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Statistically Convergent Difference Double Sequence Spaces Defined By Orlicz Functions

Ayhan Esi† **and Bipan Hazarika**‡,1

†Adıyaman University, Department of Mathematics, 02040
Adıyaman, Turkey
e-mail : aesi23@adiyaman.edu.tr

‡Department of Mathematics,Rajiv Gandhi University, Rono Hills
Doimukh-791 112, Arunachal Pradesh, India
e-mail : bh_rgu@yahoo.co.in

Abstract : In this paper we define statistically convergent, statistically null and statistically bounded generalized difference double sequence spaces on a seminenormed space via Orlicz functions. We study their different properties like solidness, denseness, symmetricitiy, completeness etc. Also we obtain some inclusion relations among them.

Keywords : Orlicz function; statistical convergence; P-convergent; solid space; complete space.
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1 Introduction

The notion of statistical convergence for single sequences was introduced by Fast [1] and Schoenberg [2] independently. Later on it was studied by Fridy and Orhan [3], Maddox [4], Salat [5], Tripathy [6, 7], Tripathy and Sen [8], Esi [9, 10] and many others. Any concept involving statistical convergence plays a vital role not only in pure mathematics but also in other branches of mathematics especially

[1]Corresponding author.

in information theory, computer science and biological science:

Example 1.1. Let n be a positive integer. In a group of n people, each person selects independently and at random one of three subgroups to which to belong, resulting in three groups with random numbers n_1, n_2, n_3 of members; $n_1 + n_2 + n_3 = n$. Each of the subgroups is then partitioned independently in the same manner to form three sub subgroups, and so forth. Subgroups having no members or having only one member are removed from the process. Denote by t_n the expected value of the number of iterations up to complete removal, starting initially with a group of n people. Then the sequence $(t_1, \frac{t_2}{2}, \frac{t_3}{3}, ..., \frac{t_k}{k}, ...)$ is a bounded non-convergent statistically convergent sequence.

Example 1.2. Let n be a positive integer. In a group of n people, each person selects at random and simultaneously another person of the group. All of the selected persons are then removed from the group, leaving a random number $n_1 < n$ of people which form a new group. The new group then repeats independently the selection and removal thus described, leaving $n_2 < n_1$ persons, and so forth until either one person remains, or no persons remain. Denote by p_n the probability that, at the end of this iteration initiated with a group of n persons, one person remains. Then the sequence $\mathbf{p}= (p_1, p_2, .., p_n, ...)$ is a statistically convergent sequence, and $\lim_n p_n$ does not exist.

Example 1.3. Statistical convergence can be used in various fields of science and engineering, e.g. computer programming, nonlinear dynamical systems, population dynamics, control of chaos, quantum physics, geoinformation systems and science(GIS), remote sensing, etc.

Example 1.4. Statistical convergence can be used in traffic control, speech analysis, bioinformatics, DNA analysis.

The notion of statistical convergence depends on the density (natural or asymptotic) of subsets of $\mathbb{N}$. A subset E of $\mathbb{N}$ is said to have natural density $\delta(E)$ if

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k)$$

exists, where χ_E denotes the characteristic function of E.

A single sequence $x = (x_k)$ is said to be *statistically convergent* to a number L if for every $\varepsilon > 0$

$$\delta\left(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}\right) = 0.$$

Throughout the paper a double sequence will be denoted by $x = (x_{k,l})$ i.e. a double infinite array to elements $x_{k,l}$ for all $k, l \in \mathbb{N}$. The notion of statistical convergence for double sequences was introduced by Tripathy [6] and Mursaleen and Edely [11] independently. Tripathy [6] introduced the notion of $\mathbb{N}\times\mathbb{N}$ as follows: A subset E of $\mathbb{N}\times\mathbb{N}$ is said to have *density* $\rho(E)$ if

$$\rho(E)=\lim_{p,q\to\infty,\infty}\frac{1}{pq}\sum_{n\leq p}\sum_{k\leq q}\chi_E(n,k) \text{ exists.}$$

Thoughout the paper (X,q) will represent a seminormed space seminormed by q.

An Orlicz function M is a mapping $M:[0,\infty)\to[0,\infty)$ such that it is continuous, non-decreasing and convex with $M(0)=0$, $M(x)>0$, for $x>0$ and $M(x)\to\infty$ as $x\to\infty$. If convexity of Orlicz function is replaced by $M(x+y)\leq M(x)+M(y)$ then this function is called *the modulus function* and characterized by Ruckle [12]. An Orlicz function M is said to satisfy $\Delta_2-condition$ if there exists a constant $K>0$ such that $M(2u)\leq KM(u)$, $for\ all\ values\ u\geq 0$. The notion of Orlicz function was used to defined sequence spaces by Lindenstrauss and Tzafriri [13], Parashar and Choudhary [14], Esi [10, 15], Esi and Et [16] and many others.

