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# Recurrence Relations of $K$-Bessel's Function

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**Abstract :** In this paper we evaluate eight differential recurrence relations and five pure recurrence relations of $K$-Bessel function.

**Keywords :** $K$-Gamma function; $K$-Pochhammer symbols; $K$-Bessel function.
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## 1 Introduction

In [1] the author introduced the generalized $K$-Gamma Function $\Gamma_k(x)$ as

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}}, k > 0, x \in C\backslash kZ^-, \tag{1.1}$$

where $(x)_{n,k}$ is the $K$-Pochhammer symbol and is given by

$$(x)_{n,k} = x(x+k)(x+2k).....(x+(n-1)k), x \in C, k \in R, n \in N^+. \tag{1.2}$$

The integral form of generalized $K$-Gamma function is given by,

$$\Gamma_k(x) = \int_0^\infty t^{x-1}e^{-\frac{t^k}{k}}dt, x \in C, k \in R, Re(x) > 0, \tag{1.3}$$

and it follows easily that

$$\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma(\frac{x}{k}). \tag{1.4}$$

and

$$\Gamma_k(x+k) = x\Gamma_k(x). \tag{1.5}$$

$K$-Bessel function defined [2], [3] as

$$J^k_{\pm\vartheta}(z) = \sum_{r=0}^{\infty} \frac{(-1)^r (\frac{z}{2})^{2r \pm \frac{\vartheta}{k}}}{\Gamma_k(rk \pm \vartheta + k)(r!)}. \tag{1.6}$$

where $k \in R^+, \vartheta \in I$ and $\vartheta > -k$

# 2 Main Results

In this section we evaluate eight differential recurrence relations and five pure recurrence relations of $K$-Bessel function.

**Theorem 2.1.** *For $k \in R^+, \vartheta \in I$ and $\vartheta > -k$, then*

$$z\dot{J}^k_{\vartheta}(z) = \frac{\vartheta}{k} J^k_{\vartheta}(z) - zJ^k_{\vartheta+k}(z). \tag{2.1}$$

*where*

$$\dot{J}^k_{\vartheta}(z) = \frac{d}{dz} J^k_{\vartheta}(z).$$

*Proof.* Consider the definition of $K$-Bessel function (1.6),

$$J^k_{\vartheta}(z) = \sum_{r=0}^{\infty} \frac{(-1)^r (\frac{z}{2})^{2r + \frac{\vartheta}{k}}}{\Gamma_k(rk + \vartheta + k)(r!)},$$

differentiating with respect to $z$, we have

$$z\dot{J}^k_{\vartheta}(z) = \sum_{r=0}^{\infty} \frac{(-1)^r (\frac{z}{2})^{2r + \frac{\vartheta}{k}} (2r + \frac{\vartheta}{k})}{\Gamma_k(rk + \vartheta + k)(r!)},$$

$$z\dot{J}^k_{\vartheta}(z) = \frac{\vartheta}{k} J^k_{\vartheta}(z) + z\sum_{r=1}^{\infty} \frac{(-1)^r (\frac{z}{2})^{2r + \frac{\vartheta}{k} - 1}}{\Gamma_k(rk + \vartheta + k)(r-1)!},$$

replacing $r$ by $r + 1$, we obtain

$$z\dot{J}^k_{\vartheta}(z) = \frac{\vartheta}{k} J^k_{\vartheta}(z) - z\sum_{r=0}^{\infty} \frac{(-1)^r (\frac{z}{2})^{2r + \frac{\vartheta + k}{k}}}{\Gamma_k(rk + (\vartheta + k) + k)(r)!},$$

$$z\dot{J}^k_{\vartheta}(z) = \frac{\vartheta}{k} J^k_{\vartheta}(z) - zJ^k_{\vartheta+k}(z).$$

Which completes the proof. ☐

**Theorem 2.2.** *For $k \in R^+, \vartheta \in I$ and $\vartheta > -k$, then*

$$zJ_{\vartheta}^{\dot{k}}(z) = -\frac{\vartheta}{k}J_{\vartheta}^k(z) + \frac{z}{k}J_{\vartheta-k}^k(z). \tag{2.2}$$

