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Relationship Between Homomorphisms of Some Groups and Hypergroups

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Abstract : We consider the hypergroup $(G/\rho, \circ)$ which is defined from an abelian group G where $x\rho y \Leftrightarrow x = y$ or $x = y^{-1}$ and $x\rho \circ y\rho = \{(xy)\rho, (xy^{-1})\rho\}$. Let $(G/\rho, \circ)$ and $(\overline{G}/\overline{\rho}, \overline{\circ})$ be the hypergroups defined respectively from abelian groups G and $\overline{G}$ as above and let $\mathrm{Hom}(G, \overline{G})$ and $\mathrm{Hom}\big((G/\rho, \circ), (\overline{G}/\overline{\rho}, \overline{\circ})\big)$ be respectively the set of all group homomorphisms from G into $\overline{G}$ and the set of all hypergroup homomorphisms from $(G/\rho, \circ)$ into $(\overline{G}/\overline{\rho}, \overline{\circ})$. Some basic properties of elements of $\mathrm{Hom}\big((G/\rho, \circ), (\overline{G}/\overline{\rho}, \overline{\circ})\big)$ are provided. The main purpose is to show that for certain G and $\overline{G}$, $\mathrm{Hom}\big((G/\rho, \circ), (\overline{G}/\overline{\rho}, \overline{\circ})\big) = \big\{\overline{\varphi} \mid \varphi \in \mathrm{Hom}(G, \overline{G})\big\}$ where $\overline{\varphi}(x\rho) = \varphi(x)\overline{\rho}$ for all $x \in G$.

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1 Introduction

By a *hyperoperation* on a nonempty set H is a function $\circ$ from $H \times H$ into $P(H) \smallsetminus \{\emptyset\}$ where $P(H)$ is the power set of H. In this case, $(H, \circ)$ is called a *hypergroupoid*. For $A, B \subseteq H$, let $A \circ B = \bigcup\limits_{\substack{a \in A \\ b \in B}} a \circ b$. A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if $x \circ (y \circ z) = (x \circ y) \circ z$ for all $x, y, z \in H$. A *hypergroup* is a semihypergroup $(H, \circ)$ satisfying the condition $H \circ x = x \circ H = H$ for all $x \in H$. Hypergroups are a generalization of groups.

Let $(G, \cdot)$ be an abelian group and ρ the equivalence relation on G defined by

$$x\rho y \Leftrightarrow x = y \text{ or } x = y^{-1},$$

that is, $x\rho = \{x, x^{-1}\}$ for all $x \in G$. Define the hyperoperation $\circ$ on G/ρ by

$$x\rho \circ y\rho = \{(xy)\rho, (xy^{-1})\rho\} \text{ for all } x, y \in G.$$

Then $(G/\rho, \circ)$ is a hypergroup ([1], page 11).

By a *homomorphism* from a hypergroup $(H, \circ)$ into a hypergroup $(H', \circ')$ we mean a mapping $\varphi : H \to H'$ satisfying the condition

$$\varphi(x \circ y) = \varphi(x) \circ' \varphi(y) \text{ for all } x, y \in H.$$

We note here that our definition of homomorphisms between hypergroups are called *good homomorphisms* in [1].

For groups G and $\overline{G}$, let $\mathrm{Hom}(G, \overline{G})$ denote the set of all (group) homomorphisms from G into $\overline{G}$. For hypergroups $(H, \circ)$ and $(\overline{H}, \bar{\circ})$, denote analogously by $\mathrm{Hom}\big((H, \circ), (\overline{H}, \bar{\circ})\big)$ the set of all (hypergroup) homomorphisms from $(H, \circ)$ into $(\overline{H}, \bar{\circ})$.

Recall that if G is a cyclic group generated by a, then $G = \langle a \rangle = \{a^n \mid n \in \mathbb{Z}\}$ where $\mathbb{Z}$ is the set of all integers. If $G = \langle a \rangle$, then every $\varphi \in \mathrm{Hom}(G, \overline{G})$ is completely determined by $\varphi(a)$.

Let $\mathbb{Z}^+ = \{x \in \mathbb{Z} \mid x > 0\}$ and $\mathbb{Z}_0^+ = \mathbb{Z}^+ \cup \{0\}$. The set of real numbers and the set of all rational numbers are denoted by $\mathbb{R}$ and $\mathbb{Q}$, respectively. Then $(\mathbb{Q}, +)$ and $(\mathbb{R}^+, \cdot)$ are abelian groups where $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$. Note that for every $n \in \mathbb{Z} \smallsetminus \{0\}$, $\dfrac{x}{n} \in \mathbb{Q}$ for all $x \in \mathbb{Q}$ and $x^{\frac{1}{n}} \in \mathbb{R}^+$ for all $x \in \mathbb{R}^+$.