Kizmaz [17] introduced the notion of difference sequence spaces as follows:

$$X(\Delta)=\{x=(x_k):\ (\Delta x_k)\in X\}$$

for $X=l_\infty, c$ and c_o. Later on, the notion was generalized by Et and Çolak [18] as follows:

$$X(\Delta^m)=\{x=(x_k):\ (\Delta^m x_k)\in X\}$$

for $X=l_\infty, c$ and c_o, where $\Delta^m x=(\Delta^m x_k)=\left(\Delta^{m-1}x_k-\Delta^{m-1}x_{k+1}\right)$, $\Delta^0 x=x$ and also this generalized difference notion has the following binomial representation:

$$\Delta^m x_k=\sum_{i=0}^{m}(-1)^i\binom{m}{i}x_{k+i} \quad \text{for all } k\in\mathbb{N}.$$

Subsequently, difference sequence spaces studied by several authors like as Esi [19], Tripathy et.al [20], Esi and Tripathy [21] and many others.

For details on double sequence spaces we refer to Gokhan and Colak ([22], [23], [24]). A double sequence $x=(x_{k,l})$ is said to be *convergent to a number* L *in the Pringsheim sense or* $P-convergent$ (see [25]) if for every $\varepsilon>0$ there exists $N\in\mathbb{N}$ such that $|x_{k,l}-L|<\varepsilon$ for $k,l>N$. We shall denote the space of all $P-convergent$ sequences by c^2. The double sequence $x=(x_{k,l})$ is bounded if and only if there exists a positive number N such that $|x_{k,l}|<N$ for all k and l. We shall denote the space of all bounded double sequences by l_∞^2.

Tripathy and Sarma [26] introduced the following difference sequence spaces over the seminormed space (X,q):

$$Z(\Delta,q)=\{x=(x_{k,l}):\ (\Delta x_{k,l})\in Z(q)\},$$

where $Z = l^2_\infty, \overline{c}^2, \overline{c}^{2B}, \overline{c}^{2R}, \overline{c}^{2BR}, \overline{c}^2_o, \overline{c}^{2B}_o, \overline{c}^{2R}_o, \overline{c}^{2BR}_o$ and $\Delta x_{k,l} = x_{k,l} - x_{k+1,l} - x_{k,l+1} + x_{k+1,l+1}$ for all $k, l \in \mathbb{N}$.

2 Definitions and Preliminaries

A double sequence $x = (x_{k,l})$ is said to be *convergence* in Pringsheim's sense to L if

$$\lim_{k,l\to\infty} x_{k,l} = L,$$

where k and l tend to infinity independent of each other. A double sequence $x = (x_{k,l})$ is said to be converge regularly if it converges in Pringsheim's sense and in addition the following limits exist:

$$\lim_{k\to\infty} x_{k,l} = L_l, \ \ (l = 1, 2, 3, ...)$$

and

$$\lim_{l\to\infty} x_{k,l} = L_k, \ \ (k = 1, 2, 3, ...),$$

Throughout the paper $l^2_\infty(\Delta^r, q), c^2(\Delta^r, q), c^2_o(\Delta^r, q), c^{2,R}(\Delta^r, q), c^{2,R}_o(\Delta^r, q)$ will denote the spaces of bounded, convergent difference sequences in Pringheim's sense, null difference in Pringheim's sense, regularly convergent difference, regularly null difference X-valued double sequence spaces, respectively.

An X-valued double sequence $x = (x_{k,l})$ is said to be *statistically difference convergent to* L if for every $\varepsilon > 0$

$$\delta\left(\{(k, l) \in \mathbb{N} \times \mathbb{N} : q\left(\Delta^r x_{k,l} - L\right) \geq \varepsilon\}\right) = 0.$$

An X-valued double sequence $x = (x_{k,l})$ is said to be *statistically difference regularly convergent to* L if it converges in Pringsheim's sense and the following statistical limits exist:

$$stat - \lim_{k\to\infty} \Delta^r x_{k,l} = L_l, \ \ (l = 1, 2, 3, ...)$$

and

$$stat - \lim_{l\to\infty} \Delta^r x_{k,l} = L_k, \ \ (k = 1, 2, 3, ...).$$

An X-valued double sequence $x = (x_{k,l})$ is said to be *statistically difference bounded* if there exists $G > 0$ such that

$$\delta\left(\{(k, l) \in \mathbb{N} \times \mathbb{N} : q\left(\Delta^r x_{k,l} - L\right) > G\}\right) = 0.$$

A double sequence space E is said to be *solid (or normal)* if $(\alpha_{k,l} x_{k,l}) \in E$ whenever $(x_{k,l}) \in E$ for all double sequences $(\alpha_{k,l})$ of scalars with $|\alpha_{k,l}| \leq 1$ for all $k, l \in \mathbb{N}$.

A double sequence space E is said to be *symmetric* if $(x_{k,l}) \in E$ implies $(x_{\pi(k),\pi(l)}) \in E$, where π is a permitation on $\mathbb{N}$.

A double sequence space E is said to be *monotone* if it contains the canonical preimages of all its step spaces.

Remark 2.1. *It is a well known result that if E is normal then it is monotone.*

In this presentation our goal is to extend a few results known in the literature from ordinary (single) difference sequences to difference double sequences.

