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On Compact Operators on Some Sequence Spaces Related to Matrix $B(r,s,t)$

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Abstract : In the present paper, we establish some identities or estimates for the operator norms and the Hausdorff measures of noncompactness of certain matrix operators on the spaces $c_0(B), \ell_\infty(B)$ and $\ell_p(B)$ which have recently been introduced in [Some new sequence spaces derived by the domain of the triple band matrix, Comput. Math. Appl. 62 (2011) 641-650]. Further, by using the Hausdorff measure of noncompactness, we characterize some classes of compact operators on these spaces.

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1 Preliminaries, Background and Notation

By e and $e^{(n)}$ $(n = 0, 1, 2, ...)$, we denote the sequences such that $e_k = 1$ for $k = 0, 1, ...,$ and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ for $k \neq n$.

An FK space is a complete linear metric sequence space with the property that convergence implies that coordinatewise convergence; a BK space is normed FK

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space. A BK space $X \supset \phi$ is said to have AK if every sequence $x = (x_k) \in X$ has a unique representation $x = \sum_{n=0}^{\infty} x_n e^{(n)}$.

Let ω be the set of all complex sequences, ϕ be the set of all finite sequences and X and Y be subsets of ω. We write ℓ_∞, c and c_0 for the sets of all bounded, convergent and null sequences. The sequence spaces ℓ_∞, c and c_0 are BK-spaces with usual sup-norm given by $\|x\|_{\ell_\infty} = \sup_k |x_k|$, where the supremum is taken over all $k \in \mathbb{N}$. Also, we write

$$\ell_1 = \left\{ x = (x_k) \in \omega : \sum_{k=0}^{\infty} |x_k| < \infty \right\}$$

and the space ℓ_1 is a BK space with the usual ℓ_1-norm defined by $\|x\|_{\ell_1} = \sum_{k=0}^{\infty} |x_k|$ [1].

By (X, Y) we denote the set of all matrices that map X into Y. Let X be a normed space. Then, we write S_X and B_X for the unit sphere and the closed unit ball in X, that is, $S_X = \{x \in X : \|x\| = 1\}$ and $B_X = \{x \in X : \|x\| \leq 1\}$. If X and Y are Banach spaces, then $B(X, Y)$ denotes the set of all bounded linear operators $L : X \to Y$. If we denote by $A = (a_{nk})_{n,k=0}^{\infty}$ an infinite matrix with complex entries and by A_n its nth row, we write

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k \quad \textit{and} \quad A(x) = (A_n(x))_{n=0}^{\infty}; \tag{1.1}$$

then $A \in (X, Y)$ if and only if $A_n(x)$ converges for all $x \in X$ and all n and $A(x) \in Y$. Furthermore,

$$X^\beta = \left\{ a \in \omega : \sum_k a_k x_k \text{ converges for all } x \in X \right\}$$

denotes the β- dual of X. The set

$$X_A = \{x \in \omega : A(x) \in X\} \tag{1.2}$$

is called the *matrix domain* of A in X.

If $X \supset \phi$ is a BK space and $a \in \omega$ we write

$$\|a\|_X^* = \sup \left\{ \left| \sum_{k=0}^{\infty} a_k x_k \right| : \|x\| = 1 \right\}. \tag{1.3}$$

Throughout, let $T = (t_{nk})_{n,k=0}^{\infty}$ be a triangle, that is $t_{nk} = 0$ for $k > n$ and $t_{nn} \neq 0$ $(n = 0, 1, 2, ...)$, and S be its inverse and $\mathcal{F}$ be a finite subset of $\mathbb{N}$.

The theory of FK spaces is the most important tool in the characterization of matrix transformations between certain sequence spaces. The most important result is that matrix transformations between FK spaces are continuous. It is quite natural to find conditions for a matrix map between FK spaces to define a compact operator. This can be achieved by applying the Hausdorff measure of noncompactness. The Hausdorff measure of noncompactness was defined by Goldenštein, Gohberg and Markus in 1957, later studied by Goldenštein and Markus in 1968 [2].

