Thai Journal of Mathematics
Volume 14 (2016) Number 3 : 667–676
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

On a Semigroup of Sets of Transformations with Restricted Range

Ananya Anantayasethi[†] **and Joerg Koppitz**[‡,1]

[†]Mathematics Department, Science Faculty, Mahasarakham University
Kantarawichai, Mahasarakham 44150, Thailand
e-mail : ananya.a@msu.ac.th

[‡]Mathematics Institute, Potsdam University
14476 Potsdam, Germany
e-mail : koppitz@uni-potsdam.de

Abstract : This paper bases on the well-studied semigroup $T(X,Y)$ of all transformations on X with restricted range $Y \subseteq X$. We introduce the semigroup $T_P(X,Y)$ of all non-empty subsets of $T(X,Y)$ under the operation $AB := \{ab : a \in A, b \in B\}$. We determine the idempotent and regular elements in $T_P(X,Y)$ for the case that $|Y| = 2$. In particular, we characterize the (maximal) regular subsemigroups of $T_P(X,Y)$, the largest semiband, and the (maximal) idempotent subsemigroups of $T_P(X,Y)$.

Keywords : transformation semigroup with restricted range; semigroups; idempotent; regular.
2010 Mathematics Subject Classification : 20M14; 20M18; 20M20.

1 Introduction

Let $X = \{1, \ldots, n\}$, we denote by $T(X)$ the monoid of all full transformations on X (functions from X to X). The operation is the composition of functions. In the paper, we will write functions from the right, $x\alpha$ rather than $\alpha(x)$ and compose from the left to the right, $x(\alpha\beta) = (x\alpha)\beta$ rather than $(\alpha\beta)(x) = \alpha(\beta(x))$, $\alpha, \beta \in T(X), x \in X$.

[1]Corresponding author.

We denote by $im\alpha$ the image (the range) of α, i.e. $im\alpha := X\alpha := \{x\alpha : x \in X\}$ and by $rank\alpha$ the cardinality of $im\alpha$, i.e. $rank\alpha := |im\alpha|$. The kernel of α is the set $\ker\alpha := \{(x,y) : x, y \in X, x\alpha = y\alpha\}$. It is an equivalence relation and thus $\ker\alpha$ corresponds uniquely to a partition of X into blocks. The transformation α is called idempotent if $\alpha\alpha = \alpha$. Notice that α is idempotent if and only if α restricted to $im\alpha$ is the identity mapping on $im\alpha$. For more information about transformation semigroups and semigroups see [1] and [2], respectively.
By several reasons, it can happen that all transformations under consideration have a range in a proper subset of X. Let Y be a non-empty subset of X, say $Y = \{y_1, \dots, y_m\}$ for some $m \in \{1, \dots, n\}$. Note that $X = Y$ if $n = m$. Let us consider the set $T(X,Y) := \{\alpha \in T(X) : im\alpha \subseteq Y\}$. In particular, $T(X,Y)$ is a subsemigroup of $T(X)$, which is a semigroup of transformations with restricted range due to J. S. V. Symons [3]. Transformation semigroups with restricted range have been widely investigated (see for example [4, 5, 6, 7, 8, 9]). If Y is a one-element set, say $Y = \{y_1\}$, then $T(X,Y)$ is an one-element set, too, since the only transformation in $T(X,Y)$ is the constant mapping with the image y_1.
Let us now consider a two-element subset Y of X, say $Y = \{y_1, y_2\}$. Then, $T(X,Y)$ consists of the constant mapping with image y_1 (denoted by c_1), the constant mapping with image y_2 (denoted by c_2), and mappings with non-trivial kernel. For any $\alpha \in T(X,Y)$, we define a mapping $\alpha^* \in T(X,Y)$ by

$$x\alpha^* := \begin{cases} y_2 & \text{if} \quad x\alpha = y_1 \\ y_1 & \text{if} \quad x\alpha = y_2. \end{cases}$$

It is easy to verify that $c_1^* = c_2$, $(\alpha^*)^* = \alpha$, $\alpha\beta = \alpha$, and $\alpha\beta^* = \alpha^*$, whenever $\beta \in T(X,Y)$ is an idempotent with rank 2. We observe that $\alpha\beta = c_i$, whenever $y_1\beta = y_2\beta = c_i$ $(i \in \{1,2\})$. For any non-empty set $A \subseteq T(X,Y)$, we put $A^* := \{\alpha^* : \alpha \in A\}$.
This motivates the consideration of the following four subsets of $T(X,Y)$:

$$\begin{aligned} T_1 &:= \{\alpha \in T(X,Y) : \alpha \text{ is idempotent with rank } 2\}; \\ T_2 &:= \{\alpha^* : \alpha \in T_1\}; \\ T_{2+i} &:= \{\alpha \in T(X,Y) : y_1\alpha = y_2\alpha = c_i\} \text{ for } i \in \{1,2\}. \end{aligned}$$

