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Pricing Discretely-Sampled Variance Swaps on Commodities

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Abstract : In this paper, we propose an analytical approach to price a discretely-sampled variance swap when the underlying asset is a commodity, with the realized variance defined in terms of squared percentage return of the underlying commodity prices. We assume that commodity price follows Schwartz (1997)'s one-factor model, which is adopted to describe the stochastic behavior of it. Furthermore, we demonstrate the validity of our closed-form solution in terms of its financial meaningfulness. Finally, a comparison between our solution and Monte Carlo simulations demonstrates the efficiency of our approach, which substantially reduces the computational burden of using Monte Carlo methods.

Keywords : variance swaps; discretely sampling; Schwartz's model; commodity prices.
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1 Introduction

Commodity trading has tended to grow tremendously over the past decade. The groups of participants in commodity markets include not only the producers, but also some financial portfolio managers, who use commodities as hedging

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instruments and even speculative assets. The liquidity of commodity trading is also boosted by global market integration and liberalization of financial market. The price of commodities becomes abruptly volatile, and it hurts economic development in terms of national income, trade balances, price levels, and nominal exchange rates [1]. In the last decade, which was a period of high energy price, there was an increasing demand for biofuels, which had a profound impact on the grain market. An escalation in price volatility of major grain commodities lead to growing concern about the security of the world food supply [2]. Commodity market participants, therefore, are constantly looking for a tool to use to hedge against the volatility of commodity price.

To hedge against volatility risk, variance swaps are the favored derivatives. Even though there are specific volatility swaps to hedge against volatility risk, investors are more familiar with variance than volatility. Variance swaps are forward contracts on the future realized variance on the specified underlying assets. Most underlying assets of variance swaps are financial assets; more specifically, indexes. Among the first of the literature that discussed variance swaps, Demeterfi et al. [3] showed that variance swaps could be replicated by a portfolio of standard put and call options, with suitably chosen exercise prices series. This methodology seemed to be acceptable in their presentation, but the assumption about standard options with a continuum of exercise prices causes it to be complicated to adopt. To solve the problem, they also pointed out in their paper that it demanded a stochastic volatility model. Howison et al. [4] proposed a closed-form pricing formula of both variance and volatility swaps. Two closed-form formulas were presented by assuming a geometric Brownian motion model and one-factor volatility process. However; they approximated the realized variance in continuous time, which is not pragmatic in the market. Differently, based on Heston [5] model, a model widely used to explain financial asset price, Swishchuk [6] defined his own discretely-sampled realized variance, which he called "a pseudo-variance", and used a probabilistic approach to approximate volatility and variance swaps price formulas in integral forms. Recently, the closed-form solutions for the fair prices of variance swaps, based on the conventional-defined realized variances, were proposed by Zhu and Lian [7, 8]. They used the Heston model to explain the underlying asset price process and introduced a new state variable, enabling them to find the fair price of variance swaps by solving the governing PDE (Partial Differential Equation) system directly with the generalized Fourier transformation. More interestingly, Rujivan and Zhu [9, 10] derived identical formulas to the ones shown by Zhu and Lian [7, 8], but used a more direct approach to price variance swaps without using their complicated procedures. Instead, Rujivan and Zhu [9, 10]'s methodology was based on the common tower property of conditional expectation.

Focusing on commodity markets, Swishchuk [11] was among the first to derive the fair price formula of volatility and variance swaps for energy (natural gas). He assumed the risk-neutral stochastic volatility process which follows the mean-reverting one-factor variance model, called the continuous-time generalized autoregressive conditional heteroskedasticity or GARCH(1,1) model. Then, he derived the price of variance swaps directly by taking integral of the first moment of the stochastic volatility over time t from the beginning until its maturity. However, his formula was for a continuously-sampled variance swaps price.

Selecting a stochastic process to describe the behavior of commodity price is crucial to the success of determining the price of variance swaps in the commodity markets. With an inappropriate process, practicality to any commodity market is lost, even with the variance swap price correctly derived. Unlike a financial asset price, a commodity price has a mean reversion; when it is low, people consume more, and the high-cost producers leave the market, leading to increased price; conversely, when it is high, people consume less, and induce many producers to the market, leading to decreased price. Moreover, there are many types of commodities; the most basic way to sort is between those that are storable and those that are not. Focusing on storable commodities, a forward price is generally explained by the theory of storage; this theory predicts the positive relationship between a spot commodity price and its forward curve slope by a key factor, called a convenience yield. A convenience yield arises from the benefits that commodity holders enjoy from their inventory holding, in terms of ease of use and providing a buffer to price volatility caused by seasonality. The Schwartz [12] one-factor model has its merits to describe commodity price. It has a mean reversion, and incorporates a convenience yield measured by the logarithm of spot price.