Let X and Y be Banach spaces. A linear operator $L : X \to Y$ is called *compact* if its domain is all of X and for every bounded sequence $(x_n)_{n=0}^{\infty}$ in X, the sequence $(L(x_n))_{n=0}^{\infty}$ has a convergent subsequence in Y. We denote the class of such operators by $K(X, Y)$.

Here, we will recall some basic definitions and results. More results about measure of noncompactness can be found in [2, 3].

By $\mathcal{M}_X$, we denote the collection of all bounded subsets of a metric space (X, d). Let Q be a bounded subset of X and $K(x, r) = \{y \in X : d(x, y) < r\}$. Then the Hausdorff measure of noncompactness of Q, denoted by $\chi(Q)$, is defined by

$$\chi(Q) = \inf\left\{\varepsilon > 0 : Q \subset \bigcup_{i=1}^{n} K(x_i, r_i),\ x_i \in X,\ r_i < \varepsilon\ (i = 1, 2, ...),\ n \in \mathbb{N}_0\right\}.$$

If Q, Q_1 and Q_2 are bounded subsets of the metric space (X, d), then we have

$\chi(Q) = 0$ if and only if Q is a totaly bounded set,

$$\chi(Q) = \chi(\overline{Q}),$$

$Q_1 \subset Q_2$ implies $\chi(Q_1) \leq \chi(Q_2)$,

$$\chi(Q_1 \cup Q_2) = \max\{\chi(Q_1), \chi(Q_2)\}$$

and

$$\chi(Q_1 \cap Q_2) \leq \min\{\chi(Q_1), \chi(Q_2)\}.$$

If Q, Q_1 and Q_2 are bounded subsets of the normed space X, then we have

$$\chi(Q_1 + Q_2) \leq \chi(Q_1) + \chi(Q_2),$$

$$\chi(Q + x) = \chi(Q) \quad (x \in X)$$

and

$\chi(\lambda Q) = |\lambda|\chi(Q)$ for all $\lambda \in \mathbb{C}$.

For our investigation we also need the following results [2, 4, 5].

Lemma 1.1. *Let X denote any of the spaces c_0, c or ℓ_∞. Then, we have $X^\beta = \ell_1$ and $\|a\|_X^* = \|a\|_{\ell_1}$ for all $a \in \ell_1$.*

Lemma 1.2. *Let $X \supset \phi$ and Y be BK spaces. Then, we have $(X,Y) \subset B(X,Y)$, that is, every matrix $A \in (X,Y)$ defines an operator $L_A \in B(X,Y)$ by $L_A(x) = Ax$ for all $x \in X$.*

Lemma 1.3. *Let $X \supset \phi$ be a BK space and Y be any of the spaces c_0, c or ℓ_∞. If $A \in (X,Y)$, then we have*

$$\|L_A\| = \|A\|_{(X,\ell_\infty)} = \sup_n \|A_n\|_X^* < \infty.$$

Lemma 1.4. *Let T be a triangle matrix. Then, we have*

(i) For arbitrary subsets X and Y of ω, $A \in (X, Y_T)$ if and only if $B = TA \in (X,Y)$.

(ii) Further, if X and Y are BK spaces and $A \in (X, Y_T)$, then $\|L_A\| = \|L_B\|$.

Lemma 1.5. *Let $Q \in \mathcal{M}_X$ and $P_r : X \to X$ $(r \in \mathbb{N})$ be the operator defined by $P_r(x) = (x_0, x_1, ..., x_r, 0, 0, ...)$ for all $x = (x_k) \in X$ where X is ℓ_p for $1 \leq p < \infty$ or c_0. Then, we have*

$$\chi(Q) = \lim_{r\to\infty}\left(\sup_{x\in Q}\|(I-P_r)(x)\|_{\ell_\infty}\right),$$

where I is the identity operator on X.