Clearly, $T_4 = \{\alpha^* : \alpha \in T_3\}$. It is easy to verify that $\{T_1, T_2, T_3, T_4\}$ is a partition of $T(X,Y)$. For any non-empty set $A \subseteq T(X,Y)$ and any $i \in \{1,2,3,4\}$, we put

$$A_i := A \cap T_i.$$

Clearly, $A = A_1 \dot\cup A_2 \dot\cup A_3 \dot\cup A_4$.
Let $A, B \subseteq T(X,Y)$ be non-empty sets. If $B_1 \neq \emptyset$, i.e. there is an idempotent $\beta \in B$ with rank 2 such that $\alpha\beta = \alpha$ for all $\alpha \in A$, then $A \subseteq AB$ and $A = AB$ if $B = B_1$. If $B_3 \neq \emptyset$, i.e. there is $\beta \in B$ with $y_1\beta = y_2\beta = c_1$ and $\alpha\beta = c_1$ for all $\alpha \in A$, then $c_1 \in AB$ and $AB = \{c_1\}$ if $B = B_3$. By the same reasons, we have $c_2 \in AB$, whenever $B_4 \neq \emptyset$ and $AB = \{c_2\}$, whenever $B = B_4$. If $B \subseteq \{c_1, c_2\}$, then $AB = B$ since $\alpha c_i = c_i$ for all $i = 1, 2$ and all $\alpha \in A$. Finally if $B_2 \neq \emptyset$, i.e.

there is an idempotent β^* with rank 2 such that $\beta \in B$ and $\alpha\beta = \alpha^*$ for all $\alpha \in A$, then $A^* \subseteq AB$ and $A^* = AB$ if $B = B_2$. So, if $B_1 \neq \emptyset$ and $B_2 \neq \emptyset$, then the set $R_A := A \cup A^*$ is contained in AB and $R_A = AB$ if $B = B_1 \cup B_2$. The notation R_A is due to the Green's relation $\mathcal{R}$. Notice that, two transformations $\alpha, \beta \in T(X,Y)$ are $\mathcal{R}$-related, in symbols $\alpha\mathcal{R}\beta$, if and only if $\ker\alpha = \ker\beta$ [2]. We observe that $\ker\alpha = \ker\beta$ if and only if $\alpha = \beta$ or $\alpha = \beta^*$, i.e. the $\mathcal{R}$-class of any $\alpha \in T(X,Y)$ is $\{\alpha, \alpha^*\}$. Throughout this paper, we will use these observations without to refer to them.

Let $T_p(X,Y)$ be the set of all non-empty subsets of $T(X,Y)$ and let us establish $T_p(X,Y)$ with a binary operation $\cdot$ defined by

$$A \cdot B := \{ab : a \in A, b \in B\}$$

for $A, B \in T_p(X,Y)$. Clearly, $\cdot$ is associative. We will write AB rather than $A \cdot B$. Notice the semigroup $T(X,Y)$ can be isomorphic embedded into $T_P(X,Y)$ by $\Phi : T(X,Y) \to T_P(X,Y)$ with $\alpha \mapsto \{\alpha\}$ for $\alpha \in T(X,Y)$. The purpose of this paper is the study of several basic properties of the semigroup $T_P(X,Y)$.

The next section deals with the set E of all idempotents in $T_P(X,Y)$, i.e. with elements $A \in T_P(X,Y)$ satisfying $AA = A$. It is easy to see that $T(X,Y)$ is not a band (take any $\beta \in T_1$ and we have $\beta^*\beta^* = (\beta^*)^* = \beta$). Therefore, $T_P(X,Y)$ is also not a band since $T(X,Y)$ can be isomorphically embedded into $T_P(X,Y)$. We will determine the set E and characterize all maximal subsemigroups consisting entirely of idempotents. Moreover, we determine the semigroup generated by E, i.e. the greatest semiband within $T_P(X,Y)$. The third section is devoted to the regular elements. A set $A \in T_P(X,Y)$ is called regular if there is $B \in T_P(X,Y)$ such that $ABA = A$. We characterize the set of all regular elements in $T_P(X,Y)$. It is a proper subset of $T_P(X,Y)$, i.e. $T_P(X,Y)$ is not regular. But we can provide all maximal regular subsemigroups of $T_P(X,Y)$. Finally, we will obtain the least subsemigroup of $T_P(X,Y)$ containing all regular elements.

2 The Idempotent Elements

Note that an idempotent element $A \in E$ of $T_P(X,Y)$ is a subsemigroup of $T(X,Y)$, i.e. $A \leq T(X,Y)$, since $AA = A$. But conversely, not each subsemigroup of $T(X,Y)$ is idempotent in $T_P(X,Y)$. For example, T_4 is a subsemigroup of $T(X,Y)$ but $T_4 \notin E$ since $T_4T_4 = \{c_2\} \subsetneqq T_4$. The following proposition characterizes all the idempotents in $T_P(X,Y)$.