Let M be an Orlicz function and $p = (p_{k,l})$ be a factorable double sequence of positive real numbers and let X be a seminormed space over the complex field $\mathbb{C}$ with the seminorm q. We now define the following new statistically convergent generalized difference double sequence spaces:

$$l_\infty^2 (M, \Delta^r, q) = \left\{ x = (x_{k,l}) : \sup_{k,l} M \left(\frac{q (\Delta^r x_{k,l})}{\rho} \right) < \infty, \text{ for some } \rho > 0 \right\},$$

$$\overline{c}^2 (M, \Delta^r, q) =$$
$$\left\{ x = (x_{k,l}) : stat - \lim_{k,l \to \infty} M \left(\frac{q (\Delta^r x_{k,l} - L)}{\rho} \right) = 0, \text{ for some } \rho > 0 \text{ and } L \in X \right\}$$

and

$$\overline{c}_o^2 (M, \Delta^r, q) = \left\{ x = (x_{k,l}) : stat - \lim_{k,l \to \infty} M \left(\frac{q (\Delta^r x_{k,l})}{\rho} \right) = 0, \text{ for some } \rho > 0 \right\},$$

where $\Delta^r x = (\Delta^r x_{k,l}) = \left(\Delta^{r-1} x_{k,l} - \Delta^{r-1} x_{k,l+1} - \Delta^{r-1} x_{k+1,l} + \Delta^{r-1} x_{k+1,l+1}\right)$, $\left(\Delta^1 x_{k,l}\right) = (\Delta x_{k,l}) = (x_{k,l} - x_{k,l+1} - x_{k+1,l} + x_{k+1,l+1})$, $\Delta^0 x = (x_{k,l})$ and also this generalized difference double notion has the following binomial representation:

$$\Delta^r x_{k,l} = \sum_{i=0}^{r} \sum_{j=0}^{r} (-1)^{i+j} \binom{r}{i} \binom{r}{j} x_{k+i,l+j}.$$

A double sequence $x = (x_{k,l}) \in \overline{c}^{2,R} (M, \Delta^r, q)$ if $x = (x_{k,l}) \in \overline{c}^2 (M, \Delta^r, q)$ then the following statistical limits exist:

$$stat - \lim_{k \to \infty} M \left(\frac{q (\Delta^r x_{k,l} - L_l)}{\rho} \right) = 0, \quad (l = 1, 2, 3, ...) \tag{2.1}$$

and

$$stat - \lim_{l \to \infty} M \left(\frac{q (\Delta^r x_{k,l} - L_k)}{\rho} \right) = 0, \quad (k = 1, 2, 3, ...) . \tag{2.2}$$

A double sequence $x = (x_{k,l}) \in \overline{c}_o^{2,R} (M, \Delta^r, q)$ if $x = (x_{k,l}) \in \overline{c}_o^2 (M, \Delta^r, q)$ and (2.1) and (2.2) hold with $L_l = L_k = \theta$, the zero element of X for all $k, l \in \mathbb{N}$.

Throughout the paper $e = (1,1,1,...)$ and $e_k = (0,0,0,...,0,1,0,...)$ where only 1 appears at the k-th place and $\overline{c}^2(M,\Delta^r,q)$, $\overline{c}_o^2(M,\Delta^r,q)$, $\overline{c}^{2,B}(M,\Delta^r,q)$, $\overline{c}_o^{2,B}(M,\Delta^r,q)$, $\overline{c}^{2,R}(M,\Delta^r,q)$, $\overline{c}_o^{2,R}(M,\Delta^r,q)$, $\overline{c}^{2,BR}(M,\Delta^r,q)$, $\overline{c}_o^{2,BR}(M,\Delta^r,q)$ denote the spaces of statistically difference convergent in Pringsheim's sense, statistically difference null in Pringsheim's sense, bounded statistically difference convergent in Prinsgheim's sense, bounded statistically difference null in Pringsheim's sense,regularly statistically difference convergent, regularly statistically null, bounded regularly statistically convergent, bounded regularly statistically nul X-valued double sequences defined by Orlicz function, respectively.

3 Main Results

Theorem 3.1. *The classes* $Z(M,\Delta^r,q)$, *where* $Z = l_\infty^2$, $\overline{c}^2$, $\overline{c}_o^2$, $\overline{c}^{2,R}$, $\overline{c}_o^{2,R}$, $\overline{c}^{2,B,R}$, $\overline{c}_o^{2,B,R}$ *are linear spaces over the complex field* $\mathbb{C}$.

Proof. We give the proof for the space $Z = l_\infty^2$ and proof of the other spaces can be obtained in a similar way. Let $x = (x_{k,l}), y = (y_{k,l}) \in l_\infty^2(M,\Delta^r,q)$. Then we have

$$\sup_{k,l} M\left(\frac{q(\Delta^r x_{k,l})}{\rho_1}\right) < \infty, \text{ for some } \rho_1 > 0 \tag{3.1}$$

and

$$\sup_{k,l} M\left(\frac{q(\Delta^r y_{k,l})}{\rho_2}\right) < \infty, \text{ for some } \rho_2 > 0. \tag{3.2}$$