Further, we know by [2, Theorem 1.10] that every $z = (z_n) \in c$ has a unique representation $z = \tilde{z}e + \sum_{n=0}^{\infty}(z_n - \tilde{z})e^{(n)}$, where $\tilde{z} = \lim_{n\to\infty} z_n$. Then, we define the projectors $P_r : c \to c$ $(r \in \mathbb{N})$ by

$$P_r(z) = \tilde{z}e + \sum_{n=0}^{r}(z_n - \tilde{z})e^{(n)}; \quad (r \in \mathbb{N}) \tag{1.4}$$

for all $z = (z_n) \in c$ with $\tilde{z} = \lim_{n\to\infty} z_n$. In this situation, the following result gives an estimate for the Hausdorff measure of noncompactness in the BK space c.

Lemma 1.6. *Let $Q \in \mathcal{M}_c$ and $P_r : c \to c$ $(r \in \mathbb{N})$ be the projector onto the linear span of $\{e, e^{(0)}, e^{(1)}, ..., e^{(r)}\}$. Then, we have*

$$\frac{1}{2}\cdot\lim_{r\to\infty}\left(\sup_{x\in Q}\|(I-P_r)(x)\|_{\ell_\infty}\right) \leq \chi(Q) \leq \lim_{r\to\infty}\left(\sup_{x\in Q}\|(I-P_r)(x)\|_{\ell_\infty}\right),$$

where I is the identity operator on c.

Moreover, we have the following result concerning with the Hausdorff measure of noncompactness in the matrix domain of triangles in normed sequence spaces.

Lemma 1.7. *Let X be a normed sequence space, T a triangle and χ_T and χ denote the Hausdorff measures of noncompactness on $\mathcal{M}_{X_T}$ and $\mathcal{M}_X$, the collections of all bounded sets in X_T and X, respectively. Then, $\chi_T(Q) = \chi(T(Q))$ for all $Q \in \mathcal{M}_{X_T}$.*

X and Y be Banach spaces. Then, the Hausdorff measure of noncompactness of L, denoted by $\|L\|_\chi$, is defined by

$$\|L\|_\chi = \chi(L(S_X)) \tag{1.5}$$

and we have

$$L \quad \text{is compact if and only if} \quad \|\mathrm{L}\|_\chi = 0. \tag{1.6}$$

2 Sequence Spaces Derived by The Domain of The Matrix $B(r,s,t)$

Let r, s and t be non-zero real numbers, and define the generalized difference matrix $B(r,s,t) = \{b_{nk}(r,s,t)\}$ by

$$b_{nk}(r,s,t) = \begin{cases} r, & (k=n) \\ s, & (k=n-1) \\ t, & (k=n-2) \\ 0, & (0 \le k < n-1 \text{ or } k > n) \end{cases}$$

for all $n, k \in \mathbb{N}$. Recently, the difference sequence spaces $c_0(B), \ell_\infty(B)$ and $\ell_p(B)$ have been introduced by Snmez [6] as follows:

$$c_0(B) = \left\{ x = (x_k) \in \omega : \lim_{k\to\infty} |rx_k + sx_{k-1} + tx_{k-2}| = 0 \right\}$$

$$\ell_\infty(B) = \left\{ x = (x_k) \in \omega : \sup_{k\in\mathbb{N}} |rx_k + sx_{k-1} + tx_{k-2}| < \infty \right\}$$

and

$$\ell_p(B) = \left\{ x = (x_k) \in \omega : \sum_k |rx_k + sx_{k-1} + tx_{k-2}|^p < \infty \right\}.$$

With the notation of (1.2), one can redefine the spaces $c_0(B), \ell_\infty(B)$ and $\ell_p(B)$ as

$$c_0(B) = \{c_0\}_{B(r,s,t)}, \quad \ell_\infty(B) = \{\ell_\infty\}_{B(r,s,t)}, \quad \ell_p(B) = \{\ell_p\}_{B(r,s,t)}.$$