Proposition 2.1. *Let $A \in T_P(X,Y)$. Then $A \in E$ if and only if the following three conditions are satisfied:*
(i) $R_A = A$ (i.e. $A = A^*$) if $A_2 \neq \emptyset$.
(ii) $A \subseteq \{c_1, c_2\}$ if $A_1 \cup A_2 = \emptyset$.
(iii) $c_i \in A$ if $A_{2+i} \neq \emptyset$, $i = 1, 2$.

Proof. Suppose that (i), (ii), and (iii) are satisfied.
Assume that $A_1 \neq \emptyset$. Then $A \subseteq AA$. Let now $\alpha \in AA$. Then there are $\alpha_1, \alpha_2 \in A$

with $\alpha = \alpha_1\alpha_2$. If $\alpha_2 \in A_1$ then $\alpha = \alpha_1\alpha_2 = \alpha_1 \in A$. If $\alpha_2 \in A_2$ then $A_2 \neq \emptyset$ and $\alpha = \alpha_1\alpha_2 = \alpha_1^* \in A^* = A$ by (i). Let $i \in \{1, 2\}$. If $\alpha_2 \in A_{2+i}$ then $A_{2+i} \neq \emptyset$ and $\alpha = \alpha_1\alpha_2 = c_i \in A$ by (iii).
Admit that $A_1 = \emptyset$. Then $A_2 = \emptyset$ by (i), i.e. $A_1 \cup A_2 = \emptyset$. Thus, $A \subseteq \{c_1, c_2\}$ by (ii) and we have $AA = A$. Suppose now that $AA = A$ and we have to show that (i), (ii), and (iii) are satisfied. Admit that $A_2 \neq \emptyset$. Then $A^* \subseteq AA = A$ and thus $A = (A^*)^* \subseteq A^*$. This shows that $A = A^*$ and we have (i). Admit that $A_1 \cup A_2 = \emptyset$, i.e. $A \subseteq A_3 \cup A_4$. Then $A = AA \subseteq \{c_1, c_2\}$. This shows (ii). Let $i \in \{1, 2\}$ and suppose that $A_{2+i} \neq \emptyset$. Then we obtain $c_i \in AA = A$, i.e. we have shown (iii). □

By Proposition 2.1, it is easy to verify that the following both sets D_1 and D_2 are subsets of E:

$$\begin{aligned} D_1 \quad &: \quad = \{R_A : \emptyset \neq A \subseteq T_1\} \text{ and} \\ D_2 \quad &: \quad = \{A \cup B \cup \{c_i\} : \emptyset \neq A \subseteq T_1, B \subseteq T_{2+i}, i = 1, 2\}. \end{aligned}$$

Lemma 2.2. *We have $D_1D_2 \cap E = \emptyset$.*

Proof. Let $A \in D_1$ and $A' \in D_2$. Then there are a non-empty set $\widehat{A} \subseteq T_1$ and a set $\widehat{B} \subseteq T_{2+i}$ such that $A' = \widehat{A} \cup \widehat{B} \cup \{c_i\}$ for some $i \in \{1, 2\}$. This gives $AA' = A \cup \{c_i\}$. We have $(AA')_2 \neq \emptyset$ (since $A_2 \neq \emptyset$) but $c_i^* \notin AA'$ (since $A \subseteq T_1 \cup T_2$). Thus, $AA' \notin E$ by Proposition 2.1. □

Lemma 2.2 shows that $T_P(X, Y)$ is not orthodox, i.e., its idempotent set does not form a subsemigroup and it arises the question for the (maximal) idempotent subsemigroups of $T_P(X, Y)$. A semigroup S is a maximal idempotent subsemigroup of $T_P(X, Y)$ if $S \subseteq E$ and each subsemigroup of $T_P(X, Y)$, which covers S properly, consists not entirely of idempotents. Let us put

$$E_1 := E \setminus D_1.$$

Lemma 2.3. *E_1 is a maximal idempotent subsemigroup of $T_P(X, Y)$.*

Proof. Let $A, B \in E_1$.
Suppose that $A_1 \neq \emptyset$ and $B_1 \neq \emptyset$. Since both A and B do not belong to D_1, we have $A_2 = \emptyset$ or $\{c_1, c_2\} \subseteq A$ as well as $B_2 = \emptyset$ or $\{c_1, c_2\} \subseteq B$. If $(AB)_2 = \emptyset$ then $B_2 = \emptyset$ since $A_1 \neq \emptyset$. Thus, $AB = A \cup C$, where $C \subseteq \{c_1, c_2\}$. Notice that $A_2 = \emptyset$ since $(AB)_2 = \emptyset$ and $AB = A \cup C$. Thus AB is an idempotent by Proposition 2.1. Because $(AB)_2 = \emptyset$, we have $AB \in E_1$. If $(AB)_2 \neq \emptyset$ then $A_2 \neq \emptyset$ or $B_2 \neq \emptyset$, i.e. $\{c_1, c_2\} \subseteq A$ (if $A_2 \neq \emptyset$) or $\{c_1, c_2\} \subseteq B$ (if $B_2 \neq \emptyset$). This provides $\{c_1, c_2\} \subseteq AB$. In both cases, we obtain $AB = R_A \cup \{c_1, c_2\}$. Thus AB is an idempotent by Proposition 2.1, and in particular, $AB \in E_1$.
Suppose that $A_1 = \emptyset$ or $B_1 = \emptyset$. Then $A \subseteq \{c_1, c_2\}$ (if $A_1 = \emptyset$, i.e. $A_1 \cup A_2 = \emptyset$) or $B \subseteq \{c_1, c_2\}$ (if $B_1 = \emptyset$, i.e. $B_1 \cup B_2 = \emptyset$) by Proposition 2.1. Hence, $AB \subseteq \{c_1, c_2\}$, i.e. AB is idempotent by Proposition 2.1 and in particular, $AB \in E_1$.
We have shown that E_1 is a semigroup and it remains to show that E_1 is maximal. But this fact becomes clear by Lemma 2.2 and the fact $D_2 \subseteq E_1$. □

