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Strong Convergence Theorems of Iterative Algorithm for Nonconvex Variational Inequalities[1]

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Abstract : In this work, we suggest and analyze an iterative scheme for solving the system of nonconvex variational inequalities by using projection technique. We prove strong convergence of iterative scheme to the solution of the system of nonconvex variational inequalities requires to the modified mapping T which is Lipschitz continuous but not strongly monotone mapping. Our result can be viewed and improvement the result of N. Petrot [1].

Keywords : Lipschitz continuous; strongly monotone mapping; nonconvex; uniformly prox-regular.
2010 Mathematics Subject Classification : 47H09; 47H10.

1 Introduction

The theory of variational inequalities is a branch of the mathematical sciences dealing with general equilibrium problems. It has a wide range of applications in economics, operations research, industry, physical, and engineering sciences. Many research papers have been written lately, both on the theory and applications of

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this field. Important connection with main areas of pure and applied science have been made, see for example [2, 3, 4] and the references cited therein.

Variational inequalities theory, which was introduce by Stampacchia [5], provides us with a simple, natural general and unified framework to study a wide class of problems arising in pure and applied science. The development of variational inequality theory can be viewed as the simultaneous pursuit of two different lines of research. On the one hand, it reveals the fundamental facts on the qualitative aspects of the solutions to important classes of problems. On the other hand, it also enables us to develop highly efficient and powerful new numerical methods for solving, for example, obstacle, unilateral, free, moving, and complex equilibrium problems.

In 2010, N. Petrot [1], introduced some existence theorems and provide the conditions for existence solutions of the variational inequalities problems in nonconvex setting and prove the strongly monotonic assumption of the mapping may not need for the existence of solutions.

In this work we consider the iterative scheme for modified mapping is Lipschitz continuous but not strongly monotone mapping and we can prove strong convergence of iterative to the solution of the system of nonconvex variational inequalities.

2 Preliminaries

Let C be a closed subset of a real Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ respectively. Let us recall the following well-known definitions and some auxiliary results of nonlinear convex analysis and nonsmooth analysis.

Definition 2.1. Let $u \in H$ be a point not lying in C. A point $v \in C$ is called a *closest point or a projection of* u *onto* C if $d_C(u) = \|u - v\|$ when d_C is a usual distance. The set of all such closest points is denoted by $P_C(u)$; that is,

$$P_C(u) = \{v \in C : d_C(u) = \|u - v\|\}. \tag{2.1}$$

Definition 2.2. Let C be a subset of H. The proximal normal cone to C at x is given by

$$N_C^P(x) = \{z \in H : \exists \rho > 0; x \in P_C(x + \rho z)\}. \tag{2.2}$$

The following characterization of $N_C^P(x)$ can be found in [6].

Lemma 2.3. *Let* C *be a closed subset of a Hilbert space* H. *Then*

$$z \in N_C^P(x) \text{ if and only if } \exists \sigma > 0, \langle z, y - x \rangle \leq \sigma \|y - x\|^2, \quad \forall y \in C. \tag{2.3}$$

Clark et al. [7] and Poliquin et al. [8] have introduced and studied a new class of nonconvex sets, which are called uniformly prox-regular sets. This class or uniformly prox-regular sets has played an important part in many nonconvex applications such as optimization, dynamic systems, and differential inclusions.

Definition 2.4. For a given $r \in (0, +\infty]$, a subset C of H is said to be *uniformly prox-regular with respect to* r if, for all $\overline{x} \in C$ and for all $0 \neq z \in N_C^P(x)$, one has

$$\langle \frac{z}{\|z\|}, x - \overline{x} \rangle \leq \frac{1}{2r}\|x - \overline{x}\|^2, \quad \forall x \in C. \tag{2.4}$$

It is well known that a closed subset of a Hilbert space is convex if and only if it is proximally smooth of radius $r > 0$. Thus, in Definition 2.4, in the case of $r = \infty$, the uniform r-prox-regularity C is equivalent to convexity of C. Then, it is clear that the class of uniformly prox-regular sets is sufficiently large to include the class p-convex sets, $C^{1,1}$ submanifolds (possibly with boundary) of H, the images under a $C^{1,1}$ diffeomorphism of convex sets, and many other nonconvex sets; see [7, 8].