*or*

$$\frac{d}{dz}[z^{\frac{\vartheta}{k}}J_{\vartheta}^k] = \frac{z^{\frac{\vartheta}{k}}}{k}J_{\vartheta-k}^k \tag{2.3}$$

*where*

$$J_{\vartheta}^{\dot{k}}(z) = \frac{d}{dz}J_{\vartheta}^k(z).$$

*Proof.* Consider the definition of $K$-Bessel function (1.6),

$$J_{\vartheta}^k(z) = \sum_{r=0}^{\infty}\frac{(-1)^r(\frac{z}{2})^{2r+\frac{\vartheta}{k}}}{\Gamma_k(rk+\vartheta+k)(r!)},$$

differentiating with respect to $z$, we have

$$zJ_{\vartheta}^{\dot{k}}(z) = \sum_{r=0}^{\infty}\frac{(-1)^r(\frac{z}{2})^{2r+\frac{\vartheta}{k}}(2r+2\frac{\vartheta}{k}-\frac{\vartheta}{k})}{\Gamma_k(rk+\vartheta+k)(r!)},$$

$$zJ_{\vartheta}^{\dot{k}}(z) = -\frac{\vartheta}{k}J_{\vartheta}^k(z) + \sum_{r=0}^{\infty}\frac{(-1)^r(\frac{z}{2})^{2r+\frac{\vartheta}{k}}(2r+2\frac{\vartheta}{k})}{\Gamma_k(rk+\vartheta+k)(r)!},$$

using (1.5), we obtain

$$zJ_{\vartheta}^{\dot{k}}(z) = -\frac{\vartheta}{k}J_{\vartheta}^k(z) + \frac{z}{k}\sum_{r=0}^{\infty}\frac{(-1)^r(\frac{z}{2})^{2r+\frac{\vartheta-k}{k}}}{\Gamma_k(rk+(\vartheta-k)+k)(r)!},$$

$$zJ_{\vartheta}^{\dot{k}}(z) = -\frac{\vartheta}{k}J_{\vartheta}^k(z) + \frac{z}{k}J_{\vartheta-k}^k(z).$$

Which completes the proof. □

**Theorem 2.3.** *For $k \in R^+, \vartheta \in I$ and $\vartheta > -k$, then*

$$2kJ_{\vartheta}^{\dot{k}}(z) = J_{\vartheta-k}^k(z) - kJ_{\vartheta+k}^k(z). \tag{2.4}$$

*where*

$$J_{\vartheta}^{\dot{k}}(z) = \frac{d}{dz}J_{\vartheta}^k(z).$$

*Proof.* Adding equation (2.1) and (2.2), we immediately have the above result. □

**Theorem 2.4.** *For $k \in R^+, \vartheta \in I$ and $\vartheta > -k$, then*

$$2\vartheta J_{\vartheta}^k(z) = z[J_{\vartheta-k}^k(z) + kJ_{\vartheta+k}^k(z)]. \tag{2.5}$$

*Proof.* Substrating (2.1) from (2.2), we immediately obtain the above result. □

**Theorem 2.5.** *For $k \in R^+, \vartheta \in I$ and $\vartheta > -k$, then*

$$z^2 \ddot{J}_\vartheta^k(z) = [(\frac{\vartheta}{k} - 1)\frac{\vartheta}{k} - \frac{z^2}{k}]J_\vartheta^k(z) + \frac{z}{k}J_{\vartheta+k}^k(z). \tag{2.6}$$