In this paper, we assume the realized variance, defined as the squared percentage returns; a natural way to compute a return variance of the spot commodity price, following Schwartz [12] one-factor model. Moreover, the way to define a realized variance in discrete sampling is more practical to the behavior of commodity prices in the markets. We then apply the Rujivan and Zhu [9, 10]'s approach to obtain solutions for the fair delivery price of variance swaps on commodities.

The remainder of the paper is organized as follows. In Section 2, we review the Schwartz [12] one-factor model. In Section 3, we discuss the concept of variance swaps and a definition of their underlying, called a realized variance. Then, we derive a simple closed-form formula for the fair delivery price of commodity variance swaps. In Section 4, we investigate the validity of our pricing formula in terms of its financial meaningfulness. In Section 5, we show a comparison of our formula to the Monte Carlo simulations. Finally, we give a brief summary in Section 6.

2 Schwartz One-Factor Model

In this section, we shall briefly review the Schwartz [12] one-factor model to describe the dynamics of commodity prices in our paper. The model is an extension of the Ornstein-Uhlenbeck (OU) model. The fundamental theorem of asset pricing [13] states that the existence of a risk-neutral probability measure guarantees that there is no arbitrage opportunity. Under a risk-neutral probability measure, the Schwartz one-factor model describes the spot commodity price at time t, denoted by S_t, follow the stochastic differential equation (SDE);

$$dS_t = \kappa\left(\mu^* - \ln S_t\right) S_t dt + \sigma S_t dz_t^*, S_0 > 0. \tag{2.1}$$

Here, κ is a degree of mean-reverting speed parameter, μ^* is the long-run value of spot commodity price, σ is the volatility, and dz_t^* is an increment to a standard Brownian motion under a risk-neutral probability space $(\Omega, \mathcal{F}, Q)$. We assume that an initial spot commodity price S_0 and parameters κ, σ are strictly positive.

The model (2.1) has a benefit of taking convenience yield into account and still preserving the "one-factor" stochastic process. The empirical study in [14] showed that the correlations between the convenience yields and returns on trading the commodities were positive. Due to the empirical study, instantaneous convenience yield at time t, denoted by δ_t, is assumed to be a stochastic process, which has a linear transformation of the logarithmic of S_t as

$$\delta_t = \kappa \ln S_t, \tag{2.2}$$

for all $t \geq 0$. Therefore, the model (2.1) can be written in terms of δ_t as

$$dS_t = \left(\kappa\mu^* - \delta_t\right) S_t dt + \sigma S_t dz_t^*. \tag{2.3}$$

For the rest of this paper, our analysis will be based on the risk-neutral probability space $(\Omega, \mathcal{F}, Q)$ with a filtration $(\mathcal{F}_t)_{t\geq 0}$. Moreover, the conditional expectation with respect to $\mathcal{F}_t$ is denoted by $E^Q[\cdot|\mathcal{F}_t] = E_t^Q[\cdot]$.

3 Pricing Discretely-Sampled Variance Swaps

In this section, we shall discuss the concept of discretely-sampled variance swaps. Then, we shall apply Rujivan and Zhu [9, 10]'s approach to derive a closed-form solution of the fair price of commodity variance swaps based on the Schwartz [12] one-factor model.

3.1 Variance Swaps

Variance swaps are actually forward contracts on the future realized variance of returns on the specified underlying asset. The long position of variance swaps pays a fixed delivery price at expiry, and receives a floating amount of annualized realized variance, whereas the short position is just the opposite. Investors who expect the increment of volatility may hold long position, while the contrary may hold the short position. With variance swaps, investors can easily gain exposure to volatility risk.

Usually the value of a variance swap at expiry can be written as

$$V_T = \left(\sigma_R^2 - K_{\text{var}}\right) \times L,$$

where σ_R^2 is an annualized realized variance over the contract life $[0, T]$, K_{var} is an annualized delivery price for the variance swap, L is a notional amount of the swap in dollars per annualized volatility point squared, and T is the contract life time. Thus, the long position of a variance swap receives L dollars for every point by which the annualized realized variance σ_R^2 exceeds the delivery price K_{var}.