In the remainder, let G and $\overline{G}$ be abelain groups with identities e and $\bar{e}$, respectively, and let $(G/\rho, \circ)$ and $(\overline{G}/\bar{\rho}, \bar{\circ})$ be the corresponding hypergroups defined as above, that is,

$$\begin{aligned} \forall x, y \in G,\ x\rho y &\Leftrightarrow x = y \text{ or } x = y^{-1}, \\ x\rho \circ y\rho &= \{(xy)\rho, (xy^{-1})\rho\}, \\ \forall x, y \in \overline{G},\ x\bar{\rho} y &\Leftrightarrow x = y \text{ or } x = y^{-1}, \\ x\bar{\rho} \mathbin{\bar{\circ}} y\bar{\rho} &= \{(xy)\bar{\rho}, (xy^{-1})\bar{\rho}\}. \end{aligned}$$

For each $\varphi \in \mathrm{Hom}(G, \overline{G})$, define $\overline{\varphi} : G/\rho \to \overline{G}/\bar{\rho}$ by

$$\overline{\varphi}(x\rho) = \varphi(x)\bar{\rho} \ \text{ for all } x \in G.$$

Our purpose is to provide some basic properties of elements in $\mathrm{Hom}\big((G/\rho, \circ), (\overline{G}/\bar{\rho}, \bar{\circ})\big)$ and show that

$$\mathrm{Hom}\big((G/\rho, \circ), (\overline{G}/\bar{\rho}, \bar{\circ})\big) = \{\overline{\varphi} \mid \varphi \in \mathrm{Hom}(G, \overline{G})\}$$

if G and $\overline{G}$ satisfy one of the following conditions.

(1) G is a cyclic group.
(2) $G = (\mathbb{Q}, +) = \overline{G}$.
(3) $G = (\mathbb{Q}, +)$ and $\overline{G} = (\mathbb{R}^+, \cdot)$.
(4) G in which every nonidentity element has order 2.

For basic concepts of groups and homorphisms, the reader is referred to [2].

2 Main Results

First, we recall that for any $\varphi \in \mathrm{Hom}(G, \overline{G}), \varphi(e) = \overline{e}$ and $\varphi(x^{-1}) = \varphi(x)^{-1}$ for all $x \in G$.

Some basic properties of elements of $\mathrm{Hom}\big((G/\rho, \circ), (\overline{G}/\overline{\rho}, \overline{\circ})\big)$ are first presented.

Proposition 2.1. *If $\psi \in Hom\big((G/\rho, \circ), (\overline{G}/\overline{\rho}, \overline{\circ})\big)$, then $\psi(e\rho) = \overline{e}\,\overline{\rho}$.*

Proof. Let $a \in \overline{G}$ be such that $\psi(e\rho) = a\overline{\rho}$. Then

$$\{a\overline{\rho}\} = \{\psi(e\rho)\} = \psi(e\rho \circ e\rho) = \psi(e\rho)\,\overline{\circ}\,\psi(e\rho) = a\overline{\rho}\,\overline{\circ}\,a\overline{\rho} = \{a^2\overline{\rho}, \overline{e}\,\overline{\rho}\}$$

which implies that $a\overline{\rho} = a^2\overline{\rho} = \overline{e}\,\overline{\rho}$. □

Proposition 2.2. *Let $\psi \in Hom\big((G/\rho, \circ), (\overline{G}/\overline{\rho}, \overline{\circ})\big)$. If $a \in G$ and $b \in \overline{G}$ are such that $\psi(a\rho) = b\overline{\rho}$, then $\psi(a^n\rho) = b^n\overline{\rho}$ for all $n \in \mathbb{Z}$.*