*where*

$$\ddot{J}_\vartheta^k(z) = \frac{d^2}{dz^2}J_\vartheta^k(z).$$

*Proof.* Differentiate (2.1) with respect to $z$, and multiply by $z$, we have

$$z^2 \ddot{J}_\vartheta^k(z) = (\frac{\vartheta}{k} - 1)z\dot{J}_\vartheta^k(z) - z(z\dot{J}_{\vartheta+k}^k(z)) - zJ_{\vartheta+k}^k(z), \tag{2.7}$$

put the value of $z\dot{J}_\vartheta^k(z)$ from (2.1) and $z\dot{J}_{\vartheta+k}^k(z)$ from (2.2) in (2.7), we obtain

$$z^2 \ddot{J}_\vartheta^k(z) = [(\frac{\vartheta}{k} - 1)\frac{\vartheta}{k} - \frac{z^2}{k}]J_\vartheta^k(z) + \frac{z}{k}J_{\vartheta+k}^k(z).$$

Which completes the proof. □

**Theorem 2.6.** *For $k \in R^+, \vartheta \in I$ and $\vartheta > -k$, then*

$$\frac{z}{2}J_{\vartheta-k}^k(z) = \sum_{r=0}^{\infty}(-1)^r(\vartheta + 2rk)k^r J_{\vartheta+2rk}^k(z). \tag{2.8}$$

*Or*

$$\frac{z}{2}J_\vartheta^k(z) = \sum_{r=0}^{\infty}(-1)^r(\vartheta + (2r+1)k)k^r J_{\vartheta+(2r+1)k}^k(z). \tag{2.9}$$

*Proof.* From (2.5), we have

$$J_{\vartheta-k}^k(z) + kJ_{\vartheta+k}^k(z) = \frac{2\vartheta}{z}J_\vartheta^k(z), \tag{2.10}$$

replacing $\vartheta$ by $\vartheta + 2k$, in (2.10) and changing the sign, we have

$$-J_{\vartheta+k}^k(z) - kJ_{\vartheta+3k}^k(z) = -\frac{2}{z}(\vartheta + 2k)J_{\vartheta+2k}^k(z), \tag{2.11}$$

replacing $\vartheta$ by $\vartheta + 4k$, in (2.10), we have

$$J_{\vartheta+3k}^k(z) + kJ_{\vartheta+5k}^k(z) = \frac{2}{z}(\vartheta + 4k)J_{\vartheta+4k}^k(z), \tag{2.12}$$

replacing $\vartheta$ by $\vartheta + 6k$, in (2.10) and changing the sign, we have

$$-J_{\vartheta+5k}^k(z) - kJ_{\vartheta+7k}^k(z) = -\frac{2}{z}(\vartheta + 6k)J_{\vartheta+6k}^k(z), etc. \tag{2.13}$$

multplying (2.10) by one, (2.11) by $k$, (2.12) by $k^2$, (2.13) by $k^3$, so on and adding, we obtain

$$\frac{z}{2}J^k_{\vartheta-k}(z)=\sum_{r=0}^{\infty}(-1)^r(\vartheta+2rk)k^rJ^k_{\vartheta+2rk}(z).$$

since $J^k_\vartheta(z)\Rightarrow 0$ as $\vartheta\Rightarrow\infty$.
Replacing $\vartheta$ by $\vartheta+k$, in (2.8), we obtain

$$\frac{z}{2}J^k_{\vartheta}(z)=\sum_{r=0}^{\infty}(-1)^r(\vartheta+(2r+1)k)k^rJ^k_{\vartheta+(2r+1)k}(z).$$

Which completes the proof. □

**Theorem 2.7.** *For $k\in R^+,\vartheta\in I$ and $\vartheta>-k$, then*

$$\dot{J}^k_\vartheta(z)=\frac{2}{zk}[\frac{\vartheta}{2}J^k_\vartheta(z)-\sum_{r=1}^{\infty}(-1)^r(\vartheta+2rk)k^rJ^k_{\vartheta+2rk}(z)]. \tag{2.14}$$

*where*

$$\dot{J}^k_\vartheta(z)=\frac{d}{dz}J^k_\vartheta(z).$$

*Proof.* Put the value of $J^k_{\vartheta-k}(z)$ from (2.8) in to (2.2), we immediately obtain the above result. □