At the beginning of a contract, the details of how the realized variance should be calculated are clearly specified. Important factors contributing to the calculation of the realized variance include the underlying assets, the observation frequency of the price of the underlying assets, the annualization factor, the contract life time, and the method of calculating the variance. Most traded contracts define the realized variance in terms of either simple returns or logarithmic returns. In this paper we will only consider the definition based on the simple returns. So, the realized variance is defined as

$$\sigma_R^2 = \frac{AF}{N} \sum_{i=1}^{N} \left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}} \right)^2 \times 100^2, \tag{3.1}$$

where S_{t_i} is a closing price of the underlying asset at the ith observation time t_i, and there are altogether N observations. AF is an annualized factor, converting this expression to an annualized variance. If the sampling frequency is every trading day, then $AF = 252$ assuming that there are 252 trading days in a year, if every week, then $AF = 52$, if every month, then $AF = 12$ and so on. We assume equally spaced discrete observations in this paper, so that the annualized factor is of a simple expression, $AF = \frac{1}{\Delta t} = \frac{N}{T}$. With the realized variance defined above, one may call it the squared percentage return.

In a risk-neutral world, the value of a variance swap at time t is the expected present value of the future payoff, $V_t = E_t^Q \left[e^{-r(T-t)} \left(\sigma_R^2 - K_{\text{var}} \right) L \right]$. This should be zero at the beginning of the contract, since there is no cost to enter into the swap. Therefore, the fair delivery price of variance swap can be defined as $K_{\text{var}} = E_0^Q \left[\sigma_R^2 \right]$, after initially setting the value of $V_0 = 0$. The variance swap valuation problem is therefore reduced to calculating the expectation value of a future realized variance in a risk-neutral world.

3.2 Our Solution Approach

In this subsection, we derive the fair price of commodity variance swaps, based on Rujivan and Zhu [9, 10]'s approach. We begin with taking the conditional expectation of σ_R^2 in (3.1) with respect to $\mathcal{F}_0$. The fair delivery price of the commodity variance swaps can be written as

$$\begin{aligned} K_{\text{var}} = E_0^Q \left[\sigma_R^2 \right] &= E_0^Q \left[\frac{1}{N \Delta t} \sum_{i=1}^{N} \left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}} \right)^2 \right] \times 100^2 \\ &= \frac{1}{N \Delta t} \sum_{i=1}^{N} E_0^Q \left[\left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}} \right)^2 \right] \times 100^2. \end{aligned} \tag{3.2}$$

From (3.2), the problem of pricing commodity variance swaps is reduced to evaluating the N conditional expectations of the form:

$$E_0^Q\left[\left(\frac{S_{t_i}-S_{t_{i-1}}}{S_{t_{i-1}}}\right)^2\right], \tag{3.3}$$

for some fixed equal time period Δt and N at different tenors $t_i = i\Delta t$; $(i = 1, 2, ..., N)$. Next, we apply the tower property of conditional expectation to (3.3), and obtain

$$\begin{aligned} E_0^Q\left[\left(\frac{S_{t_i}-S_{t_{i-1}}}{S_{t_{i-1}}}\right)^2\right] &= E_0^Q\left[E_{t_{i-1}}^Q\left[\left(\frac{S_{t_i}-S_{t_{i-1}}}{S_{t_{i-1}}}\right)^2\right]\right] \\ &= E_0^Q\left[\frac{1}{S_{t_{i-1}}^2}\left(E_{t_{i-1}}^Q\left[S_{t_i}^2\right]-2S_{t_{i-1}}E_{t_{i-1}}^Q\left[S_{t_i}\right]\right)+1\right]. \end{aligned} \tag{3.4}$$

The following theorem provides a closed-form formula for the γ^{th} conditional moment of S_t, based on the model (2.1), for any real number γ. Moreover, we shall adopt the theorem to derive $E_{t_{i-1}}^Q\left[S_{t_i}\right]$ and $E_{t_{i-1}}^Q\left[S_{t_i}^2\right]$in the RHS of (3.4) later on.