Proof. By Proposition 2.1, $\psi(a^0\rho) = \psi(e\rho) = \overline{e}\,\overline{\rho} = b^0\overline{\rho}$. Since

$$\{\psi(a^2\rho), \overline{e}\,\overline{\rho}\} = \psi\{a^2\rho, e\rho\} = \psi(a\rho \circ a\rho) = \psi(a\rho)\,\overline{\circ}\,\psi(a\rho) = b\overline{\rho}\,\overline{\circ}\,b\overline{\rho} = \{b^2\overline{\rho}, \overline{e}\,\overline{\rho}\},$$

it follows that $\psi(a^2\rho) = b^2\overline{\rho}$. Assume that $k \geq 2$ and $\psi(a^n\rho) = b^n\overline{\rho}$ for all $n \in \{0, 1, \ldots, k\}$. Then

$$\begin{aligned} \{\psi(a^{k+1}\rho), b^{k-1}\overline{\rho}\} &= \psi\{a^{k+1}\rho, a^{k-1}\rho\} \\ &= \psi(a^k\rho \circ a\rho) \\ &= \psi(a^k\rho)\,\overline{\circ}\,\psi(a\rho) \\ &= b^k\overline{\rho}\,\overline{\circ}\,b\overline{\rho} \\ &= \{b^{k+1}\overline{\rho}, b^{k-1}\overline{\rho}\} \end{aligned}$$

which implies that $\psi(a^{k+1}\rho) = b^{k+1}\overline{\rho}$. This shows that

$$\psi(a^n\rho) = b^n\overline{\rho} \text{ for all } n \in \mathbb{Z}_0^+.$$

If $n \in \mathbb{Z}$ is such that $n < 0$, then $-n > 0, a^n\rho = a^{-n}\rho$ and $b^n\overline{\rho} = b^{-n}\overline{\rho}$, and hence $\psi(a^n\rho) = \psi(a^{-n}\rho) = b^{-n}\overline{\rho} = b^n\overline{\rho}$. Therefore the proposition is proved. □

Corollary 2.3. *Let $\psi \in Hom\big((G/\rho, \circ), (\overline{G}/\overline{\rho}, \overline{\circ})\big)$, $a \in G$, $b \in \overline{G}$ and $\psi(a\rho) = b\overline{\rho}$. If $a = a^{-1}$ in G, then $b = b^{-1}$ in $\overline{G}$.*

Proof. Assume that $a = a^{-1}$. Then $a^2 = e$, so by Proposition 2.1 ans Proposition 2.2,

$$\overline{e}\,\overline{\rho} = \psi(e\rho) = \psi(a^2\rho) = b^2\overline{\rho}.$$

But $\overline{e}\,\overline{\rho} = \{\overline{e}\}$, so $b^2 = \overline{e}$, that is, $b = b^{-1}$. □

Proposition 2.4. *If $\varphi \in Hom(G, \overline{G})$, then $\overline{\varphi} : G/\rho \to \overline{G}/\overline{\rho}$ defined by $\overline{\varphi}(x\rho) = \varphi(x)\overline{\rho}$ for all $x \in G$ belongs to $Hom\big((G/\rho, \circ), (\overline{G}/\overline{\rho}, \overline{\circ})\big)$. Hence*

$$\{\overline{\varphi} \mid \varphi \in Hom(G, \overline{G})\} \subseteq Hom\big((G/\rho, \circ), (\overline{G}/\overline{\rho}, \overline{\circ})\big).$$

Proof. To show that $\overline{\varphi}$ is well-defined, let $x, y \in G$ be such that $x\rho = y\rho$. Then $y = x$ or $y = x^{-1}$, so $\varphi(y) = \varphi(x)$ or $\varphi(y) = \varphi(x^{-1}) = \varphi(x)^{-1}$. Hence $\varphi(x)\overline{\rho} = \varphi(y)\overline{\rho}$.

Also, for any $x, y \in G$,

$$\begin{aligned}
\overline{\varphi}(x\rho \circ y\rho) &= \overline{\varphi}\{(xy)\rho, (xy^{-1})\rho\} \\
&= \{\overline{\varphi}((xy\rho)), \overline{\varphi}((xy^{-1})\rho)\} \\
&= \{\varphi(xy)\overline{\rho}, \varphi(xy^{-1})\overline{\rho}\} \\
&= \{(\varphi(x)\varphi(y))\overline{\rho}, (\varphi(x)\varphi(y)^{-1})\overline{\rho}\} \\
&= \varphi(x)\overline{\rho}\,\overline{\circ}\,\varphi(y)\overline{\rho}.
\end{aligned}$$

□

Now we are ready to provide our main results.

Theorem 2.5. *If G is a cyclic group, then*

$$Hom\big((G/\rho, \circ), (\overline{G}/\overline{\rho}, \overline{\circ})\big) = \{\overline{\varphi} \mid \varphi \in Hom(G, \overline{G})\}$$

where for $\varphi \in Hom(G, \overline{G})$, $\overline{\varphi}$ is defined as above.