Let $\alpha,\beta \in \mathbb{C}$ be scalars and $\rho = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since M is non-decreasing convex function by using inequalities (3.1) and (3.2) we have

$$M\left(\frac{q(\Delta^r(\alpha x_{k,l}+\beta y_{k,l}))}{\rho}\right)$$

$$\leq M\left(\frac{q(\alpha\Delta^r x_{k,l})}{\rho}\right) + M\left(\frac{q(\beta\Delta^r y_{k,l})}{\rho}\right)$$

$$\leq \frac{1}{2}M\left(\frac{q(\Delta^r x_{k,l})}{\rho_1}\right) + \frac{1}{2}M\left(\frac{q(\Delta^r y_{k,l})}{\rho_2}\right) < M\left(\frac{q(\Delta^r x_{k,l})}{\rho_1}\right) + M\left(\frac{q(\Delta^r y_{k,l})}{\rho_2}\right).$$

Thus

$$\sup_{k,l} M\left(\frac{q(\Delta^r(\alpha x_{k,l}+\beta y_{k,l}))}{\rho}\right) < \infty.$$

Therefore $\alpha x + \beta y \in l_\infty^2(M,\Delta^r,q)$. Hence $l_\infty^2(M,\Delta^r,q)$ is a linear space. □

Theorem 3.2. *The spaces* $Z(M,\Delta^r,q)$, *where* $Z = l_\infty^2$, $\overline{c}^2$, $\overline{c}_o^2$, $\overline{c}^{2,R}$, $\overline{c}_o^{2,R}$, $\overline{c}^{2,B}$, $\overline{c}_o^{2,B}$, $\overline{c}^{2,BR}$, $\overline{c}_o^{2,BR}$ *are seminormed spaces, seminormed by*

$$f((x_{k,l})) = \sum_{k=1}^{n} q(x_{k,1}) + \sum_{l=1}^{n} q(x_{1,l}) + \inf\left\{\rho > 0 : \sup_{k,l} M\left(\frac{q(\Delta^n x_{k,l})}{\rho}\right) \leq 1\right\}.$$

Proof. Since q is a seminorm, so we have $f((x_{k,l})) \geq 0$ for all $x = (x_{k,l})$; $f(\theta) = 0$ and $f((\lambda x_{k,l})) = |\lambda| f((x_{k,l}))$ for all scalars λ.

Now, let $x = (x_{k,l})$, $y = (y_{k,l}) \in \overline{c}^{2,B}(M, \Delta^r, q)$. Then there exist $\rho_1, \rho_2 > 0$ such that

$$\sup_{k,l} M\left(\frac{q(\Delta^n x_{k,l})}{\rho_1}\right) \leq 1 \text{ and } \sup_{k,l} M\left(\frac{q(\Delta^n y_{k,l})}{\rho_2}\right) \leq 1.$$

Let $\rho = \rho_1 + \rho_2$. Then we have

$$\sup_{k,l} M\left(\frac{q(\Delta^n (x_{k,l} + y_{k,l}))}{\rho}\right)$$

$$\leq \left(\frac{\rho_1}{\rho_1+\rho_2}\right) \sup_{k,l} M\left(\frac{q(\Delta^n x_{k,l})}{\rho_1}\right) + \left(\frac{\rho_2}{\rho_1+\rho_2}\right) \sup_{k,l} M\left(\frac{q(\Delta^n y_{k,l})}{\rho_2}\right) \leq 1.$$

Since ρ_1,$\rho_2 > 0$, so we have

$$\begin{aligned} f((x_{k,l}) + (y_{k,l})) &= \sum_{k=1}^{n} q(x_{k,1} + y_{k,1}) + \sum_{l=1}^{n} q(x_{1,l} + y_{1,l}) \\ &\quad + \inf\left\{\rho > 0 : \sup_{k,l} M\left(\frac{q(\Delta^n (x_{k,l} + y_{k,l}))}{\rho}\right) \leq 1\right\} \end{aligned}$$

$$\begin{aligned} &\leq \sum_{k=1}^{n} q(x_{k,1}) + \sum_{l=1}^{n} q(x_{1,l}) + \inf\left\{\rho_1 > 0 : \sup_{k,l} M\left(\frac{q(\Delta^n x_{k,l})}{\rho_1}\right) \leq 1\right\} \\ &+ \sum_{k=1}^{n} q(y_{k,1}) + \sum_{l=1}^{n} q(y_{1,l}) + \inf\left\{\rho_2 > 0 : \sup_{k,l} M\left(\frac{q(\Delta^r y_{k,l})}{\rho_2}\right) \leq 1\right\} \\ &= f((x_{k,l})) + f((y_{k,l}))\,. \end{aligned}$$

Therefore f is a seminorm. □

Theorem 3.3. *Let (X, q) be a complete seminormed space. Then the spaces $Z(M, \Delta^r, q)$, where $Z = l_\infty^2, \overline{c}^{2,B}, \overline{c}_o^{2,B}, \overline{c}^{2,BR}, \overline{c}_o^{2,BR}$ are complete seminormed spaces seminormed by f.*