Now we put

$$E_2 := E \setminus D_2.$$

It is easy to check that $R_A, R_A \cup \{c_1, c_2\} \in E_2$, for any $A \subseteq T_1 \cup T_2$.

Lemma 2.4. *E_2 is a maximal idempotent subsemigroup of $T_P(X,Y)$.*

Proof. Let $A, B \in E_2$. If $A \subseteq \{c_1, c_2\}$ or $B \subseteq \{c_1, c_2\}$, then $AB \subseteq \{c_1, c_2\}$, and $AB \in E_2$. Suppose now that $A, B \nsubseteq \{c_1, c_2\}$. Then $\{c_1, c_2\} \subseteq A$ or $A \cap \{c_1, c_2\} = \emptyset$ and the same for B.
Suppose that $A_1 \neq \emptyset$ and $B_1 \neq \emptyset$. Then $AB = A \cup A^* \cup C$ (if $B_2 \neq \emptyset$) or $AB = A \cup C$ (if $B_2 = \emptyset$), where $C = \emptyset$ or $C = \{c_1, c_2\}$. Since $A \in E_2$, it is easy to verify that AB is idempotent by Proposition 2.1 . Clearly, $A \cup A^* \cup C \notin D_2$ and $A \cup C \notin D_2$. Thus, $AB \in E_2$.
Suppose now that $B_1 = \emptyset$. Then $B \subseteq \{c_1, c_2\}$ by Proposition 2.1 and thus $AB = B \in E_2$.
Suppose that $A_1 = \emptyset$ but $B_1 \neq \emptyset$. Then $A \subseteq \{c_1, c_2\}$ by Proposition 2.1 and thus $AB = A \in E_2$. or $AB = \{c_1, c_2\}$. Notice that $\{c_1, c_2\} \in E_2$. Therefore, we have $AB \in E_2$.
So, we have shown that E_2 is a semigroup. It remains to show that E_2 is maximal, which is clear by Lemma 2.2 and the fact that $D_1 \subseteq E_2$. □

Theorem 2.5. *Let $S \subseteq E$. Then S is a maximal idempotent subsemigroup of $T_P(X,Y)$ if and only if $S = E_1$ or $S = E_2$.*

Proof. One direction is clear by Lemma 2.3 and Lemma 2.4. Suppose now that S is a maximal idempotent subsemigroup. Since $S \subset E$, we obtain $S \subseteq E \setminus D_1 = E_1$ or $S \subseteq E \setminus D_2 = E_2$ by Lemma 2.2. Hence, $S = E_1$ or $S = E_2$ because of the maximality of S. □

Finally, we determine the greatest semiband in $T_P(X,Y)$. Let

$$\begin{aligned} D_3 &:= \{R_A \cup \{c_i\} : \emptyset \neq A \subseteq T_1, i = 1,2\} \text{ and} \\ E_3 &:= E \cup D_3. \end{aligned}$$

Proposition 2.6. *E_3 is the greatest semiband in $T_P(X,Y)$.*

Proof. We have to show that E_3 is the least subsemigroup of $T_P(X,Y)$ containing E.
Let $A \subseteq T_1$ and let $i \in \{1,2\}$. Then $R_A(T_1 \cup \{c_i\}) = R_A \cup \{c_i\}$, where both elements R_A and $(T_1 \cup \{c_i\})$ are idempotent. This shows that D_3 belongs to the least subsemigroup of $T_P(X,Y)$ containing E.
For the converse direction, we have to show that the product of elements in E_3 belongs to E_3. First, we check $EE \subseteq E_3$. By Lemma 2.3, Lemma 2.4 and the fact that $D_1 \cap D_2 = \emptyset$, it is enough to check the case that one factor belongs to D_1 and the other factor belongs to D_2. For this let $\emptyset \neq A \subseteq T_1$, $k \in \{1,2\}$, and $B \subseteq T_{k+2}$. Further, let $R_G \in D_1$, where $\emptyset \neq G \subseteq T_1$. Then we get that