Proof. From Proposition 2.4, $\{\overline{\varphi} \mid \varphi \in \mathrm{Hom}(G, \overline{G})\} \subseteq \mathrm{Hom}\big((G/\rho, \circ), (\overline{G}/\overline{\rho}, \overline{\circ})\big)$. For the reverse inclusion, let $\psi \in \mathrm{Hom}\big((G/\rho, \circ), (\overline{G}/\overline{\rho}, \overline{\circ})\big)$. Let G be generated by $a \in G$. Then $G = \{a^n \mid n \in \mathbb{Z}\}$. Let $b \in \overline{G}$ be such that $\psi(a\rho) = b\overline{\rho}$. Therefore by Proposition 2.2, $\psi(a^n\rho) = b^n\overline{\rho}$. Define $\varphi \in \mathrm{Hom}(G, \overline{G})$ by $\varphi(a) = b$, so $\varphi(a^n) = b^n$ for all $n \in \mathbb{Z}$. Hence for every $n \in \mathbb{Z}$, $\psi(a^n\rho) = \varphi(a^n)\overline{\rho} = \overline{\varphi}(a^n\rho)$. Hence the theorem is proved. □

Recall that $(\mathbb{Q}, +)$ is not a cyclic group. The next theorem shows that the converse of Theorem 2.5 is not true in general.

Theorem 2.6. *If G is $(\mathbb{Q}, +)$ and $\overline{G}$ is either $(\mathbb{Q}, +)$ or $(\mathbb{R}^+, \cdot)$, then*

$$Hom\big((G/\rho, \circ), (\overline{G}/\overline{\rho}, \overline{\circ})\big) = \{\overline{\varphi} \mid \varphi \in Hom(G, \overline{G})\}.$$

Proof. Let $\psi \in \mathrm{Hom}\,\big((G/\rho, \circ), (\overline{G}/\overline{\rho}, \overline{\circ})\big)$. Let $b \in \overline{G}$ be such that $\psi(1\rho) = b\overline{\rho}$.

Case 1: $\overline{G} = (\mathbb{Q}, +)$. Then $b \in \mathbb{Q}$. Let $k \in \mathbb{Z}$ and $m \in \mathbb{Z} \smallsetminus \{0\}$. Then $\psi\big(\frac{1}{m}\rho\big) = c\rho$ for some $c \in \mathbb{Q}$. Hence by Proposition 2.2,

$$b\overline{\rho} = \psi\big((m(\frac{1}{m}))\rho\big) = (mc)\overline{\rho}$$

which implies that $mc = b$ or $mc = -b$. Thus $c = \frac{b}{m}$ or $c = -\frac{b}{m}$. It follows that $\psi\big(\frac{1}{m}\rho\big) = \frac{b}{m}\overline{\rho}$. Again, by Proposition 2.2,

$$\psi\big(\frac{k}{m}\rho\big) = \psi\big((k\frac{1}{m})\rho\big) = \big(k(\frac{b}{m})\big)\overline{\rho} = \big(\frac{k}{m}b\big)\overline{\rho}.$$

This shows that $\psi(x\rho) = (xb)\overline{\rho}$ for all $x \in \mathbb{Q}$. Define $\varphi : \mathbb{Q} \to \mathbb{Q}$ by $\varphi(x) = bx$ for all $x \in \mathbb{Q}$. Then $\varphi \in \mathrm{Hom}((\mathbb{Q}, +), (\mathbb{Q}, +))$ and $\psi(x\rho) = \varphi(x)\overline{\rho} = \overline{\varphi}(x\rho)$ for all $x \in \mathbb{Q}$.