Let C_r be a uniformly r-prox-regular(nonconvex) set. For given nonlinear mappings $T : C_r \to H$, we consider the problem of finding $x^*, y^* \in C_r$ such that

$$\begin{aligned} &\langle \rho T y^* + x^* - y^*, x - x^* \rangle \geq 0, \forall x \in C_r, \rho > 0 \\ &\langle \eta T x^* + y^* - x^*, x - y^* \rangle \geq 0, \forall x \in C_r, \eta > 0, \end{aligned} \tag{2.5}$$

which is called the *system of nonconvex variational inequalities.*

It is worth mentioning that if $x^* = y^* = u$ and $\rho = \eta$, then problem (2.5) is equivalent to finding $u \in C_r$ such that

$$\langle Tu, v - u \rangle \geq 0, \forall v \in C_r, \tag{2.6}$$

which is known as *nonconvex variational inequalities* introduced and studied by Bounkhel et. al. [9] and Noor [10, 11].

It is known that problem (2.6) is equivalent to finding $u \in C_r$ such that

$$0 \in Tu + N_{C_r}^P(u), \tag{2.7}$$

which $N_{C_r}^P(u)$ denote the normal cone of C_r at u. The problem (2.7) is called the *variational inclusion associated with nonconvex variational inequalities* (2.6).

Lemma 2.5 ([1])**.** *Let C be a nonempty closed subset of H, $r \in (0, +\infty]$ and set $C_r; = \{x \in H : d(x, C) < r\}$. If C is uniform r-uniformly prox-regular, then the following hold:*

(1) *for all $x \in C_r, P_C(x) \neq \emptyset$,*
(2) *for all $s \in (0, r), P_C$ is Lipschitz continuous with constant $t_s = \frac{r}{r-s}$ on C_s,*
(3) *the proximal normal cone is closed as a set-valued mapping.*

Let C be a closed subset of a real Hilbert space H. A mapping $T : C \to H$ is called $\gamma -$ *strongly monotone* if there exists a constant $\gamma > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq \gamma \|x - y\|^2, \tag{2.8}$$

for all $x, y \in C$. A mapping T is called $\mu -$ *Lipschitz* if there exists a constant $\mu > 0$ such that

$$\|Tx - Ty\| \leq \mu \|x - y\|, \tag{2.9}$$

for all $x, y \in C$.

Lemma 2.6. *In a real Hilbert space H, there holds the inequality*

1. $\|x+y\|^2 \leq \|x\|^2+2\langle y, x+y\rangle \quad x, y \in H$ *and* $\|x-y\|^2 = \|x\|^2-2\langle x, y\rangle+\|y\|^2$,
2. $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2, \forall t \in [0, 1]$.

3 Main Results

In this section we first establish the equivalent between the system of nonconvex variational inequalities (2.5) and the fixed point problem with the projection technique.

Lemma 3.1. *For given $x^*, y^* \in C_r$ is a solution of system of nonconvex variational inequalities (2.5), if and only if*

$$x^* = P_C[y^* - \rho Ty^*],$$
$$y^* = P_C[x^* - \eta Tx^*], \tag{3.1}$$

where P_C is the projection of H onto the uniformly prox-regular set C_r.

Proof. Let $x^*, y^* \in C_r$ be a solution of (2.5), from (2.7), for a constant $\rho > 0$, we have

$$0 \in \rho Ty^* + x^* - y^* + \rho N^P_{C_r}(x^*) = (I + \rho N^P_{C_r})(x^*) - [y^* - \rho Ty^*]$$

if and only if

$$x^* = (I + \rho N^P_{C_r})^{-1}[y^* - \rho Ty^*] = P_C[y^* - \rho Ty^*],$$

where we have used the well-known fact that $P_C = (I + \rho N^P_{C_r})^{-1}$.

Similarly, we obtain

$$y^* = P_C[x^* - \eta Tx^*].$$

This prove our assertions. □

algorithm 3.2. *For arbitrarily chosen initial points $x_0, y_0 \in C_r$, $T_1, T_2 : C \to H$ with $T = T_1 + T_2$, the sequence $\{x_n\}$ and $\{y_n\}$ in the following way:*

$$\begin{aligned} y_n &= P_C[x_n - \eta Tx_n], \eta > 0 \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n P_C[y_n - \rho Ty_n], \rho > 0, \end{aligned} \tag{3.2}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$.