**Theorem 2.8.** *For $k\in R^+,\vartheta\in I$ and $\vartheta>-k$, then*

$$\frac{d}{dz}[(J^k_\vartheta(z))^2+k(J^k_{\vartheta+k}(z))^2]=\frac{2}{z}[\frac{\vartheta}{k}(J^k_\vartheta(z))^2-(\vartheta+k)(J^k_{\vartheta+k}(z))^2] \tag{2.15}$$

*Proof.* Consider the left hand side

$$A\equiv\frac{d}{dz}[(J^k_\vartheta(z))^2+k(J^k_{\vartheta+k}(z))^2],$$

$$A\equiv 2J^k_\vartheta(z)\dot{J}^k_\vartheta(z)+2kJ^k_{\vartheta+k}(z)\dot{J}^k_{\vartheta+k}(z), \tag{2.16}$$

put the value of $\dot{J}^k_\vartheta(z)$ from (2.1) and $\dot{J}^k_{\vartheta+k}(z)$ from (2.2) in (2.16), we obtain the desire result. □

**Theorem 2.9.** *For $k\in R^+,\vartheta\in I$ and $\vartheta>-k$, then*

$$\frac{d}{dz}[zJ^k_\vartheta(z)J^k_{\vartheta+k}(z)]=z[\frac{1}{k}(J^k_\vartheta(z))^2-(J^k_{\vartheta+k}(z))^2] \tag{2.17}$$

*Proof.* Consider the left hand side

$$A\equiv\frac{d}{dz}[zJ^k_\vartheta(z)J^k_{\vartheta+k}(z)],$$

$$A\equiv J^k_\vartheta(z)J^k_{\vartheta+k}(z)+z[J^k_\vartheta(z)\dot{J}^k_{\vartheta+k}(z)+J^k_{\vartheta+k}(z)\dot{J}^k_\vartheta(z)], \tag{2.18}$$

put the value of $\dot{J}^k_\vartheta(z)$ from (2.1) and $\dot{J}^k_{\vartheta+k}(z)$ from (2.2) in (2.18), we obtain the desire result. □

**Theorem 2.10.** *For $k \in R^{+}, \vartheta \in I$ and $\vartheta > -k$, then*

$$(J_0^k(z))^2 + 2[k(J_k^k(z))^2 + k^2(J_{2k}^k(z))^2 + k^3(J_{3k}^k(z))^2 + ...] = 1 \tag{2.19}$$

*Proof.* Put $\vartheta = 0$ in (2.15), we have

$$\frac{d}{dz}[(J_0^k(z))^2 + k(J_k^k(z))^2] = \frac{2}{z}[0 - (k)(J_k^k(z))^2], \tag{2.20}$$

put $\vartheta = k$ in (2.15), we have

$$\frac{d}{dz}[(J_k^k(z))^2 + k(J_{2k}^k(z))^2] = \frac{2}{z}[(J_k^k(z))^2 - (2k)(J_{2k}^k(z))^2], \tag{2.21}$$

put $\vartheta = 2k$ in (2.15), we have

$$\frac{d}{dz}[(J_{2k}^k(z))^2 + k(J_{3k}^k(z))^2] = \frac{2}{z}[2(J_{2k}^k(z))^2 - (3k)(J_{3k}^k(z))^2],\ etc \tag{2.22}$$

adding (2.20), $k$ times (2.21), $k^2$ times (2.22) etc, we obtain

$$\frac{d}{dz}[(J_0^k(z))^2 + 2[k(J_k^k(z))^2 + k^2(J_{2k}^k(z))^2 + k^3(J_{3k}^k(z))^2 + ...]] = 0, \tag{2.23}$$

since $J_\vartheta^k(z) \Rightarrow 0$ as $\vartheta \Rightarrow \infty$.
Integrate (2.23), we obtain

$$(J_0^k(z))^2 + 2[k(J_k^k(z))^2 + k^2(J_{2k}^k(z))^2 + k^3(J_{3k}^k(z))^2 + ...] = c\,(constant), \tag{2.24}$$

putting $z = 0$ in (2.24), we have $c = 1$
there for (2.24) becomes

$$(J_0^k(z))^2 + 2[k(J_k^k(z))^2 + k^2(J_{2k}^k(z))^2 + k^3(J_{3k}^k(z))^2 + ...] = 1.$$