Theorem 3.1. *Suppose S_t follows the dynamics described in (2.1). Then,*

$$E_{t_{i-1}}^Q\left[S_{t_i}^\gamma\right] = S_{t_{i-1}}^{\gamma A_2(\Delta t)} \exp\{A_1(\gamma, \Delta t)\}, \tag{3.5}$$

for all $(t_i, S_{t_i}, \gamma) \in (0, T] \times (0, \infty) \times (-\infty, \infty)$, where $\Delta t = t_i - t_{i-1}$; $i = 1, ..., N$, and

$$A_1(\gamma, \Delta t) = \gamma(1-\exp\{-\kappa\Delta t\})\alpha^* + \gamma^2(1-\exp\{-2\kappa\Delta t\})\frac{\sigma^2}{4\kappa}, \tag{3.6}$$

$$A_2(\Delta t) = \exp\{-\kappa\Delta t\}, \tag{3.7}$$

where $\alpha^* = \mu^* - \frac{\sigma^2}{2\kappa}$.

Proof. We first define

$$Y_t := S_t^\gamma, \tag{3.8}$$

for all $t \geq 0$. Applying It ô's Lemma to (3.8), we obtain

$$dY_t = \left(\gamma\kappa\mu^* + \frac{\gamma(\gamma-1)}{2}\sigma^2 - \kappa \ln Y_t\right) Y_t dt + \gamma\sigma Y_t dz_t^*. \tag{3.9}$$

Let

$$U_i(t, y) = E^Q\left[Y_{t_i} | Y_{t_{i-1}} = y\right], \tag{3.10}$$

for all $(t, y) \in [t_{i-1}, t_i) \times \mathbb{R}^+$. According to the Feynman-Kac theorem, $U_i(t, y)$ satisfies the PDE

$$\frac{\partial}{\partial t}U_i(t,y)+\frac{1}{2}\gamma^2\sigma^2y^2\frac{\partial^2}{\partial y^2}U_i(t,y)+\left(\gamma\kappa\mu^*+\frac{\gamma(\gamma-1)}{2}\sigma^2-\kappa\ln y\right)y\frac{\partial}{\partial y}U_i(t,y)=0, \tag{3.11}$$

subject to the terminal condition

$$U_i\left(t_i, y\right) = y, \tag{3.12}$$

for all $y \in \mathbb{R}^+$.

Next, we assume that

$$U_i\left(t, y\right) = y^{A_2(t_i - t)} \exp\left\{A_1\left(\gamma, t_i - t\right)\right\}, \tag{3.13}$$

where $A_1\left(\gamma, t_i - t\right)$ and $A_2\left(t_i - t\right)$ are deterministic functions to be determined later on. Let $\tau = t_i - t$, and substitute (3.13) into the PDE (3.11); we get a system of ordinary differential equations (ODEs):

$$\frac{d}{d\tau} A_2\left(\tau\right) = -\kappa A_2\left(\tau\right), \tag{3.14}$$

$$\frac{d}{d\tau} A_1\left(\gamma, \tau\right) = -\gamma\left(\kappa\mu^* + \frac{\sigma^2}{2}\right) A_2\left(\tau\right) - \frac{1}{2}\gamma^2\sigma^2 A_2(\tau)^2, \tag{3.15}$$

subject to initial conditions

$$A_2\left(0\right) = 1, \tag{3.16}$$

$$A_1\left(\gamma, 0\right) = 0. \tag{3.17}$$

Setting $t = t_{i-1}$, and $\tau = \Delta t = t_i - t_{i-1}$, the solutions of the system of ODEs can be expressed as written in (3.6) and (3.7). □

From Theorem 3.1, substitute for $\gamma = 1$ and $\gamma = 2$ into (3.5); we obtain

$$E^Q_{t_{i-1}}\left[S_{t_i}\right] = S_{t_{i-1}}^{\exp\{-\kappa\Delta t\}} \exp\left\{A_1\left(1, \Delta t\right)\right\}, \tag{3.18}$$

$$E^Q_{t_{i-1}}\left[S^2_{t_i}\right] = S_{t_{i-1}}^{2\exp\{-\kappa\Delta t\}} \exp\left\{A_1\left(2, \Delta t\right)\right\}. \tag{3.19}$$