Remark 3.3 ([1])**.** *Let C be a uniformly r-prox-regular closed subset of a Hilbert space H, and let $T_1, T_2 : C \to H$ be such that T_1 is a μ_1-Lipschitz continuous and γ-strongly monotone mapping, T_2 is a μ_2-Lipschitz continuous mapping. Let*

$$\xi = r[\mu_1^2 - \gamma\frac{\mu_2 - \sqrt{(\mu_1^2 - \gamma\mu_2)^2 - \mu_1^2(\gamma - \mu_2)^2}}{\mu_1^2}] \tag{3.3}$$

then for each $s \in (0, \xi)$, *we have*

$$\gamma t_s - \mu_2 > \sqrt{(\mu_1^2 - \mu_2^2)(t_s^2 - 1)}, \tag{3.4}$$

where $t_s = \frac{r}{r-s}$.

In this paper, we may assume that $M^{\rho,\eta}\delta_{T(C)} < \xi$, *we see that for any* $s \in (M^{\rho,\eta}\delta_{T(C)}, \xi)$ *it satisfy the inequality (3.4) too. where* $M^{\rho,\eta} = \min\{\rho, \eta\}, \delta_{T(C)} = \sup\{\|u - v\| : u, v \in T(C)\}$.

Now, we suggest and analyze the following explicit projection method (3.2) for solving the system of nonconvex variational inequalities (2.5). Thus, from now on, without loss of generality we will always assume that $\mu_2 < \mu_1$.

Theorem 3.4. *Let* C *be a uniformly* r*-prox-regular closed subset of a Hilbert space* H, *and let* $T_1, T_2 : C \to H$ *be such that* T_1 *is a* μ_1*-Lipschitz continuous and* γ*-strongly monotone mapping,* T_2 *is a* μ_2*-Lipschitz continuous mapping. If* $T = T_1 + T_2$ *and there exists constant* $\rho, \eta > 0$ *and* $s \in (M^{\rho,\eta}\delta_{T(C)}, \xi)$, *such that*

$$\frac{\gamma t_s - \mu_2}{t_s(\mu_1^2 - \mu_2^2)} - \triangle_{t_s} < \rho, \eta < \min\{\frac{\gamma t_s - \mu_2}{t_s(\mu_1^2 - \mu_2^2)} + \triangle_{t_s}, \frac{1}{t_s\mu_2}\}, \tag{3.5}$$

where $\triangle_{t_s} = \frac{\sqrt{(\gamma t_s - \mu_2)^2 - (\mu_1^2 - \mu_2^2)(t_s^2 - 1)}}{t_s(\mu_1^2 - \mu_2^2)}$. *If the sequence of positive real number* $\alpha_n \in [0, 1]$ *with* $\Sigma_{n=0}^{\infty}\alpha_n = \infty$, *then the sequences* $\{x_n\}$ *and* $\{y_n\}$ *obtained from Algorithm 3.2 converge to a solution of the system of nonconvex variational inequalities (2.5).*

Proof. Let $x^*, y^* \in C_r$ be a solution of (2.5) and from Lemma 3.1, we have

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|(1 - \alpha_n)x_n + \alpha_n P_C[y_n - \rho T y_n] - x^*\| \\ &= \|(1 - \alpha_n)(x_n - x^*) + \alpha_n(P_C[y_n - \rho T y_n] - P_C[y^* - \rho T y^*])\| \\ &\le (1 - \alpha_n)\|x_n - x^*\| + \alpha_n\|P_C[y_n - \rho T y_n] - P_C[y^* - \rho T y^*]\| \\ &\le (1 - \alpha_n)\|x_n - x^*\| + \alpha_n t_s\|(y_n - \rho T y_n) - (y^* - \rho T y^*)\| \\ &\le (1 - \alpha_n)\|x_n - x^*\| + \alpha_n t_s[\|(y_n - y^*) - \rho(T_1 y_n - T_1 y^*)\| \\ &\quad + \rho\|(T_2 y_n - T_2 y^*)\|.] \end{aligned} \tag{3.6}$$