It is obvious that $c_0(B)$ and $\ell_\infty(B)$ are BK spaces with the norm given by

$$\|x\|_B = \|B(r,s,t)(x)\|_{\ell_\infty} = \sup_n |B_n(r,s,t)(x)|. \tag{2.1}$$

Throughout for any sequence $x = (x_k) \in \omega$, we define the associated sequence $y = (y_k)$, which will frequently be used, as the $B(r,s,t)-$transform of x, that is $y = B(r,s,t)(x)$. Then, it can easily be shown that

$$y_k = rx_k + sx_{k-1} + tx_{k-2}, \quad (k \in \mathbb{N}) \tag{2.2}$$

and hence,

$$x_k = \sum_{j=1}^{k}\left(\frac{1}{r}\sum_{v=0}^{j-1}\left(\frac{-s+\sqrt{s^2-4tr}}{2r}\right)^{k-v-1}\left(\frac{-s-\sqrt{s^2-4tr}}{2r}\right)^{v}\right)y_{k-j+1}; \quad (k\in\mathbb{N}). \tag{2.3}$$

λ be any of the spaces c_0 and ℓ_∞. If the sequences x and y are connected by the relation (2.2), then $x\in\lambda(B)$ if and only if $y\in\lambda$; furthermore if $x\in\lambda(B)$, then, $\|x\|_B = \|y\|_{\ell_\infty}$. In fact, since $B(r,s,t)$ is a triangle, the linear operator $L_B : X\to Y$, which maps every sequence in X to its associated sequence in Y, is bijective and norm preserving; where $X=\lambda(B)$ and $Y=\lambda$.

If we take $t=0$, then we get $B(r,s,t)=B(r,s)$. We should record here that the matrix $B(r,s,t)$ can be reduced to the difference matrices $\Delta^{(2)}$ and $\Delta^{(1)}$ in case $r=1, s=-2, t=1$ and $r=1, s=-1, t=0$, respectively. So, the results related to the matrix domain of the matrix $B(r,s,t)$ are more general and more comprehensive than the consequences of the matrices domain of $B(r,s), \Delta^{(2)}$ and $\Delta^{(1)}$, and include them, where $B(r,s)=\{b_{nk}(r,s)\}$ and $\Delta^{(1)}=(\delta_{nk})$ are defined by

$$b_{nk}(r,s)=\begin{cases} r, & (k=n)\\ s, & (k=n-1)\\ 0, & (\text{otherwise})\end{cases} \qquad \delta_{nk}=\begin{cases} (-1)^{n-k}, & (n-1\le k\le n)\\ 0, & (0\le k<n-1 \text{ or } k>n).\end{cases}$$

In [6, Corollary 3.5 and Corollary 4.2], the $\beta-$ and $\gamma-$ duals of the spaces $c_0(B), \ell_\infty(B)$ and $\ell_p(B)$ have been determined and some related matrix transformations have also been characterized.

The following results will be needed in establishing our results.

Lemma 2.1. *Let X denote any of the spaces $c_0(B)$ and $\ell_\infty(B)$. If $a=(a_k)\in X^\beta$, then $\tilde{a}=(\tilde{a}_k)\in\ell_1$ and the equality*

$$\sum_{k=0}^{\infty} a_k x_k = \sum_{k=0}^{\infty}\tilde{a}_k y_k \tag{2.4}$$

holds for every $x=(x_k)\in X$, where $y=B(r,s,t)(x)$ is the associated sequence defined by (2.2) and

$$\tilde{a}_k = \frac{1}{r}\sum_{j=k}^{\infty}\sum_{v=0}^{j-k}\left(\frac{-s+\sqrt{s^2-4tr}}{2r}\right)^{j-k-v}\left(\frac{-s-\sqrt{s^2-4tr}}{2r}\right)^{v} a_j; \quad (k\in\mathbb{N}).$$