Proof. We prove the theorem for the space $\overline{c}^{2,B}(M, \Delta^r, q)$. The other cases can be establish following similar technique. Let $x^i = \left(x^i_{k,l}\right)$ be a Cauchy sequence in $\overline{c}^{2,B}(M, \Delta^r, q)$ .Let $\varepsilon > 0$ be given and for $b > 0$, choose x_o fixed such that $M\left(\frac{bx_o}{2}\right) \geq 1$ and there exists $m_o \in \mathbb{N}$ such that

$$f(x^i - x^j) = f\left(\left(x^i_{k,l}\right) - \left(x^j_{k,l}\right)\right) < \frac{\varepsilon}{bx_o} \text{ for all } i, j \geq m_o.$$

By definition of seminorm, we have

$$\begin{aligned}&\sum_{k=1}^{r} q\left(x_{k,1}^{i}-x_{k,1}^{j}\right)+\sum_{l=1}^{r} q\left(x_{1,l}^{i}-x_{1,l}^{j}\right)\\&\quad+\inf\left\{\rho>0:\ \sup_{k,l} M\left(\frac{q(\Delta^{r}x_{k,l}^{i}-\Delta^{r}x_{k,l}^{j})}{\rho}\right)\leq 1\right\}<\frac{\varepsilon}{bx_{o}}.\end{aligned} \tag{3.3}$$

This implies that

$$\sum_{k=1}^{r} q\left(x_{k,1}^{i}-x_{k,1}^{j}\right)+\sum_{l=1}^{r} q\left(x_{1,l}^{i}-x_{1,l}^{j}\right)<\varepsilon.$$

This shows that $\left(x_{k,1}^{i}\right)$ and $\left(x_{1,l}^{j}\right)$ $(k,l\leq n-r)$ are Cauchy sequences in (X,q). Since (X,q) is complete, so there exists $x_{k,1}, x_{1,l}\in X$ such that

$$\lim_{i\to\infty} x_{k,1}^{i}=x_{k,1} \text{ and } \lim_{j\to\infty} x_{1,l}^{j}=x_{1,l} \quad (k,l\leq n).$$

Now from (3.3) we have

$$M\left(\frac{q\left(\Delta^{r}\left(x_{k,l}^{i}-x_{k,l}^{j}\right)\right)}{f\left(\left(x_{k,l}^{i}\right)-\left(x_{k,l}^{j}\right)\right)}\right)\leq 1\leq M\left(\frac{bx_{o}}{2}\right) \text{ for all } i,j\geq m_{o}. \tag{3.4}$$

This implies

$$q\left(\Delta^{r}\left(x_{k,l}^{i}-x_{k,l}^{j}\right)\right)\leq \frac{bx_{o}}{2}.\frac{\varepsilon}{bx_{o}}=\frac{\varepsilon}{2} \text{ for all } i,j\geq m_{o}.$$

So $\left(\Delta^{r}\left(x_{k,l}^{i}\right)\right)$ is a Cauchy sequence in (X,q). Since (X,q) is complete, there exists $x_{k,l}\in X$ such that $\lim_{i}\Delta^{r}\left(x_{k,l}^{i}\right)=x_{k,l}$ for all $k,l\in\mathbb{N}$. Since M is continuous, so for $i\geq m_{o}$, on taking limit as $j\to\infty$, we have from (3.4)

$$M\left(\frac{q\left(\Delta^{r}\left(x_{k,l}^{i}\right)-\lim_{j\to\infty}\Delta^{r}x_{k,l}^{j}\right)}{\rho}\right)\leq 1\Rightarrow M\left(\frac{q\left(\Delta^{r}\left(x_{k,l}^{i}\right)-x_{k,l}\right)}{\rho}\right)\leq 1.$$

On taking the infimum of such $\rho's$ we have

$$f\left(\left(x_{k,l}^{i}-x_{k,l}\right)\right)<\varepsilon, \text{ for all } i\geq m_{o}.$$

Thus $\left(x_{k,l}^{i}-x_{k,l}\right)\in\overline{c}^{2,B}\left(M,\Delta^{r},q\right)$. By linearity of the space $\overline{c}^{2,B}\left(M,\Delta^{r},q\right)$, we have for all $i\geq m_{o}$

$$(x_{k,l})=\left(x_{k,l}^{i}\right)-\left(x_{k,l}^{i}-x_{k,l}\right)\in\overline{c}^{2,B}\left(M,\Delta^{r},q\right).$$

Thus $\overline{c}^{2,B}\left(M,\Delta^{r},q\right)$ is a complete semi-normed space. □

The proof of the following result is a routine work in view of the techniques used for establishing the below result.