From T_1 are both μ_1-Lipschitz continuous and γ-strongly monotone mapping and from Lemma 2.6, we obtain

$$\begin{aligned} \|(y_n - y^*) - \rho(T_1 y_n - T_1 y^*)\|^2 &= \|y_n - y^*\|^2 - 2\rho\langle y_n - y^*, T_1 y_n - T_1 y^*\rangle \\ &\quad + \rho^2\|T_1 y_n - T_1 y^*\|^2 \\ &\le \|y_n - y^*\|^2 - 2\rho\gamma\|y_n - y^*\|^2 + \rho^2\mu_1^2\|y_n - y^*\|^2 \\ &= (1 - 2\rho\gamma + \rho^2\mu_1^2)\|y_n - y^*\|^2. \end{aligned}$$

It follows that

$$\|(y_n - y^*) - \rho(T_1 y_n - T_1 y^*)\| \le \sqrt{1 - 2\rho\gamma + \rho^2\mu_1^2}\|y_n - y^*\|. \tag{3.7}$$

On the other hand, from T_2 is μ_2-Lipschitz continuous, we have

$$\|T_2 y_n - T_2 y^*\| \le \mu_2 \|y_n - y^*\|. \tag{3.8}$$

Thus, by (3.6), (3.7) and (3.8), we have

$$\|x_{n+1} - x^*\| \le (1-\alpha_n)\|x_n - x^*\| + \alpha_n t_s(\rho\mu_2 + \sqrt{1 - 2\rho\gamma + \rho^2\mu_1^2})\|y_n - y^*\|. \tag{3.9}$$

Similarly, we have

$$\begin{aligned}
\|y_n - y^*\| &= \|P_C[x_n - \eta T x_n] - y^*\| \\
&= \|P_C[x_n - \eta T x_n] - P_C[x^* - \eta T x^*]\| \\
&\le t_s\|(x_n - \eta T x_n) - (x^* - \eta x^*)\| \\
&\le t_s[\|(x_n - x^*) - \eta(T_1 x_n - T_1 x^*)\| + \eta\|T_2 x_n - T_2 x^*\|].
\end{aligned} \tag{3.10}$$

Similarly, from T_1 are both μ_1-Lipschitz continuous and γ-strongly monotone mapping, we obtain

$$\begin{aligned}
\|(x_n - x^*) - \eta(T_1 x_n - T_1 x^*)\|^2 &= \|x_n - x^*\|^2 - 2\eta\langle x_n - x^*, T_1 x_n - T_1 x^*\rangle \\
&\quad + \eta^2\|T_1 x_n - T_1 x^*\|^2 \\
&\le \|x_n - x^*\|^2 - 2\eta\gamma\|x_n - x^*\|^2 + \eta^2\mu_1^2\|x_n - x^*\|^2 \\
&= (1 - 2\eta\gamma + \eta^2\mu_1^2)\|x_n - x^*\|^2.
\end{aligned}$$

It follows that

$$\|(x_n - x^*) - \eta(T_1 x_n - T_1 x^*)\| \le \sqrt{1 - 2\eta\gamma + \eta^2\mu_1^2}\|x_n - x^*\|. \tag{3.11}$$

On the other hand, from T_2 is μ_2-Lipschitz continuous, we have

$$\|T_2 x_n - T_2 x^*\| \le \mu_2\|x_n - x^*\|. \tag{3.12}$$

Thus, by (3.10), (3.11) and (3.12), we have

$$\|y_n - y^*\| \le t_s(\eta\mu_2 + \sqrt{1 - 2\eta\gamma + \eta^2\mu_1^2})\|x_n - x^*\|. \tag{3.13}$$

Moreover, from (3.9) and (3.13) we put $\theta_1 = t_s(\rho\mu_2 + \sqrt{1 - 2\rho\gamma + \rho^2\mu_1^2}), \theta_2 = t_s(\eta\mu_2 + \sqrt{1 - 2\eta\gamma + \eta^2\mu_1^2})$, it follows that

$$\begin{aligned}
\|x_{n+1} - x^*\| &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\theta_1\theta_2\|x_n - x^*\| \\
&= (1 - (1 - \theta_1\theta_2)\alpha_n)\|x_n - x^*\| \\
&\le \prod_{i=0}^{n}(1 - (1 - \theta_1\theta_2)\alpha_i)\|x_0 - x^*\|.
\end{aligned} \tag{3.14}$$

