatorname{int} C.$$

Remark 4.9. *Definitions 4.7-4.8 generalize and extend the concepts of B-pseudomonotonicity given in [16,5,18,10,15].*

Theorem 4.10. *For each $s\in I$, let the mapping $G_s : K\times K_s \to 2^Y$ is relatively generalized B-pseudomonotone such that, for each $A \in \mathcal{F}(K)$, $u \to \sum_{s\in I}\alpha_s G_s(u,v_s)$ is lower semicontinuous on CoA. Assume that there exists a nonempty, convex and compact subset D of K and a point $\tilde{v}\in D$ such that for each $u \in K\backslash D$, $\sum_{s\in I}\alpha_s G_s(u,\tilde{v}_s) \subseteq -intC$. Then SGVEP (5) is solvable.*

Proof. Define a set-valued mapping $T : K \to 2^K$ by

$$T(u) = \{v \in K : \sum_{s\in I}\alpha_s G_s(u,v_s) \subseteq -\operatorname{int} C\}.$$

For each $u \in K$, it is clear that $T(u)$ is convex. Let $A \in \mathcal{F}(K)$, then for all $v \in CoA$

$$[(T^{-1}(v))^c]\bigcap CoA = \{u\in CoA : \sum_{s\in I}\alpha_s G_s(u,v_s) \nsubseteq -\operatorname{int} C\}$$

is closed in CoA by the lower semicontinuity of the mapping $u \to \sum_{s\in I}\alpha_s G_s(u,v_s)$ on CoA. Hence $\big(T^{-1}(u)\big)\bigcap CoA$ is open in CoA. Next, suppose that $u, v \in CoA$ and $\{u^\theta\}$ is a net in K converging to u such that

$$\sum_{s\in I}\alpha_s G_s(u^\theta, tu_s+(1-t)v_s) \nsubseteq -\operatorname{int} C, \quad \forall t\in[0,1].$$

By relatively generalized B-pseudomonotonicity of G_s, we have

$$\sum_{s\in I}\alpha_s G_s(u,v_s) \subseteq -\operatorname{int} C,$$

that is, $v \notin T(u)$.

Assume that $T(u)$ is nonempty for $u \in K$. Thus all the conditions of Theorem 2.2 are satisfied. Hence there exists $u \in K$ such that $u \in T(u)$, that is,

$$0 = \sum_{s \in I} \alpha_s G_s(u, v_s) \subseteq -\operatorname{int} C,$$

which is a contradiction. Hence there exists $u \in K$ such that $T(u) = \emptyset$ which implies that

$$\sum_{s \in I} \alpha_s G_s(u, v_s) \nsubseteq -\operatorname{int} C \quad \forall v_s \in K_s,\ s \in I.$$

This completes the proof. □

Finally, we have the following consequences of Theorems 4.5 and 4.10.

Theorem 4.11. *For each $s \in I$, let K_s be convex; $T_s : K \to 2^{L(X_s,Y)}$ be u-hemicontinuous and relatively w-pseudomonotone and that has nonempty compact values. Suppose that there exists a nonempty, convex and compact subset D of K and a point $\tilde{v} \in D$ such that for all $u \in K \backslash D$, $\sum_{s \in I} \alpha_s T_s(u)(\tilde{v}_s - u_s) \subseteq -intC$. Then the system of generalized vector variational inequality problems over product sets (for short, SGVVIP): Find $u \in K$ such that*

$$T_s(u)(v_s - u_s) \nsubseteq -intC, \quad \forall v_s \in K_s,\ s \in I \tag{12}$$

is solvable.

Proof. Setting: $G_s(u, v_s) = T_s(u)(v_s - u_s)$ in the Theorem 4.5 and Lemmas 4.3-4.4, the result follows. □

Theorem 4.12. *For each $s \in I$, the mapping $T_s : K \to 2^{L(X_s,Y)}$ is relatively B-pseudomonotone such that for each $A \in \mathcal{F}(K)$, $u \to \sum_{s \in I} \alpha_s T_s(u)(v_s - u_s)$ is lower-semicontinuous on CoA. Assume that there exists a nonempty, convex and compact subset D of K and a point $\tilde{v} \in D$ such that for each $u \in K \backslash D$, $\sum_{s \in I} \alpha_s T_s(u)(v_s - u_s) \subseteq -intC$. Then SGVVIP (12) is solvable.*

Proof. The result follows from Theorem 4.10 with $G_s(u, v_s) = T_s(u)(v_s - u_s)$. □

It is of further research effort to study GVEP (3) and generalized vector quasi-equilibrium problem over the cartesian product of a countable number of sets with moving cone.