We would like to point out that, using (3.5), the conditional variance, conditional skewness and conditional kurtosis of S_{t_i} with respect to $\mathcal{F}_{t_{i-1}}$ can be easily found, with $\gamma = 1, 2, 3, 4$, as

$$\begin{aligned} \text{Var}\left[S_{t_i} | \mathcal{F}_{t_{i-1}}\right] &= E^Q_{t_{i-1}}[S^2_{t_i}] - (E^Q_{t_{i-1}}[S_{t_i}])^2, \\ &= S_{t_{i-1}}^{2A_2(\Delta t)} \exp\left\{A_1\left(2, \Delta t\right)\right\}\left(1 - \exp\left\{-2\tilde{C}\left(\Delta t\right)\sigma^2\right\}\right), \end{aligned} \tag{3.20}$$

$$\begin{aligned} \text{Skew}[S_{t_i} | \mathcal{F}_{t_{i-1}}] &= \frac{E^Q_{t_{i-1}}[(S_{t_i} - E^Q_{t_{i-1}}[S_{t_i}])^3]}{(\text{Var}[S_{t_i} | \mathcal{F}_{t_{i-1}}])^{\frac{3}{2}}}, \\ &= \frac{\exp\{A_1(1,\Delta t)\}\left(\exp\left\{3\tilde{C}(\Delta t)\sigma^2\right\} + \exp\left\{\tilde{C}(\Delta t)\sigma^2\right\} - 2\exp\left\{-\tilde{C}(\Delta t)\sigma^2\right\}\right)}{\left(1 - \exp\left\{-2\tilde{C}(\Delta t)\sigma^2\right\}\right)^{1/2}}, \end{aligned} \tag{3.21}$$

$$\begin{aligned}\mathrm{Kurt}[S_{t_i}|\mathcal{F}_{t_{i-1}}] &= \frac{E^{Q}_{t_{i-1}}[(S_{t_i} - E^{Q}_{t_{i-1}}[S_{t_i}])^4]}{(\mathrm{Var}[S_{t_i}|\mathcal{F}_{t_{i-1}}])^2},\\ &= \exp\left\{8\tilde{C}(\Delta t)\sigma^2\right\} + 2\exp\left\{6\tilde{C}(\Delta t)\sigma^2\right\}\\ &\quad + 3\exp\left\{4\tilde{C}(\Delta t)\sigma^2\right\} - 3, \end{aligned} \tag{3.22}$$

where $\tilde{C}(\Delta t) = \frac{1-\exp\{-2\kappa\Delta t\}}{4\kappa}$ for all $\Delta t > 0$. The conditional expectation of the underlying stock price at a given future time such as [9] depends only on its own value at current time, whereas the conditional expectation of the underlying commodity price based on Schwartz [12] one-factor model as shown in (3.18) depends on both its own value and the process variance. Moreover, the conditional variance of the underlying commodity price, based on Schwartz one-factor model, at a given future time as shown in (3.20), depends on both the underlying spot commodity price at the current time and the process variance. On the other hand, (3.21) and (3.22) indicate that both the conditional skewness and conditional kurtosis depend only on the process variance.

By utilizing (3.18) and (3.19) to compute the conditional expectations in the RHS of (3.4), we promptly attain the fair price of variance swaps on commodities based on the Schwartz one-factor model in the following theorem.

Theorem 3.2. *The conditional expectation in (3.4) can be written in terms of an initial spot commodity price as*

$$\begin{aligned}E^{Q}_{0}\left[\left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}\right)^2\right] &= S_0^{2\tilde{A}_1(\Delta t, t_{i-1})}\exp\left\{-2\tilde{A}_1(\Delta t, t_{i-1})\,\alpha^* + \tilde{A}_2\,(\Delta t, t_{i-1})\,\frac{\sigma^2}{\kappa}\right\}\\ &\quad -2S_0^{\tilde{A}_1(\Delta t, t_{i-1})}\exp\left\{-\tilde{A}_1(\Delta t, t_{i-1})\,\alpha^* + \tilde{A}_2\,(\Delta t, t_{i-1})\,\frac{\sigma^2}{4\kappa}\right\}\\ &\quad + 1, \end{aligned} \tag{3.23}$$

for all $\Delta t = t_i - t_{i-1}; i = 1, 2, ..., N$ *and* $t_i \in (0, T]$, *where*

$$\tilde{A}_1\,(\Delta t, t_{i-1}) = (\exp\{-\kappa\Delta t\} - 1)\exp\{-\kappa t_{i-1}\}, \tag{3.24}$$

$$\tilde{A}_2\,(\Delta t, t_{i-1}) = \left(1 - \exp\{-2\kappa\Delta t\} + (\exp\{-\kappa\Delta t\} - 1)^2(1 - \exp\{-2\kappa t_{i-1}\})\right). \tag{3.25}$$