Since $\Sigma_{n=0}^{\infty}\alpha_n = \infty$ and conditions (3.5), we obtain

$$\lim_{n\to\infty}\prod_{i=0}^{n}(1 - (1 - \theta_1\theta_2)\alpha_i) = 0. \tag{3.15}$$

It follows from (3.15) and (3.14), we have

$$\lim_{n\to\infty} \|x_n - x^*\| = 0. \tag{3.16}$$

From (3.13) and (3.16), we have

$$\lim_{n\to\infty} \|y_n - y^*\| = 0. \tag{3.17}$$

Which is $x^*, y^* \in C_r$ satisfying the system of nonconvex variational inequalities (2.5). This completes the proof. □

Corollary 3.5. *Let C be a uniformly r-prox-regular closed subset of a Hilbert space H, and let $T : C \to H$ be such that T is a μ-Lipschitz continuous and γ-strongly monotone mapping. If there exists constant $\rho, \eta > 0$ and $s \in (M^{\rho,\eta}\delta_{T(C)}, \xi)$, such that*

$$\frac{\gamma}{\mu^2} - \triangle_{t_s} < \rho, \eta < \frac{\gamma}{\mu^2} + \triangle_{t_s}, \tag{3.18}$$

where $\triangle_{t_s} = \frac{\sqrt{(\gamma t_s)^2-(\mu_1^2)(t_s^2-1)}}{t_s(\mu_1^2)}$. If the sequence of positive real number $\alpha_n \in [0,1]$ with $\Sigma_{n=0}^{\infty}\alpha_n = \infty$, and $\alpha_n \in [0,1]$ with $\Sigma_{n=0}^{\infty}\alpha_n = \infty$, then the sequence $\{x_n\}$ and $\{y_n\}$ is generated by for $x_0, y_0 \in C_r$,

$$\begin{aligned} y_n &= P_C[x_n - \eta T x_n], \eta > 0 \\ x_{n+1} &= P_C[y_n - \rho T y_n], \rho > 0, \end{aligned} \tag{3.19}$$

strongly converge to a solution of the system of nonconvex variational inequalities (2.5).

Proof. From Theorem 3.4, if $T_2 \equiv 0$ and $\alpha_n = 1$ for any $n \geq 0$, we have a result. □

4 Applications

In this section, we can applied Theorem 3.4 to the system of general of nonconvex variational inequalities, for given nonlinear mappings $T, g : C_r \to H$, we consider the problem of finding $x^*, y^* \in C_r$ such that

$$\begin{aligned} &\langle \rho T g(y^*) + g(x^*) - g(y^*), x - g(x^*)\rangle \geq 0, \forall x \in C_r, \rho > 0 \\ &\langle \eta T g(x^*) + g(y^*) - g(x^*), x - g(y^*)\rangle \geq 0, \forall x \in C_r, \eta > 0, \end{aligned} \tag{4.1}$$

which is called the *system of general nonconvex variational inequalities*. Similar of the proof of Lemma 3.1, we can proof that

Lemma 4.1. *For given $x^*, y^* \in C_r$ is a solution of system of nonconvex variational inequalities (4.1), if and only if*

$$g(x^*) = P_C[g(y^*) - \rho T g(y^*)],$$
$$g(y^*) = P_C[g(x^*) - \eta T g(x^*)], \tag{4.2}$$

where P_C is the projection of H onto the uniformly prox-regular set C_r.

Theorem 4.2. *Let C be a uniformly r-prox-regular closed subset of a Hilbert space H, let $g : C \to H$ is injective mapping and let $T_1, T_2 : C \to H$ be such that T_1 is a μ_1-Lipschitz continuous and γ-strongly monotone mapping, T_2 is a μ_2-Lipschitz continuous mapping. If $T = T_1 + T_2$ and there exists constant $\rho, \eta > 0$ and $s \in (M^{\rho,\eta}\delta_{T(C)}, \xi)$, such that*

$$\frac{\gamma t_s - \mu_2}{t_s(\mu_1^2 - \mu_2^2)} - \triangle_{t_s} < \rho, \eta < \min\{\frac{\gamma t_s - \mu_2}{t_s(\mu_1^2 - \mu_2^2)} + \triangle_{t_s}, \frac{1}{t_s \mu_2}\}, \tag{4.3}$$