Proof. This follows immediately by [7, Theorem 5.6]. ☐

Lemma 2.2. *Let X denote any of the spaces $c_0(B)$ and $\ell_\infty(B)$. Then, we have*

$$\|a\|_X^* = \|\tilde{a}\|_{\ell_1} = \sum_{k=0}^{\infty}|\tilde{a}_k| < \infty$$

for all $a=(a_k)\in X^\beta$, where $\tilde{a}=(\tilde{a}_k)$ is as in Lemma 2.1.

Proof. Let Y be the respective one of the spaces c_0 or ℓ_∞, and take any $a=(a_k)\in X^\beta$. Then, we have by Lemma 2.1 that $\tilde{a}=(\tilde{a}_k)\in\ell_1$ and the equality (2.4) holds for all sequences $x=(x_k)\in X$ and $y=(y_k)\in Y$ which are connected by the relation (2.2). Further, it follows by (2.1) that $x\in S_X$ if and only if $y\in S_Y$. Therefore, we derive from (1.3) and (2.4) that

$$\|a\|_X^* = \sup_{x\in S_X}\left|\sum_{k=0}^{\infty} a_k x_k\right| = \sup_{y\in S_Y}\left|\sum_{k=0}^{\infty}\tilde{a}_k y_k\right| = \|\tilde{a}\|_Y^*$$

and since $\tilde{a}\in\ell_1$, we obtain from Lemma 1.1 that

$$\|a\|_X^* = \|\tilde{a}\|_Y^* = \|\tilde{a}\|_{\ell_1} < \infty$$

which concludes the proof. □

Lemma 2.3. *Let X be any of the spaces $c_0(B)$ and $\ell_\infty(B)$, Y the respective one of the spaces c_0 or ℓ_∞, Z a sequence space and $A=(a_{nk})$ an infinite matrix. If $A\in(X,Z)$, then $\tilde{A}\in(Y,Z)$ such that $Ax=\tilde{A}y$ for all sequences $x\in X$ and $y\in Y$ which are connected by the relation (2.2), where $\tilde{A}=(\tilde{a}_{nk})$ is the associated matrix defined by*

$$\tilde{a}_{nk} = \frac{1}{r}\sum_{j=k}^{\infty}\sum_{v=0}^{j-k}\left(\frac{-s+\sqrt{s^2-4tr}}{2r}\right)^{j-k-v}\left(\frac{-s-\sqrt{s^2-4tr}}{2r}\right)^{v} a_{nj}; \quad (k\in\mathbb{N}). \tag{2.5}$$

provided the series on the right converge for all $n,k\in\mathbb{N}$.

Proof. Let $x\in X$ and $y\in Y$ be connected by the relation (2.2) and suppose that $A\in(X,Z)$. Then $A_n\in X^\beta$ for all $n\in\mathbb{N}$. Thus, it follows by Lemma 2.1 that $\tilde{A}_n\in\ell_1=Y^\beta$ for all $n\in\mathbb{N}$, and the equality $Ax=\tilde{A}y$ holds, hence $\tilde{A}y\in Z$. Further, we have by (2.3) that every $y\in Y$ is the associated sequence of some $x\in X$. Hence, we deduce that $\tilde{A}\in(Y,Z)$. This completes the proof. □

Lemma 2.4. *Let X be any of the spaces $c_0(B)$ and $\ell_\infty(B)$, $A=(a_{nk})$ an infinite matrix and $\tilde{A}=(\tilde{a}_{nk})$ the associated matrix. If A is any of the classes $(X,c_0),(X,c)$ or (X,ℓ_∞), then*

$$\|L_A\| = \|A\|_{(X,\ell_\infty)} = \sup_n\left(\sum_{k=0}^{\infty}|\tilde{a}_{nk}|\right) < \infty.$$

Proof. This is immediate by combining Lemmas 1.3 and 2.2. □

3 Compact Operators on The Spaces $c_0(B), \ell_\infty(B)$ and $\ell_p(B)$

In this section, we determine the Hausdorff measures of noncompactness of certain matrix operators on the spaces $c_0(B), \ell_\infty(B)$ and $\ell_p(B)$, and apply our results to characterize some classes of compact operators on those spaces. For the most recent works on this topic, we refer to [8] - [19].