Theorem 3.4. *The double sequence spaces* $Z(M,\Delta^r,q)$, *where* $Z = l^2_\infty$, $\overline{c}^{2,B}$, $\overline{c}^{2,B}_o$, $\overline{c}^{2,BR}$, $\overline{c}^{2,BR}_o$ *are K-spaces.*

Theorem 3.5. $Z\left(M,\Delta^{r-1},q\right) \subset Z(M,\Delta^r,q)$ *for* $Z = l^2_\infty$, $\overline{c}^2$, $\overline{c}^{2,B}$, $\overline{c}^{2,R}$, $\overline{c}^{2,BR}$,$\overline{c}^2_o$, $\overline{c}^{2,B}_o$, $\overline{c}^{2,R}_o$, $\overline{c}^{2,BR}_o$ *and the inclusions are strict.*

Proof. We give the proof for the $\overline{c}^2\left(M,\Delta^{r-1},q\right) \subset \overline{c}^2(M,\Delta^r,q)$ only. The rest of the results follows similar way. Let $x = (x_{k,l}) \in \overline{c}^2\left(M,\Delta^{r-1},q\right)$. Then for some $\rho > 0$ we have

$$stat - \lim_{k,l\to\infty} M\left(\frac{q\left(\Delta^r x_{k,l} - L\right)}{\rho}\right) = 0. \tag{3.5}$$

Since $\Delta^r x_{k,l} = \Delta^{r-1}x_{k,l} - \Delta^{r-1}x_{k,l+1} - \Delta^{r-1}x_{k+1,l} + \Delta^{r-1}x_{k+1,l+1} + L - L + L - L$. Then from the equation (3.5) and the continuity of M we have

$$\begin{aligned}
& stat - \lim_{k,l\to\infty} M\left(\frac{q\left(\Delta^r x_{k,l} - L\right)}{\rho}\right) \\
= \ & stat - \lim_{k,l\to\infty} M\left(\frac{\Delta^{r-1}x_{k,l} - \Delta^{r-1}x_{k,l+1} - \Delta^{r-1}x_{k+1,l} + \Delta^{r-1}x_{k+1,l+1} + L - L + L - L}{\rho}\right) \\
= \ & stat - \lim_{k,l\to\infty} M\left(\frac{\Delta^{r-1}x_{k,l} - L}{\rho}\right) + stat - \lim_{k,l\to\infty} M\left(\frac{\Delta^{r-1}x_{k,l+1} - L}{\rho}\right) \\
& + stat - \lim_{k,l\to\infty} M\left(\frac{\Delta^{r-1}x_{k+1,l} - L}{\rho}\right) + stat - \lim_{k,l\to\infty} M\left(\frac{\Delta^{r-1}x_{k+1,l+1} - L}{\rho}\right) \\
= \ & 0.
\end{aligned}$$

This shows that $x = (x_{k,l}) \in \overline{c}^2(M,\Delta^r,q)$.
To show that the inclusions are strict, consider the following example. ☐

Example 3.6. Let $X = \mathbb{C}, M(x) = x^2, r = 2, q(x) = |x|$ and consider the double sequence $x_{k,l} = k + l$ for all $k,l \in \mathbb{N}$. We have $\Delta^2 x_{k,l} = 16$ for all $k,l \in \mathbb{N}$. Hence $x = (x_{k,l}) \in \overline{c}^2(M,\Delta,q)$ but $x = (x_{k,l}) \notin \overline{c}^2(M,q)$.

The proof of the following result is easy and thus omitted.

Proposition 3.7. $Z\left(M,\Delta^{r-1},q\right) \subset l^2_\infty(M,\Delta^r,q)$ *for* $Z = \overline{c}^{2,B}$, $\overline{c}^{2,BR}$, $\overline{c}^{2,B}_o$, $\overline{c}^{2,BR}_o$ *and the inclusions are strict.*

The following result follows from the Theorem 3.3 and Proposition 3.7.

Theorem 3.8. *The double spaces* $Z(M,\Delta^r,q)$ *for* $Z = \overline{c}^{2,B}$, $\overline{c}^{2,BR}$, $\overline{c}^{2,B}_o$, $\overline{c}^{2,BR}_o$ *are nowhere dense subsets of* $l^2_\infty(M,\Delta^r,q)$.

Proposition 3.9. *The double spaces* $Z(M,\Delta^r,q)$ *for* $Z = l^2_\infty$,$\overline{c}^2$, $\overline{c}^{2,B}$, $\overline{c}^{2,R}$,$\overline{c}^{2,BR}$, $\overline{c}^2_o$, $\overline{c}^{2,B}_o$, $\overline{c}^{2,R}_o$, $\overline{c}^{2,BR}_o$ *are not solid.*

Proof. To show the result, we consider following examples. ☐

Example 3.10. Let $X=\mathbb{C}, M(x)=x$ and $q(x)=|x|$. Consider the sequence $x=(x_{k,l})$ is defined by $x=(x_{k,l})=((kl)^r)$ for all $k,l\in\mathbb{N}$. Consider the sequence of scalars $(\alpha_{k,l})$ is defined by $\alpha_{k,l}=(-1)^l$ for all $k,l\in\mathbb{N}$. Then $x=(x_{k,l})\in Z(M,\Delta^r,q)$, but $(\alpha_{k,l}x_{k,l})\notin Z(M,\Delta^r,q)$ for $Z=\overline{c}_o^2,\overline{c}_o^{2,B},\overline{c}_o^{2,R},\ \overline{c}_o^{2,BR}$.