$(A \cup B \cup \{c_k\})R_G = R_{A\cup B\cup\{c_k\}} \in E$ and $R_G(A \cup B \cup \{c_k\}) = R_G \cup \{c_k\} \in D_3$. It is easy to verify that the product of an idempotent with an element from D_3 is equal to a product $e_1e_2e_3$ of three idempotents e_1, e_2, e_3 and thus, it is equal to a product of two idempotents (if e_1e_2 or e_2e_3 is an idempotent) or of two elements in D_3 (if both e_1e_2 and e_2e_3 are in D_3). So, it remains to show that $D_3D_3 \subseteq E$. Indeed, let $\emptyset \neq A, A' \subseteq T_1$ and $i, i' \in \{1, 2\}$. Then $(R_A \cup \{c_i\})(R_{A'} \cup \{c_{i'}\}) = R_A \cup \{c_1, c_2\} \in E$. □

3 The Regular Elements

This section is devoted to the regular subsemigroups of $T_p(X, Y)$. Clearly, each idempotent is regular. Hence, we have still to find the regular elements in $T_p(X, Y)$ which are not idempotent. For this let

$$\widehat{T}_{i+2} := \{A \subseteq T_{i+2} : c_i \in A\} \cup \{\emptyset\} \text{ for } i = 1, 2.$$

Lemma 3.1. *If $\emptyset \neq A \subseteq T_2$, $B \in \widehat{T}_3$, and $C \in \widehat{T}_4$ then $A \cup B \cup C$ is regular in $T_p(X, Y)$.*

Proof. We can calculate $(A \cup B \cup C)T_2 = A^* \cup B^* \cup C^*$ and $(A^* \cup B^* \cup C^*)A = A \cup B \cup C$. Because of $(A^* \cup B^* \cup C^*)B = \{c_1\} \subseteq B$ if $B \neq \emptyset$ and $(A^* \cup B^* \cup C^*)C = \{c_2\} \subseteq C$ if $C \neq \emptyset$, we obtain $(A \cup B \cup C)T_2\ (A \cup B \cup C) = A \cup B \cup C$. Therefore, $A \cup B \cup C$ is regular in $T_p(X, Y)$. □

We observe that, if $\emptyset \neq A \subseteq T_2$, $B \in \widehat{T}_3$, and $C \in \widehat{T}_4$, then $A \cup B \cup C$ is not idempotent by Proposition 2.1 . Hence the set

$$D_4 := \{A \cup B \cup C : \emptyset \neq A \subseteq T_2, B \in \widehat{T}_3, C \in \widehat{T}_4\}$$

is a set of non-idempotent regular elements in $T_p(X, Y)$. Moreover, we have:

Lemma 3.2. *If $A \in T_p(X, Y)$ is regular then $A \in E \cup D_4$.*

Proof. Let $A \in T_p(X, Y)$ be regular. Then there is $B \in T_p(X, Y)$ such that $ABA = A$.
If $A_3 \neq \emptyset$ then $c_1 \in ABA = A$. This shows that $A_3 \in \widehat{T}_3$. By the same reason, we obtain that $A_4 \in \widehat{T}_4$ if $A_4 \neq \emptyset$.
Suppose now that $A_2 = \emptyset$ and $A_1 = \emptyset$. Then $A \subseteq T_3 \cup T_4$ and $A = ABA \subseteq \{c_1, c_2\}$, i.e. $A \in E$ by Proposition 2.1. Suppose that $A_2 = \emptyset$ and $A_1 \neq \emptyset$. Then by the previous observations concerning A_3 and A_4, we obtain $A \in E$ by Proposition 2.1. Admit now that $A_2 \neq \emptyset$. Clearly, then $B_1 \cup B_2 \neq \emptyset$. If $A_1 \neq \emptyset$ then $(BA)_2 \neq \emptyset$. Thus, $A^* \subseteq ABA = A$, i.e. $A = A^*$ (it follows from $A = (A^*)^* \subseteq A^*$) and we obtain $A \in E$ by Proposition 2.1. If $A_1 = \emptyset$ then $A = A_2 \cup A_3 \cup A_4 \in D_4$. □

Proposition 3.3. *Any $A \in T_p(X, Y)$ is regular if and only if $A \in E \cup D_4$.*

Proof. Lemma 3.1 and Lemma 3.2 give the assertion. □

It is easy to verify that $D_3 \cap D_4 = \emptyset$. Hence, the regular elements do not form a semigroup. We are asking for the maximal regular subsemigroups of $T_p(X,Y)$. A semigroup S is a maximal regular subsemigroup of $T_P(X,Y)$ if S is regular and each subsemigroup of $T_P(X,Y)$, which covers S properly, is not regular.