We begin with the following lemma [20, Lemma 3.1] which will be used in proving results.

Lemma 3.1. *Let X denote any of the spaces c_0 or ℓ_∞. If $A \in (X, c)$, then we have*

$$\alpha_k = \lim_{n\to\infty} a_{nk} \text{ exists for every } k \in \mathbb{N},$$
$$\alpha = (\alpha_k) \in \ell_1,$$
$$\sup_n \left(\sum_{k=0}^{\infty} |a_{nk} - \alpha_k| \right) < \infty,$$
$$\lim_{n\to\infty} A_n(x) = \sum_{k=0}^{\infty} \alpha_k x_k \text{ for all } x = (x_k) \in X.$$

Now, let $A = (a_{nk})$ be an infinite matrix and $\tilde{A} = (\tilde{a}_{nk})$ the associated matrix defined by (2.5). Then, we have the following result on the Hausdorff measures of noncompactness.

Theorem 3.2. *Let X denote any of the spaces $c_0(B)$ and $\ell_\infty(B)$. Then, we have*

(i) If $A \in (X, c_0)$, then

$$\|L_A\|_\chi = \limsup_{n\to\infty} \left(\sum_{k=0}^{\infty} |\tilde{a}_{nk}| \right). \tag{3.1}$$

(ii) If $A \in (X, c)$, then

$$\frac{1}{2} \cdot \limsup_{n\to\infty} \left(\sum_{k=0}^{\infty} |\tilde{a}_{nk} - \tilde{\alpha}_k| \right) \le \|L_A\|_\chi \le \limsup_{n\to\infty} \left(\sum_{k=0}^{\infty} |\tilde{a}_{nk} - \tilde{\alpha}_k| \right), \tag{3.2}$$

where $\tilde{\alpha}_k = \lim_{n\to\infty} \tilde{a}_{nk}$ for all $k \in \mathbb{N}$.

(iii) If $A \in (X, \ell_\infty)$, then

$$0 \le \|L_A\|_\chi \le \limsup_{n\to\infty} \left(\sum_{k=0}^{\infty} |\tilde{a}_{nk}| \right). \tag{3.3}$$

Proof. Let us remark that the expressions in (3.1) and (3.3) exists by Lemma 2.4. Also, by combining Lemmas 2.3 and 3.1, we deduce that the expressions in (3.2) exists.

We write $S = S_X$, for short. Then, we obtain by (1.5) and Lemma 1.2 that

$$\|L_A\|_\chi = \chi(AS). \tag{3.4}$$

For (i), we have $AS \in \mathcal{M}_{c_0}$. Thus, it follows by applying Lemma 1.5 that

$$\chi(AS) = \lim_{r\to\infty}\left(\sup_{x\in S}\|(I-P_r)(Ax)\|_{\ell_\infty}\right), \tag{3.5}$$

where $P_r : c_0 \to c_0$ $(r \in \mathbb{N})$ is the operator by $P_r(x) = (x_0, x_1, ..., x_r, 0, 0, ...)$ for all $x = (x_k) \in c_0$. This yields that $\|(I-P_r)(Ax)\|_{\ell_\infty} = \sup_{n>r}|A_n(x)|$ for all $x \in X$ and every $r \in \mathbb{N}$. Therefore, by using (1.1) and (1.3) and Lemma 2.2, we have for every $r \in \mathbb{N}$ that

$$\sup_{x\in S}\|(I-P_r)(Ax)\|_{\ell_\infty} = \sup_{n>r}\|A_n\|_X^* = \sup_{n>r}\|\tilde{A}_n\|_{\ell_1}.$$

This and (3.5) imply that

$$\chi(AS) = \lim_{r\to\infty}\left(\sup_{n>r}\|\tilde{A}_n\|_{\ell_1}\right) = \limsup_{n\to\infty}\|\tilde{A}_n\|_{\ell_1}.$$

Hence, we get (3.1) by (3.4).