Example 3.11. Let $X=\mathbb{C}$, $M(x)=x$ and $q(x)=|x|$. Consider the sequence $x=(x_{k,l})$ is defined by $x=(x_{k,l})=(k^r)$ for all $k,l\in\mathbb{N}$. Consider the sequence of scalars $(\alpha_{k,l})$ is defined by $\alpha_{k,l}=(-1)^l$ for all $k,l\in\mathbb{N}$. Then $x=(x_{k,l})\in Z(M,\Delta^r,q)$, but $(\alpha_{k,l}x_{k,l})\notin Z(M,\Delta^r,q)$ for $Z=l_\infty^2,\overline{c}^2,\overline{c}^{2,B},\overline{c}^{2,R},\ \overline{c}^{2,BR}$.

Proposition 3.12. *The double spaces $Z(M,\Delta^r,q)$ for $Z=l_\infty^2,\overline{c}^2,\ \overline{c}^{2,B},\ \overline{c}^{2,R},\ \overline{c}^{2,BR},\ \overline{c}_o^2,\ \overline{c}_o^{2,B},\ \overline{c}_o^{2,R},\ \overline{c}_o^{2,BR}$ are not symmetric in general.*

Proof. To show the result, we consider following examples. □

Example 3.13. Let $X=\mathbb{C}$, $M(x)=x, r=1$ and $q(x)=|x|$. Consider the sequence $x=(x_{k,l})$ is defined by

$$x_{k,l}=\begin{cases}1, & \text{if } k \text{ is even and for all } l;\\ 2 & \text{otherwise.}\end{cases}$$

Then $x=(x_{k,l})\in Z(M,\Delta^r,q)$ for $Z=\overline{c}_o^2,\overline{c}_o^{2,B},\overline{c}_o^{2,R},\ \overline{c}_o^{2,BR}$. Consider a sequence $y=(y_{k,l})$ be the rearrangement of the sequence $x=(x_{k,l})$ defined by

$$y_{k,l}=\begin{cases}1, & \text{if } k+l \text{ is odd;}\\ 2 & \text{otherwise.}\end{cases}$$

Then $(y_{k,l})\notin Z(M,\Delta^r,q)$ for $Z=\overline{c}_o^2,\overline{c}_o^{2,B},\overline{c}_o^{2,R},\ \overline{c}_o^{2,BR}$. Hence the double spaces $Z(M,\Delta^r,q)$ for $Z=\overline{c}_o^2,\overline{c}_o^{2,B},\overline{c}_o^{2,R},\ \overline{c}_o^{2,BR}$ are not symmetric.

Example 3.14. Let $X=\mathbb{C}$, $M(x)=x$, $r=1$ and $q(x)=|x|$. Consider the sequence $x=(x_{k,l})$ is defined by $x=(x_{k,l})=(kl)$ for all $k,l\in\mathbb{N}$. Then $x=(x_{k,l})\in Z(M,\Delta,q)$ for $Z=\overline{c}^2,\overline{c}^{2,B},\overline{c}^{2,R},\ \overline{c}^{2,BR}$. Consider a sequence $y=(y_{k,l})$ be the rearrangement of the sequence $x=(x_{k,l})$ defined by

$$y_{k,l}=\begin{cases}l^2, & \text{if } l\text{=1 and for all k,}\\ k, & \text{if } k=l,\\ kl & \text{otherwise.}\end{cases}$$

Then $(y_{k,l})\notin Z(M,\Delta,q)$ for $Z=\overline{c}^2,\overline{c}^{2,B},\overline{c}^{2,R},\ \overline{c}^{2,BR}$. Hence the double spaces $Z(M,\Delta,q)$ for $Z=\overline{c}^2,\overline{c}^{2,B},\overline{c}^{2,R},\ \overline{c}^{2,BR}$ are not symmetric.

Example 3.15. Let $X=\mathbb{C}$, $M(x)=x$, $r=1$ and $q(x)=|x|$. Consider the sequence $x=(x_{k,l})$ is defined by $x=(x_{k,l})=(k)$ for all $k,l\in\mathbb{N}$. Then $x=(x_{k,l})\in l_\infty^2(M,\Delta,q)$. Consider a sequence $y=(y_{k,l})$ be the rearrangement of the sequence $x=(x_{k,l})$ defined by

$$y_{k,l}=\begin{cases}k, & \text{if } \textit{both } k \text{ and } l \text{ are even or odd },\\ 0 & \text{otherwise.}\end{cases}$$

Then $(y_{k,l})\notin l_\infty^2(M,\Delta,q)$. Hence the double space $l_\infty^2(M,\Delta,q)$ is not symmetric.

Theorem 3.16. *Let M, M_1 and M_2 be Orlicz functions, q, q_1 and q_2 be seminorms. Then*

(i) $Z(M_1, \Delta^r, q) \subset Z(MoM_1, \Delta^r, q)$,

(ii) $Z(M_1, \Delta^r, q) \cap Z(M_2, \Delta^r, q) \subset Z(M_1 + M_2, \Delta^r, q)$,

(iii) $Z(M, \Delta^r, q_1) \cap Z(M, \Delta^r, q_2) \subset Z(M_1, \Delta^r, q_1 + q_2)$,

(iv) If q_1 is stronger than q_2 then $Z(M_1, \Delta^r, q_1) \subset Z(M_1, \Delta^r, q_2)$, *for* $Z = l^2_\infty, \overline{c}^2, \overline{c}^{2,B}, \overline{c}^{2,R}, \overline{c}^{2,BR}, \overline{c}^2_o, \overline{c}^{2,B}_o, \overline{c}^{2,R}_o, \overline{c}^{2,BR}_o$.