Lemma 3.4. *$E_1 \cup D_4$ is a semigroup.*

Proof. We have to show that the product of two elements in $E_1 \cup D_4$ belongs to $E_1 \cup D_4$. By Lemma 2.3, it is enough to verify the case that at least one of the factors belongs to D_4. For this let $G \in E_1$, $\emptyset \neq A, A' \subseteq T_2$, $B, B' \in \widehat{T}_3$, and $C, C' \in \widehat{T}_4$. Then $(A \cup B \cup C)(A' \cup B' \cup C') = A^* \cup B^* \cup C^* \cup D$ with $D \subseteq \{c_1, c_2\}$, where $A^* \cup B^* \cup C^* \cup D \in E$ by Proposition 2.1. Because $(A^* \cup B^* \cup C^* \cup D) \cap T_2 = \emptyset$, we have $A^* \cup B^* \cup C^* \cup D \in E_1$.
Suppose that $G_2 \neq \emptyset$. Then $G = G^*$ and $\{c_1, c_2\} \subseteq G$. This implies $(A \cup B \cup C)G = R_{A \cup B \cup C} \cup \{c_1, c_2\} \in E_1$ and $G(A \cup B \cup C) = G^* = G \in E_1$.
Suppose that $G_2 = \emptyset$. If $G_1 = \emptyset$, then $G \subseteq \{c_1, c_2\}$ and thus $(A \cup B \cup C)G, G(A \cup B \cup C) \subseteq \{c_1, c_2\}$, i.e. $(A \cup B \cup C)G, G(A \cup B \cup C) \in E_1$. Admit now that $G_1 \neq \emptyset$. Then $(A \cup B \cup C)G = A \cup B \cup C \cup D'$ with $D' := (A \cup B \cup C)(G_3 \cup G_4) \subseteq \{c_1, c_2\}$. Let $B' := B \in \widehat{T}_3$ if $c_1 \notin D'$ and let $B' := B \cup \{c_1\} \in \widehat{T}_3$ if $c_1 \in D'$. In the same matter, we define $C' \in \widehat{T}_4$. Then $A \cup B \cup C \cup D' = A \cup B' \cup C' \in D_4$. Finally, we have $G(A \cup B \cup C) = G^* \cup ((B \cup C) \cap \{c_1, c_2\})$. Notice, we have $G^* = G_1^* \cup G_3^* \cup G_4^*$, where $\emptyset \neq G_1^* \subseteq T_2$ and $G_3^* \in \widehat{T}_4$ as well as $G_4^* \in \widehat{T}_3$. Thus, $G(A \cup B \cup C) = G^* \cup ((B \cup C) \cap \{c_1, c_2\}) \in D_4$ by the same argumentation as above. □

Proposition 3.5. *$E_1 \cup D_4$ is a maximal regular subsemigroup of $T_p(X,Y)$.*

Proof. $E_1 \cup D_4$ is a semigroup by Lemma 3.4. This semigroup is regular since for any $A \in D_4$, we have $AT_2A = A$ (see the proof of Lemma 3.1), where $T_2 \in D_4$.
It remains to show that $E_1 \cup D_4$ is maximal. It is easy to see that D_1 is the set of all regular elements in $T_p(X,Y)$ which not belong to $E_1 \cup D_4$. By Lemma 2.2 and Proposition 2.6, we have $D_1 D_2 \subseteq D_3$, where $D_2 \subseteq E_1$ and $D_3 \cap (E \cup D_4) = \emptyset$. This shows that $E_1 \cup D_4$ is a maximal regular subsemigroup of $T_p(X,Y)$ by Lemma 3.2. □

Let us denote by D_5 the set of all $A \in D_4$ with $A_3 \neq \emptyset$ if and only if $A_4 \neq \emptyset$.

Lemma 3.6. *$E_2 \cup D_5$ is a semigroup.*

Proof. We have to show that the product of two elements in $E_2 \cup D_5$ belongs to $E_2 \cup D_5$ again. It is enough to verify the case that at least one of the both factors belongs to D_5. For this let $G \in E_2$, $\emptyset \neq A, A' \subseteq T_2$, $B, B' \in \widehat{T}_3$, and $C, C' \in \widehat{T}_4$ such that $B \neq \emptyset$ if and only if $C \neq \emptyset$ and $B' \neq \emptyset$ if and only if $C' \neq \emptyset$. Then $(A \cup B \cup C)(A' \cup B' \cup C') = A^* \cup B^* \cup C^*$ or $(A \cup B \cup C)(A' \cup B' \cup C') = A^* \cup B^* \cup C^* \cup \{c_1, c_2\}$, where $A^* \subseteq T_1$, $C^*, C^* \cup \{c_1\} \in \widehat{T}_3$, and $B^*, B^* \cup \{c_2\} \in \widehat{T}_4$ such that $B^* \neq \emptyset$ if and only if $C^* \neq \emptyset$. Then, $(A \cup B \cup C)(A' \cup B' \cup C')$ is

idempotent by Proposition 2.1 and moreover, $(A\cup B\cup C)(A'\cup B'\cup C')\in E_2$ since $B^*\neq\emptyset$ if and only if $C^*\neq\emptyset$.
If $G_1=\emptyset$, then $G\subseteq\{c_1,c_2\}$. It follows that $G(A\cup B\cup C),(A\cup B\cup C)G\subseteq\{c_1,c_2\}$. Therefore, $G(A\cup B\cup C),(A\cup B\cup C)G\in E_2$. If $G_1\neq\emptyset$, we notice that $G_3\neq\emptyset$ if and only if $G_4\neq\emptyset$. It follows that

$$G(A\cup B\cup C)=\begin{cases} G^* & \text{if } B\cup C=\emptyset \\ G^*\cup\{c_1,c_2\} & \text{otherwise.}\end{cases}$$