In addition, the fair price of a commodity variance swap can be written as

$$\begin{aligned}K_{\mathrm{var}}(S_0, T, \Delta t) &= \frac{100^2}{T}\sum_{i=1}^{N} S_0^{2\tilde{A}_1(\Delta t, t_{i-1})}\exp\left\{\tilde{A}_3(\Delta t, t_{i-1})\right\}\\ &\quad - 2S_0^{\tilde{A}_1(\Delta t, t_{i-1})}\exp\left\{\tilde{A}_4(\Delta t, t_{i-1})\right\} + 1, \end{aligned} \tag{3.26}$$

where

$$\tilde{A}_3\,(\Delta t, t_{i-1}) = -2\tilde{A}_1\,(\Delta t, t_{i-1})\,\alpha^* + \tilde{A}_2\,(\Delta t, t_{i-1})\,\frac{\sigma^2}{\kappa}, \tag{3.27}$$

$$\tilde{A}_4\left(\Delta t, t_{i-1}\right) = -\tilde{A}_1\left(\Delta t, t_{i-1}\right)\alpha^* + \tilde{A}_2\left(\Delta t, t_{i-1}\right)\frac{\sigma^2}{4\kappa}. \tag{3.28}$$

Furthermore, $K_{\text{var}}\left(S_0, T, \Delta t\right)$ *can also be written in terms of the initial convenience yield as*

$$K_{\text{var}}\left(\delta_0, T, \Delta t\right) = \frac{100^2}{T}\sum_{i=1}^{N}\exp\left\{2\frac{\tilde{A}_1\left(\Delta t, t_{i-1}\right)}{\kappa}\delta_0 + \tilde{A}_3\left(\Delta t, t_{i-1}\right)\right\} - 2\exp\left\{\frac{\tilde{A}_1\left(\Delta t, t_{i-1}\right)}{\kappa}\delta_0 + \tilde{A}_4\left(\Delta t, t_{i-1}\right)\right\} + 1, \tag{3.29}$$

where $\delta_0 = \kappa \ln S_0$.

Proof. Substituting (3.18) and (3.19) into the RHS of (3.4), we obtain

$$E_0^Q\left[\left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}\right)^2\right] = \exp\left\{A_1\left(2, \Delta t\right)\right\}E_0^Q\left[S_{t_{i-1}}^{2(\exp\{-\kappa\Delta t\}-1)}\right] - 2\exp\left\{A_1\left(1, \Delta t\right)\right\}E_0^Q\left[S_{t_{i-1}}^{\exp\{-\kappa\Delta t\}-1}\right] + 1. \tag{3.30}$$

We next apply Theorem 3.1 in order to derive the conditional expectations $E_0^Q\left[S_{t_{i-1}}^{2(\exp\{-\kappa\Delta t\}-1)}\right]$ and $E_0^Q\left[S_{t_{i-1}}^{\exp\{-\kappa\Delta t\}-1}\right]$. By setting

$$\gamma = 2\left(\exp\left\{-\kappa\Delta t\right\} - 1\right) \text{ and } \gamma = \left(\exp\left\{-\kappa\Delta t\right\} - 1\right)$$

into (3.5), respectively, we thus obtain

$$E_0^Q\left[S_{t_{i-1}}^{2(\exp\{-\kappa\Delta t\}-1)}\right] = S_0^{2(\exp\{-\kappa\Delta t\}-1)\exp\{-\kappa t_{i-1}\}}\exp\left\{A_1\left(2\left(\exp\left\{-\kappa\Delta t\right\} - 1\right), t_{i-1}\right)\right\}, \tag{3.31}$$

$$E_0^Q\left[S_{t_{i-1}}^{(\exp\{-\kappa\Delta t\}-1)}\right] = S_0^{(\exp\{-\kappa\Delta t\}-1)\exp\{-\kappa t_{i-1}\}}\exp\left\{A_1\left(\left(\exp\left\{-\kappa\Delta t\right\} - 1\right), t_{i-1}\right)\right\}. \tag{3.32}$$

Substitute (3.31) and (3.32) into (3.30), we can derive the conditional expectation in the LHS of (3.30) in closed-form, as shown in (3.23). □

According to [13], there are two ways to compute a derivative security price - specifically a variance swap price in our paper: (1) using Monte Carlo (MC) simulation to generate path of the underlying commodity price and use these paths to estimate the expected realized variance; or (2) numerically solve a partial differential equation (PDE) governing the realized variance according to Feynman-Kac theorem. In our paper, we focus on the second method. And, instead of numerically solving the PDE, we provide an analytical approach. Unlike Zhu and Lian [7, 8] analytically solving the governing PDE by utilizing the generalized Fourier transformation, in our paper we apply Rujivan and Zhu [9, 10]'s methodology, called the common tower property of conditional expectation. Theorem 3.1 shows that, with their technique, in place of solving the governing PDE directly, we can simplify the problem to solve the system of ODEs instead. And Theorem 3.2 shows the derivation of closed-form variance swaps pricing formula.