Which completes the proof. □

**Theorem 2.11.** *For $k \in R^{+}, \vartheta \in I$ and $\vartheta > -k$, then*

$$z = 2[J_0^k(z)J_k^k(z) + 3k^2 J_k^k(z)J_{2k}^k(z) + 5k^3 J_{2k}^k(z)J_{3k}^k(z) + ... + (2r+1)J_{rk}^k(z)J_{(r+1)k}^k(z) + ...]. \tag{2.25}$$

*Proof.* Put $\vartheta = 0$ in (2.17), we have

$$\frac{d}{dz}[zJ_0^k(z)J_k^k(z)] = z[\frac{1}{k}(J_0^k(z))^2 - (J_k^k(z))^2], \tag{2.26}$$

Put $\vartheta = k$ in (2.17), we have

$$\frac{d}{dz}[zJ_k^k(z)J_{2k}^k(z)] = z[\frac{1}{k}(J_k^k(z))^2 - (J_{2k}^k(z))^2], \tag{2.27}$$

Put $\vartheta = 2k$ in (2.17), we have

$$\frac{d}{dz}[zJ^k_{2k}(z)J^k_{3k}(z)] = z[\frac{1}{k}(J^k_{2k}(z))^2 - (J^k_{3k}(z))^2],\ etc \tag{2.28}$$

multiply (2.26) by $k$, (2.27) by $3k^2$, (2.28) by $5k^3$, etc. and adding, we obtain

$$\frac{d}{dz}[zJ^k_0(z)J^k_k(z) + 3zk^2J^k_k(z)J^k_{2k}(z) + 5zk^3J^k_{2k}(z)J^k_{3k}(z) + ...] =$$

$$z[(J^k_0(z))^2 + 2[k(J^k_k(z))^2 + k^2(J^k_{2k}(z))^2 + k^3(J^k_{3k}(z))^2 + ...]] \tag{2.29}$$

using (2.19), we have

$$\frac{d}{dz}[zJ^k_0(z)J^k_k(z) + 3zk^2J^k_k(z)J^k_{2k}(z) + 5zk^3J^k_{2k}(z)J^k_{3k}(z) + ...] = z, \tag{2.30}$$

integrate (2.30), we obtain

$$[zJ^k_0(z)J^k_k(z) + 3zk^2J^k_k(z)J^k_{2k}(z) + 5zk^3J^k_{2k}(z)J^k_{3k}(z) + ....] = \frac{z^2}{2} + c\,(constant), \tag{2.31}$$

putting $z = 0$ in (2.31), we have $c = 0$

$$z = 2[J^k_0(z)J^k_k(z) + 3k^2J^k_k(z)J^k_{2k}(z) + 5k^3J^k_{2k}(z)J^k_{3k}(z) + ... + (2r+1)J^k_{rk}(z)J^k_{(r+1)k}(z) + ...].$$

Which completes the proof. □

**Theorem 2.12.** *For $k \in R^+, \vartheta \in I$ and $\vartheta > -k$, then*

$$2^rk^r\frac{d^r}{dz^r}J^k_\vartheta(z) = \sum_{n=0}^{\infty}(-1)^nk^nJ^k_{\vartheta-rk+2nk}(z). \tag{2.32}$$

*Proof.* Differentiate (2.4) with respect to $z$ and multiply by $2k$, we have

$$2^2k^2\ddot{J}^k_\vartheta(z) = 2k\dot{J}^k_{\vartheta-k}(z) - k.2k\dot{J}^k_{\vartheta+k}(z), \tag{2.33}$$

substitute the value of $\dot{J}^k_{\vartheta-k}(z)$ and $\dot{J}^k_{\vartheta+k}(z)$ from (2.4), we obtain

$$2^2k^2\ddot{J}^k_\vartheta(z) = J^k_{\vartheta-2k}(z) - 2kJ^k_\vartheta(z) + k^2J^k_{\vartheta+2k}(z), \tag{2.34}$$