We observe that $G^*=G$ (if $G_2\neq\emptyset$) and $G^*\subseteq T_2\cup T_3\cup T_4$ such that $(G^*)_2\neq\emptyset$ and $(G^*)_3\neq\emptyset$ if and only if $(G^*)_4\neq\emptyset$ (if $G_2=\emptyset$). Therefore, $G(A\cup B\cup C)\in E_2$ if $G_2\neq\emptyset$ and $G(A\cup B\cup C)\in D_5$ if $G_2=\emptyset$. On the other hand, we have

$$(A\cup B\cup C)G=\begin{cases} A\cup B\cup C\cup\{c_1,c_2\} & \in D_5 \quad \text{if} \quad G_2=\emptyset \text{ and } G_3\neq\emptyset \\ A\cup B\cup C & \in D_5 \quad \text{if} \quad G_2=\emptyset \text{ and } G_3=\emptyset \\ R_{A\cup B\cup C} & \in E_2 \quad \text{if} \quad G_2\neq\emptyset \text{ and } G_3=\emptyset \\ R_{A\cup B\cup C}\cup\{c_1,c_2\} & \in E_2 \quad \text{if} \quad G_2\neq\emptyset \text{ and } G_3\neq\emptyset.\end{cases}$$

This shows that $(A\cup B\cup C)G\in E_2\cup D_5$. ☐

Proposition 3.7. *$E_2\cup D_5$ is a maximal regular subsemigroup of $T_p(X,Y)$.*

Proof. By Lemma 3.6, we know that $E_2\cup D_5$ is a semigroup. Since $T_2\in D_5$ and $AT_2A=A$ for all $A\in D_5\subseteq D_4$ (see the proof of Lemma 3.1), the semigroup $E_2\cup D_5$ is regular.
It remains to show that $E_2\cup D_5$ is maximal. We show that any semigroup which covers $E_2\cup D_5$ properly, contains non-regular elements. For this let A be a regular element, which not belongs to $E_2\cup D_5$. It is easy to verify that $A\in D_2\cup(D_4\setminus D_5)$. Suppose that $A\in D_2$. Then $BA\in D_3$ for all $B\in D_1$ by Lemma 2.2, where $D_1\subseteq E_2$ and $D_3\cap D_4=\emptyset$, i.e. BA is not regular for all $B\in D_1\subseteq E_2$.
Suppose now that $A\in D_4\setminus D_5$. Then there are $\emptyset\neq A'\subseteq T_2$ and $B\in\widehat{T}_{2+k}$ for some $k\in\{1,2\}$ such that $A=A'\cup B$. We have $T_2\in D_5$, we calculate $AT_2=(A')^*\cup B^*$, where $B^*\in\begin{cases}\widehat{T}_3 & \text{if} \quad k=2\\ \widehat{T}_4 & \text{if} \quad k=1\end{cases}$ and $\emptyset\neq(A')^*\subseteq T_1$, i.e. $AT_2\in D_2$. ☐

Theorem 3.8. *Let $S\leq T_p(X,Y)$. Then S is a maximal regular subsemigroup of $T_p(X,Y)$ if and only if $S=E_1\cup D_4$ or $S=E_2\cup D_5$.*

Proof. One direction is clear by Proposition 3.5 and Proposition 3.7. Suppose now that S is a maximal regular subsemigroup of $T_p(X,Y)$. By Lemma 3.2, we have $S\subseteq E\cup D_4$. Assume that $S\nsubseteq E_1\cup D_4$ and $S\nsubseteq E_2\cup D_5$. Then there are $A\in S\setminus(E_1\cup D_4)$ and $B\in S\setminus(E_2\cup D_5)$. Since $A,B\in E\cup D_4$, we have $A\in D_1$ and $B\in D_2\cup(D_4\setminus D_5)$. If $B\in D_2$ then $AB\in D_3$ by Lemma 2.2.
Admit now that $B\in D_4\setminus D_5$. Then there are $\emptyset\neq C'\subseteq T_2$ and $C\in\widehat{T}_{2+i}$ for some $i\in\{1,2\}$ such that $B=C\cup C'$. Since $A^*=A=R_{A_1}$ we can calculate $AB=A\cup\{c_i\}$, i.e. $AB\in D_3$, too. But $D_3\cap(E\cup D_4)=\emptyset$. This contradicts $AB\in S\subseteq E\cup D_4$. Consequently, $S\subseteq E_1\cup D_4$ or $S\subseteq E_2\cup D_5$. Finally, the maximality of S provides $S=E_1\cup D_4$ or $S=E_2\cup D_5$. ☐

Finally, we determine the least semigroup containing all regular elements in $T_p(X,Y)$, i.e. the least semigroup containing $E \cup D_4$.