4 Validity of Our Solution

In this section, we provide an interesting discussion in terms of the validity of our current solution, as written in (3.29). The purpose of such an investigation is to guarantee that one of the fundamental assumptions, that the fair delivery price of a variance swap should be of finite and positive value for a given set of parameters determined from market data, i.e.,
$0 \leq K_{\text{var}} < \infty$, is indeed satisfied.

Proposition 4.1. *Suppose $T > 0$ and $\Delta t > 0$. Then,*

$$0 < K_{\text{var}}\left(\delta_0, T, \Delta t\right) < \infty, \tag{4.1}$$

for all $\delta_0 \in \mathbb{R}$.

Proof. Since $\kappa > 0$, from (3.27) and (3.28), $\tilde{A}_3\left(\Delta t, t_{i-1}\right)$ and $\tilde{A}_4\left(\Delta t, t_{i-1}\right)$ are finite for all $\Delta t \geq 0$ and $t_{i-1}; i = 1, .., N$. These results imply that $K_{\text{var}}\left(\delta_0, T, \Delta t\right) < \infty$ for all $\delta_0 \in \mathbb{R}$. Next, in order to show that $K_{\text{var}}\left(\delta_0, T, \Delta t\right)$ is strictly positive, we use the fact that

$$0 \leq \left(\exp\left\{a + b\right\} - 1\right)^2 < \exp\left\{2a + 4b\right\} - 2\exp\left\{a + b\right\} + 1, \tag{4.2}$$

for all $a \in \mathbb{R}$ and $b \in \mathbb{R}^+$. By setting

$$a_i = \frac{\tilde{A}_1\left(\Delta t, t_{i-1}\right)}{\kappa}\delta_0 - \tilde{A}_1\left(\Delta t, t_{i-1}\right)\alpha^*, \tag{4.3}$$

$$b_i = \tilde{A}_2\left(\Delta t, t_{i-1}\right)\frac{\sigma^2}{4\kappa}, \tag{4.4}$$

for all $i = 1, 2, ..., N$, from (3.29), we have

$$K_{\text{var}}\left(\delta_0, T, \Delta t\right) = \frac{100^2}{T}\sum_{i=1}^{N}\exp\left\{2a_i + 4b_i\right\} - 2\exp\left\{a_i + b_i\right\} + 1. \tag{4.5}$$

Notice from (3.25) that $\tilde{A}_2\left(\Delta t, t_{i-1}\right) > 0$ for all $\Delta t \geq 0$ and $t_{i-1}; i = 1, ..., N$. Hence, $b_i > 0$ for all $i = 1, ..., N$. Applying the inequality (4.2) to formula (4.5), we immediately find that $K_{\text{var}}\left(\delta_0, T, \Delta t\right)$ is strictly positive as desired. □

5 A Comparison to Monte Carlo Simulations

In this section, we conduct Monte Carlo (MC) simulations to illustrate the accuracy of the closed-form formula in (3.29). Although, theoretically, there would no need to discuss the accuracy and present the numerical results of our formula, some comparisons with MC simulations may give readers a sense of verification for the newly found solution. This is particularly so for some practitioners who are very used to MC simulations and would not trust analytical solutions that may contain algebraic errors unless they have seen numerical evidence of such a comparison.

In our numerical test, we use the following parameters, $\mu^* = 3.177$; $\sigma = 0.129$; and $\kappa = 0.099$, calibrated from oil market as proposed by [12]. To ensure the correctness of our solution, we have employed the MC method to simulate the underlying process (S_t) and calculate realized variance according to definition (3.1). In our MC simulations, we have used the Euler-Maruyama discretization for the underlying process (S_t)

$$S_{t_i} = S_{t_{i-1}} + \kappa\left(\mu^* - \ln S_{t_{i-1}}\right) S_{t_{i-1}}\Delta t + \sigma S_{t_{i-1}}\sqrt{\Delta t}\varepsilon_{t_i}, \tag{5.1}$$