Case 2: $\overline{G} = (\mathbb{R}^+, \cdot)$. Then $b \in \mathbb{R}^+$, so $b^{\frac{1}{m}} \in \mathbb{R}^+$ for all $m \in \mathbb{Z} \setminus \{0\}$. Let $k \in \mathbb{Z}$ and $m \in \mathbb{Z} \setminus \{0\}$, and let $c \in \mathbb{R}^+$ be such that $\psi\big(\frac{1}{m}\rho\big) = c\overline{\rho}$. From Proposition 2.2, we have

$$b\overline{\rho} = \psi\big((m\frac{1}{m})\rho\big) = c^m\overline{\rho},$$

so $c^m = b$ or $c^m = b^{-1}$. Hence $c = b^{\frac{1}{m}}$ or $c = b^{-\frac{1}{m}}$. Thus $\psi\big(\frac{1}{m}\rho\big) = b^{\frac{1}{m}}\overline{\rho}$. Also, by Proposition 2.2, we have

$$\psi\big(\frac{k}{m}\rho\big) = \psi\big((k\frac{1}{m})\rho\big) = \big(b^{\frac{1}{m}}\big)^k\overline{\rho} = b^{\frac{k}{m}}\overline{\rho}.$$

This proves that $\psi(x\rho) = b^x\overline{\rho}$ for all $x \in \mathbb{Q}$. Define $\varphi : \mathbb{Q} \to \mathbb{R}^+$ by $\varphi(x) = b^x$ for all $x \in \mathbb{Q}$. Then $\varphi \in \mathrm{Hom}((\mathbb{Q}, +), (\mathbb{R}^+, \cdot))$ and $\psi(x\rho) = \varphi(x)\overline{\rho} = \overline{\varphi}(x\rho)$ for all $x \in \mathbb{Q}$.

From Case 1 and Case 2, we have that $\mathrm{Hom}\big((G/\rho, \circ), (\overline{G}/\overline{\rho}, \overline{\circ})\big) \subseteq \{\overline{\varphi} \mid \varphi \in \mathrm{Hom}(G, \overline{G})\}$. Hence these two sets are identical by Proposition 2.4. $\square$

If every nonidentity element of G has order 2, then $x = x^{-1}$ for all $x \in G$, and hence $x\rho = \{x\}$. Some examples of such a group G are the following ones.

(1) $G = \prod_{i \in I} G_i$ with componentwise operation

where $G_i = (\mathbb{Z}_2, +)$ for every $i \in I$.

(2) X is a set and $G = P(X)$, the power set of X, with the operation $*$ defined by

$$A * B = (A \setminus B) \cup (B \setminus A) \text{ for all } A, B \in P(X).$$

The last theorem shows that if G has this property, the same result is also obtained.

Theorem 2.7. *If every nonidentity element of G has order 2, then*

$$\mathit{Hom}\big((G/\rho, \circ), (\overline{G}/\overline{\rho}, \overline{\circ})\big) = \{\overline{\varphi} \mid \varphi \in \mathit{Hom}(G, \overline{G})\}.$$

Proof. Let $\psi \in \operatorname{Hom}\big((G/\rho, \circ), (\overline{G}/\overline{\rho}, \overline{\circ})\big)$. For each $x \in G$, there is an element $x' \in \overline{G}$ such that $\psi(x\rho) = x'\overline{\rho}$. Since $x^2 = e$ for all $x \in G$, by Corollary 2.3, $(x')^2 = \overline{e}$. Thus $x\rho = \{x\}$ and $x'\overline{\rho} = \{x'\}$ for all $x \in G$. Define $\varphi : G \to \overline{G}$ by $\varphi(x) = x'$ for all $x \in G$. Then $\psi(x\rho) = \varphi(x)\overline{\rho}$ for all $x \in G$. If $x, y \in G$, then $\psi(x\rho \circ y\rho) = \psi(x\rho) \circ \psi(y\rho)$, so

$$\begin{aligned}
\{(xy)'\overline{\rho}\} &= \{(xy)'\overline{\rho}, (xy^{-1})'\overline{\rho}\} && \text{since } y = y^{-1} \\
&= \{\psi((xy)\rho), \psi((xy^{-1})\rho)\} \\
&= \psi\{(xy)\rho, (xy^{-1})\rho\} \\
&= \psi(x\rho \circ y\rho) \\
&= \psi(x\rho) \mathbin{\overline{\circ}} \psi(y\rho) \\
&= x'\overline{\rho} \mathbin{\overline{\circ}} y'\overline{\rho} \\
&= \{(x'y')\overline{\rho}, (x'(y')^{-1})\overline{\rho}\} \\
&= \{(x'y')\overline{\rho}\} && \text{since } (y')^{-1} = y'.
\end{aligned}$$

Hence $(xy)'\overline{\rho} = (x'y')\overline{\rho}$. But $(xy)'\overline{\rho} = \{(xy)'\}$, so we have $(xy)' = x'y'$. Hence $\varphi(xy) = \varphi(x)\varphi(y)$. By Proposition 2.4, the result is obtained, as before. □

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