differentiate (2.34) with respect to $z$ and multiply by $2k$, we have

$$2^3k^3\dddot{J}^k_\vartheta(z) = 2k\dot{J}^k_{\vartheta-2k}(z) - 2k.2k\dot{J}^k_\vartheta(z) + k^22k\dot{J}^k_{\vartheta+2k}(z), \tag{2.35}$$

substitute the value of $\dot{J}^k_{\vartheta-2k}(z)$ , $\dot{J}^k_\vartheta(z)$ and $\dot{J}^k_{\vartheta+2k}(z)$ from (2.4), we obtain

$$2^3k^3\dddot{J}^k_\vartheta(z) = J^k_{\vartheta-3k}(z) - 3kJ^k_{\vartheta-k}(z) + 3k^2J^k_{\vartheta+k}(z) - k^3J^k_{\vartheta+3k}(z), \tag{2.36}$$

applying the same process again and again, we have

$$2^rk^r\frac{d^r}{dz^r}J^k_\vartheta(z) = \sum_{n=0}^{\infty}(-1)^nk^nJ^k_{\vartheta-rk+2nk}(z).$$

Which completes the proof. □

**Theorem 2.13.** *For $k \in R^+, \vartheta \in I$ and $\vartheta > -k$, then*

$$\frac{J^k_{\vartheta+k}}{J^k_\vartheta} = \cfrac{\frac{z}{2}}{(\vartheta+k) - \cfrac{k(\frac{z}{2})^2}{(\vartheta+2k) - \cfrac{k(\frac{z}{2})^3}{(\vartheta+3k) - \cfrac{k(\frac{z}{2})^4}{(\vartheta+4k) - \dots\dots\dots}}}} \tag{2.37}$$

*Proof.* From (2.5), we have

$$J^k_\vartheta(z) = \frac{2(\vartheta+k)}{z} J^k_{\vartheta+k}(z) - kJ^k_{\vartheta+2k}(z),$$

divide by $J^k_{\vartheta+k}(z)$

$$\frac{J^k_\vartheta}{J^k_{\vartheta+k}} = \frac{2(\vartheta+k)}{z} - k\frac{J^k_{\vartheta+2k}(z)}{J^k_{\vartheta+k}(z)}, \tag{2.38}$$

$$\frac{J^k_{\vartheta+k}}{J^k_\vartheta} = \cfrac{1}{\frac{2(\vartheta+k)}{z} - \cfrac{k}{\cfrac{J^k_{\vartheta+k}(z)}{J^k_{\vartheta+2k}(z)}}}, \tag{2.39}$$

put the value of $\dfrac{J^k_{\vartheta+k}(z)}{J^k_{\vartheta+2k}(z)}$ from (2.38), we obtain

$$\frac{J^k_{\vartheta+k}}{J^k_\vartheta} = \cfrac{1}{\frac{2(\vartheta+k)}{z} - \cfrac{k}{\frac{2(\vartheta+2k)}{z} - \cfrac{k}{\cfrac{J^k_{\vartheta+3k}(z)}{J^k_{\vartheta+4k}(z)}}}}, \tag{2.40}$$

continuing this process, we obtain

$$\frac{J^k_{\vartheta+k}}{J^k_\vartheta} = \cfrac{\frac{z}{2}}{(\vartheta+k) - \cfrac{k(\frac{z}{2})^2}{(\vartheta+2k) - \cfrac{k(\frac{z}{2})^3}{(\vartheta+3k) - \cfrac{k(\frac{z}{2})^4}{(\vartheta+4k) - \dots}}}}$$

Which completes the proof. □

**Particular Cases**

For $k = 1$, all the results given by equations (2.1), (2.2)(2.4), (2.5), (2.6), (2.8), (2.14), (2.15), (2.17), (2.19), (2.25), (2.32) and (2.37) will reduces for well known Recurrence Relations of Bessel's Function given by ([4], pages 110-111).

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