Proposition 3.9. *The least semigroup containing all regular elements in $T_p(X,Y)$ is $E \cup D_3 \cup D_4$.*

Proof. Notice, $E \cup D_3$ is the greatest semiband in $T_p(X,Y)$ by Proposition 2.6. Therefore, it is clear that the least semigroup containing all regular elements of $T_p(X,Y)$ covers $E \cup D_3 \cup D_4$. So, it remains to show that $E \cup D_3 \cup D_4$ is a semigroup. Notice that $E_1 \cup D_4$ is a regular semigroup by Lemma 3.4. Thus, it is enough to check that D_1D_4 and D_4D_1 as well as D_3D_4 and D_4D_3 are subsets of $E \cup D_3 \cup D_4$.
Let $R_{A'} \in D_1$ with $\emptyset \neq A' \subseteq T_1$ and let $A \cup B \cup C \in D_4$ with $\emptyset \neq A \subseteq T_2$, $B \in \widehat{T}_3$, and $C \in \widehat{T}_4$. Then $R_{A'}(A \cup B \cup C) = \begin{cases} R_{A'} \cup \{c_1\} & \in D_3 \quad \text{if } B \neq \emptyset \text{ and } C = \emptyset \\ R_{A'} \cup \{c_2\} & \in D_3 \quad \text{if } B = \emptyset \text{ and } C \neq \emptyset \\ R_{A'} \cup \{c_1, c_2\} & \in E \quad \text{if } B \neq \emptyset \text{ and } C \neq \emptyset \\ R_{A'} & \in E \quad \text{if } B = \emptyset \text{ and } C = \emptyset. \end{cases}$
This shows that $D_1D_4 \in E \cup D_3 \subseteq E \cup D_3 \cup D_4$. On the other hand, we have $(A \cup B \cup C)R_{A'} = R_{A \cup B \cup C}$. By Proposition 2.1, we can check that $R_{A \cup B \cup C}$ is idempotent. This provides $D_4D_1 \subseteq E \subseteq E \cup D_3 \cup D_4$. Let additional $i \in \{1,2\}$. Then $\{c_i\}(A \cup B \cup C) = D$ for some $\emptyset \neq D \subseteq \{c_1, c_2\}$ and $(A \cup B \cup C)\{c_i\} = \{c_i\}$. Hence, $(R_{A'} \cup \{c_i\})(A \cup B \cup C) = R_{A'}(A \cup B \cup C) \cup D$ and $(A \cup B \cup C)(R_{A'} \cup \{c_i\}) = (A \cup B \cup C)R_{A'} \cup \{c_i\}$. By the previous observations, we obtain $(R_{A'} \cup \{c_i\})(A \cup B \cup C) = R_{A'} \cup D' \in E \cup D_3$, where $D \subseteq D' \subseteq \{c_1, c_2\}$, and $(A \cup B \cup C)(R_{A'} \cup \{c_i\}) = R_{A \cup B \cup C} \cup \{c_i\} \in E \cup D_3$. So, we have shown that $D_3D_4, D_4D_3 \subseteq E \cup D_3$. □

Acknowledgements : We would like to thank the referees for their comments and suggestions on the manuscript. We also would like to thank the Faculty of Science and Division of the Development Fund of Mahasarakham University, Thailand for the financial support.

References

[1] J.M. Howie, Fundamentals in Semigroup Theory, Oxford University Press, London, 2003.

[2] O. Ganyushkin, V. Mazorchuk, Classical Finite Transformation Semigroups-An Introduction, Series: Algebra and Applications, Springer-Verlag, 2008.

[3] J.S.V. Symons, Some Results Concerning a Transformation Semigroups, Australian Mathematical Society, Series A (19) 4 (1975) 413-425.

[4] V.H. Fernandes, P. Honyam, T.M. Quinteiro, B. Singha, On Semigroups of Orientation-preserving Transformations with Restriced Range, Communications in Algebra 44 (1) (2016) 253-264.

[5] V.H. Fernandes, J. Sanwong, On the Ranks of Semigroups of Transformations on a Finite Set with Restricted Range, Algebra Colloq 21 (3) (2014) 497-510.

[6] S. Mendes-Goncalves, R.P. Sullivan, The Ideal Structure of Transformation Semigroups with Restricted Range, Bull. Aust. Math. Soc. 83 (2001) 289-300.

[7] S. Nenthein, P. Youngkhong, Y. Kemprasit, The Regular Part of Some Transformation Semigroups, Pure Math. Appl. 16 (3) (2005) 307-314.

[8] J. Sanwong, W. Sommanee, Regularrity and Green's Relations of Transformation Semigroups with Restricted Range, Int. Journal of Mathematics and Mathematical Sciences (2008) doi:10.1155/2008/794013.

[9] R.P. Sullivan, Semigroups of Linear Transformations with Restricted Range, Bull. Austral. Math. Soc. 77 (2008) 441-453.

(Received 4 August 2016)
(Accepted 20 December 2016)