Acknowledgement: The author would like to thank the referee for his/her very constructive and useful suggestions and comments.

References

[1] E. Blum,W. Oettli, From optimization and variational inequalities to equilibrium problem, Math. Stud. 63 (1994) 123-145.

[2] S.A. Khan, Generalized vector complementarity-type problems in topological vector spaces, Comput. Math. Appl. 59 (2010) 3595-3602.

[3] S.A. Khan, Generalized vector implicit quasi complementarity problems, J. Global Optim. 49 (2011) 695-705.

[4] S.A. Khan, B.S. Lee, F. Suhel, Vector mixed quasi complementarity problems in Banach spaces, Math. Comput. Modelling 55 (2012) 983-988.

[5] Q.H. Ansari, Z. Khan, Relatively B-pseudomonotone variational inequalities over product sets, J. Ineq. Pure Appl. Math. 4(1) Art. 6 (2003) 1-8.

[6] Q.H. Ansari, S. Schaible, J.C. Yao, Generalized vector quasi-variational inequalities over product sets, J. Global Optim. 32 (2005) 437-449.

[7] Q.H. Ansari, S. Schaible, J.C. Yao, The system of generalized vector equilibrium problems with applications, J. Optim. Theory Appl. 22 (2002) 3-16.

[8] Q.H. Ansari, S. Schaible, J.C. Yao, The system of vector equilibrium problems and its applications, J. Optim. Theory Appl. 107 (3) (2000) 547-557.

[9] M. Bianchi, N. Hadjisavvas, S. Schaible, Vector equilibrium problems with generalized monotone bifunctions, J. Optim. Theory Appl. 92 (1997) 527-542.

[10] A. Domokos, J. Kolumban, Variational inequalities with operator solutions, J. Global Optim. 23 (2002) 99-110.

[11] K. Fan, A generalization of Tychonoffs fixed-point theorem, Math. Annalen (1961) 305-310.

[12] J.-Y. Fu, Vector equilibrium problems, existence theorems and convexity of solution set, J. Global Optim. 31 (2005) 109-119.

[13] J.-Y. Fu, Generalized vector quasi-equilibrium problems, Math. Method Oper. Res. 52 (2000) 57-64.

[14] I.V. Konnov, J.C. Yao, Existence of solutions for generalized vector equilibrium problems, J. Math. Anal. Appl. 233 (1999) 328-335.

[15] I.V. Konnov, Relatively monotone variational inequalities over product sets, Oper. Res. Lett. 28 (2001) 21-26.

[16] E. Allevi, A. Gnudi, I.V. Konnov, Generalized vector variational inequalities over product of sets, Nonlinear Analysis 47 (2001) 573-582.

[17] E. Allevi, A. Gnudi, I.V. Konnov, Generalized vector variational inequalities over countable product of sets, J. Global Optim. 30 (2004) 155-167.

[18] H. Brezis, Equations et Inequations non lineaires dans les espaces vectoriels en dualite, Ann. Inst. Fourier 18 (1968) 115-175.

[19] M.S.R. Chowdhury, K.K. Tan, Generalized variational inequalities for quasi-monotone operators and application, Bull. Polish Acad. Sci. (Math.) 45 (1997) 25-54.

[20] M.S.R. Chowdhury, K.K. Tan, Generalization of Ky Fan minimax inequality with applications to generalized variational inequalities for pseudomonotone operators and fixed-point theorems, J. Math. Anal. Appl. 204 (1996) 910-926.

[21] T.C. Lin, Convex sets, fixed points, variational and minimax inequalities, Bull. Austral. Math. Soc. 34 (1986) 107-117.

[22] F. Giannessi, Vector Variational Inequalities and Vector Equilibria: Mathematical Theories, Kluwer Academic Publications, Dordrecht-Boston, London, 2000.

[23] N. Hadjisavvas, S. Schaible, From scalar to vector equilibrium problems in the quasimonotone case, J. Optim. Theory Appl. 96 (1998) 297-309.

(Received 17 September 2014)
(Accepted 30 August 2015)