where $\triangle_{t_s} = \frac{\sqrt{(\gamma t_s - \mu_2)^2 - (\mu_1^2 - \mu_2^2)(t_s^2 - 1)}}{t_s(\mu_1^2 - \mu_2^2)}$. If the sequence of positive real number $\alpha_n \in [0,1]$ with $\Sigma_{n=0}^{\infty}\alpha_n = \infty$, then the sequence $\{x_n\}$ and $\{y_n\}$ is generated by for $x_0, y_0 \in C_r$,

$$\begin{aligned} g(y_n) &= P_C[g(x_n) - \eta T g(x_n)], \eta > 0 \\ g(x_{n+1}) &= (1 - \alpha_n) g(x_n) + \alpha_n P_C[g(y_n) - \rho T g(y_n)], \rho > 0, \end{aligned} \tag{4.4}$$

strongly converge to a solution of the system of nonconvex variational inequalities (4.1).

Proof. Similar the proof in Theorem 3.4, let $x^*, y^* \in C_r$ be a solution of (4.1) and from Lemma 4.1, we can compute that

$$\|g(x_{n+1}) - g(x^*)\| \le \prod_{i=0}^{n}(1 - (1 - \theta_1\theta_2)\alpha_i)\|g(x_0) - g(x^*)\|. \tag{4.5}$$

where $\theta_1 = t_s(\rho\mu_2 + \sqrt{1 - 2\rho\gamma + \rho^2\mu_1^2})$, $\theta_2 = t_s(\eta\mu_2 + \sqrt{1 - 2\eta\gamma + \eta^2\mu_1^2}$. From $\Sigma_{n=0}^{\infty}\alpha_n = \infty$ and conditions (4.3), we obtain

$$\lim_{n\to\infty} \prod_{i=0}^{n}(1 - (1 - \theta_1\theta_2)\alpha_i) = 0. \tag{4.6}$$

It follows from (4.5) and (4.6), we have

$$\lim_{n\to\infty} \|g(x_n) - g(x^*)\| = 0. \tag{4.7}$$

And we can compute that

$$\|g(y_n) - g(y^*)\| \le \theta_2\|g(x_n) - g(x^*)\|, \tag{4.8}$$

where $\theta_2 = t_s(\eta\mu_2 + \sqrt{1 - 2\eta\gamma + \eta^2\mu_1^2})$, it follows that

$$\lim_{n\to\infty} \|g(y_n) - g(y^*)\| = 0. \tag{4.9}$$

From g is injective mapping, we have $\lim_{n\to\infty} \|x_n - x^*\| = 0$ and $\lim_{n\to\infty} \|y_n - y^*\| = 0$ satisfying the system of general nonconvex variational inequalities (4.1). This complete the proof. □

Corollary 4.3. *Let C be a uniformly r-prox-regular closed subset of a Hilbert space H, let $g : C \to H$ is injective mapping and let $T : C \to H$ be such that T is a μ-Lipschitz continuous and γ-strongly monotone mapping. If there exists constant $\rho, \eta > 0$ and $s \in (M^{\rho,\eta}\delta_{T(C)}, \xi)$, such that*

$$\frac{\gamma}{\mu^2} - \triangle_{t_s} < \rho, \eta < \frac{\gamma}{\mu^2} + \triangle_{t_s}, \tag{4.10}$$

where $\triangle_{t_s} = \frac{\sqrt{(\gamma t_s)^2 - (\mu_1^2)(t_s^2 - 1)}}{t_s(\mu_1^2)}$. If the sequence of positive real number $\alpha_n \in [0,1]$ with $\Sigma_{n=0}^{\infty}\alpha_n = 0$, and $\alpha_n \in [0,1]$ with $\Sigma_{n=0}^{\infty}\alpha_n = \infty$, then the sequence $\{x_n\}$ and $\{y_n\}$ is generated by for $x_0, y_0 \in C_r$,

$$\begin{aligned} g(y_n) &= P_C[g(x_n) - \eta T g(x_n)], \eta > 0 \\ g(x_{n+1}) &= P_C[g(y_n) - \rho T g(y_n)], \rho > 0, \end{aligned} \tag{4.11}$$

strongly converge to a solution of the system of nonconvex variational inequalities (4.1).

Proof. From Theorem 3.4, if $T_2 \equiv 0$ and $\alpha_n = 1$ for any $n \geq 0$, we have a result. □

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