Proof. (i) We prove the result for $Z = \overline{c}^2$ and the rest cases will follow similarly. Let $x = (x_{k,l}) \in \overline{c}^2(M_1, \Delta^r, q)$, so that

$$stat - \lim_{k,l\to\infty} M_1\left(\frac{q(\Delta^r x_{k,l} - L)}{\rho}\right) = 0. \tag{3.6}$$

Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M(t) < \varepsilon$ for $0 \le t \le \delta$. Now we write

$$T_1 = \left\{(k,l) \in \mathbb{N}\times\mathbb{N} : M_1\left(\frac{q(\Delta^r x_{k,l} - L)}{\rho}\right) \le \delta\right\},$$

$$T_2 = \left\{(k,l) \in \mathbb{N}\times\mathbb{N} : \ M_1\left(\frac{q(\Delta^r x_{k,l} - L)}{\rho}\right) > \delta\right\}.$$

If $x = (x_{k,l}) \in x = (x_{k,l}) \in \overline{c}^2(M_1, \Delta^r, q)$, then for $M_1\left(\frac{q(\Delta^r x_{k,l} - L)}{\rho}\right) > \delta$ we have

$$M_1\left(\tfrac{q(\Delta^r x_{k,l} - L)}{\rho}\right) < M_1\left(\tfrac{q(\Delta^r x_{k,l} - L)}{\rho}\right)\delta^{-1} < 1 + \left[\left|M_1\left(\tfrac{q(\Delta^r x_{k,l} - L)}{\rho}\right)\delta^{-1}\right|\right],$$

where $(k,l) \in T_2$ and $[|u|]$ denotes the integer part of u. Given $\varepsilon > 0$ by the definition of Orlicz function M, we have for $M_1\left(\frac{q(\Delta^r x_{k,l} - L)}{\rho}\right) > \delta$

$$M\left(M_1\left(\frac{q(\Delta^r x_{k,l} - L)}{\rho}\right)\right) \le \left(1 + \left[\left|M_1\left(\frac{q(\Delta^r x_{k,l} - L)}{\rho}\right)\delta^{-1}\right|\right]\right) M(1)$$

$$\le 2M(1)\left(M_1\left(\frac{q(\Delta^r x_{k,l} - L)}{\rho}\right)\delta^{-1}\right) < \varepsilon$$

for $(k,l) \in I_2$ and $k,l > n_1 \in \mathbb{N}$ using (3.6). Again for $x = (x_{k,l}) \in \overline{c}^2(M_1, \Delta^r, q)$, then for $M_1\left(\frac{q(\Delta^r x_{k,l} - L)}{\rho}\right) \le \delta$ we have

$$M\left(M_1\left(\frac{q(\Delta^r x_{k,l} - L)}{\rho}\right)\right) < \varepsilon$$

where $(k,l) \in T_1$ and $k,l > n_2 \in \mathbb{N}$ using (3.6). Thus for $k,l > \max\{n_1, n_2\}$ we have

$$M\left(M_1\left(\frac{q(\Delta^r x_{k,l} - L)}{\rho}\right)\right) < \varepsilon.$$

Hence $x=(x_{k,l})\in \overline{c}^{2}(MoM_1,\Delta^r,q)$. Thus $\overline{c}^{2}(M_1,\Delta^r,q)\subset \overline{c}^{2}(MoM_1,\Delta^r,q)$.

(ii) It follows from the following inequality:

$$(M_1+M_2)\left(\frac{q(\Delta^r x_{k,l}-L)}{\rho}\right)$$

$$\leq M_1\left(\frac{q(\Delta^r x_{k,l}-L)}{\rho}\right)+M_2\left(\frac{q(\Delta^r x_{k,l}-L)}{\rho}\right).$$

The proofs of (iii) and (iv) are obvious. □

The proof of the following results are also routine work.

Proposition 3.17. *Let M_1 and M_2 be two Orlicz functions which satisfying $\Delta_2-condition$. If $\beta=\lim_{t\to\infty}\frac{M_2(t)}{t}\geq 1$ then $Z(M_1,\Delta^r,q)=Z(M_1oM_2,\Delta^r,q)$, where $Z=l_\infty^2,\overline{c}^2,\overline{c}^{2,B},\overline{c}^{2,R},\ \overline{c}^{2,BR},\ \overline{c}_o^2,\overline{c}_o^{2,B},\overline{c}_o^{2,R},\ \overline{c}_o^{2,BR}$.*

Proposition 3.18. *For any Orlicz function M, if $q_1\approxeq$ (equivalent to) q_2, then $Z(M,\Delta^r,q_1)=Z(M,\Delta^r,q_2)$, for $Z=l_\infty^2,\overline{c}^2,\overline{c}^{2,B},\overline{c}^{2,R},\ \overline{c}^{2,BR},\ \overline{c}_o^2,\overline{c}_o^{2,B},\overline{c}_o^{2,R},\ \overline{c}_o^{2,BR}$.*

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