where ε_{t_i} is a standard normal random variable. We generate sample paths of S_t on $[0, T]$ where $T = 1$. For the spot commodity prices obtained by using (5.1), we define

$$K_{\text{var}}^{\text{MC}}(N_p) := \frac{\sum_{p=1}^{N_p}\left(\frac{1}{N\Delta t}\sum_{i=1}^{N}\left(\frac{S_{t_i}(\omega_p)-S_{t_{i-1}}(\omega_p)}{S_{t_{i-1}}(\omega_p)}\right)^2 \times 100^2\right)}{N_p} \tag{5.2}$$

where $N = 252$, $\Delta t = {}^1/_N$, $S_{t_i}(\omega_p)$ is the commodity price at time t_i obtained by using (5.1) for path ω_p, and N_p is the number of paths. By the law of large number, $K_{\text{var}}^{\text{MC}}(N_p) \to K_{\text{var}}$ as $N_p \to \infty$. In other words, we can estimate K_{var} by $K_{\text{var}}^{\text{MC}}(N_p)$ when N_p is sufficient large. Thus, we choose $N_p = 10^5$ to obtain a good approximation for the fair delivery price.

By choosing 17 values of δ_0 varying from -2.07 to 2.73, we plot $K_{\text{var}}^{MC}(N_p)$ against K_{var} for all δ_0 as shown in Figure 1. One can clearly see from the figure that the results from our closed-form solution (3.29) perfectly match the results from the MC simulation for $N_p = 10^5$.

N_p	Relative Error (%)	Computation Time (seconds)
1,000	0.3679	454.7128
10,000	0.2736	4,204.9451
100,000	0.1555	43,750.3681

Table 1: Relative errors and computation time of MC simulations, where computation time of using our formula is 0.2189 seconds.

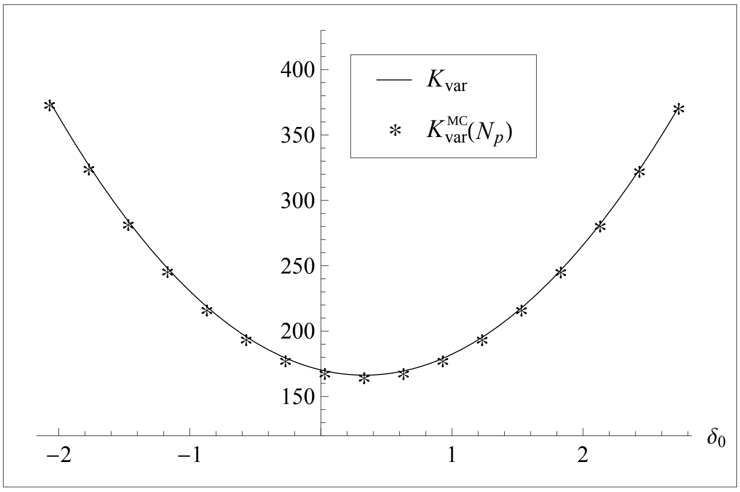

Figure 1: A Comparison between computed K_{var} from our formula and from MC simulations.

Furthermore, we compute averages of percentage relative errors of $K_{\text{var}}^{MC}(N_p)$ for $N_p = 10^3, 10^4$, and 10^5 tabulated in Table 1, in order to show a convergence of $K_{\text{var}}^{MC}(N_p)$ to K_{var} as N_p approaches infinity. We clearly see from Table 1 that the MC simulation takes a much longer time to reach 0.1555% average of percentage relative errors than using our closed-form formula which consumes just 0.2189 seconds; a roughly 200 thousand fold reduction in computation time. It is clear that our approach substantially reduces the computation time burden of using the MC simulation and can be implemented efficiently.

6 Conclusion

In this paper, we have presented an analytical approach to price discretely-sampled variance swaps when the underlying asset is a commodity. By assuming that commodity price follows the Schwartz [12] one-factor model and defining discretely-sampled realized variance in terms of squared percentage return of the underlying commodity price, we have derived the closed-form formula of a fair delivery price of variance swaps on commodities based on the Schwartz [12] model. Moreover, we have proved that our pricing formula has financial meaningfulness, such that the fair delivery price of commodity variance swaps computed with our formula is finite and has a positive value in the parameter space. Furthermore, we have demonstrated that the fair delivery prices computed from our formula perfectly match with those from Monte Carlo simulations, but using our pricing formula substantially reduces the computation time burden of using the MC simulations.

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