To prove (ii), we write $AS \in \mathcal{M}_c$. Thus, we are going to apply Lemma 1.6 to get an estimate for the value $\chi(AS)$ in (3.4). For this, let $P_r : c \to c$ $(r \in \mathbb{N})$ be the projectors defined by (1.4). Then, we have every $r \in \mathbb{N}$ that $(I-P_r)(z) = \sum_{n=r+1}^{\infty}(z_n - \tilde{z})e^{(n)}$ and hence,

$$\|(I-P_r)(z)\|_{\ell_\infty} = \sup_{n>r}|z_n - \tilde{z}| \tag{3.6}$$

for all $z = (z_n) \in c$ and every $r \in \mathbb{N}$, where $\tilde{z} = \lim_{n\to\infty} z_n$ and I is the identity operator on c. Now, by using (3.4), we obtain by applying Lemma 1.6 that

$$\frac{1}{2}\cdot\lim_{r\to\infty}\left(\sup_{x\in S}\|(I-P_r)(Ax)\|_{\ell_\infty}\right) \le \|L_A\|_\chi \le \lim_{r\to\infty}\left(\sup_{x\in S}\|(I-P_r)(Ax)\|_{\ell_\infty}\right). \tag{3.7}$$

On the other hand, it is given that $X = c_0(B)$ or $X = \ell_\infty(B)$, and let Y be the respective one of the spaces c_0 or ℓ_∞. Also, for every given $x \in X$, let $y \in Y$ be the associated sequence defined by (2.2). Since $A \in (X, c)$, we have by Lemma 2.3 that $\tilde{A} \in (Y, c)$ and $Ax = \tilde{A}y$. Further, it follows from Lemma 3.1 that the limits $\tilde{\alpha}_k = \lim_{n\to\infty}\tilde{a}_{nk}$ exists for all k, $\tilde{\alpha} = (\tilde{\alpha}_k) \in \ell_1 = Y^\beta$ and

$\lim_{n\to\infty}\tilde{A}_n(y)=\sum_{k=0}^{\infty}\tilde{\alpha}_k y_k$. Consequently, we derive from (3.6) that

$$\begin{aligned}\|(I-P_r)(Ax)\|_{\ell_\infty} &= \|(I-P_r)(\tilde{A}y)\|_{\ell_\infty}\\ &= \sup_{n>r}\left|\tilde{A}_n(y)-\sum_{k=0}^{\infty}\tilde{\alpha}_k y_k\right|\\ &= \sup_{n>r}\left|\sum_{k=0}^{\infty}(\tilde{a}_{nk}-\tilde{\alpha}_k)y_k\right|\end{aligned}$$

for all $r\in\mathbb{N}$. Moreover, since $x\in S$ if and only if $y\in S_Y$, we obtain by (1.3) and Lemma 1.1 that

$$\begin{aligned}\sup_{x\in S}\|(I-P_r)(Ax)\|_{\ell_\infty} &= \sup_{n>r}\left(\sup_{y\in S_Y}\left|\sum_{k=0}^{\infty}(\tilde{a}_{nk}-\tilde{\alpha}_k)y_k\right|\right)\\ &= \sup_{n>r}\|\tilde{A}_n-\tilde{\alpha}\|_Y^*\\ &= \sup_{n>r}\|\tilde{A}_n-\tilde{\alpha}\|_{\ell_1}\end{aligned}$$

for all $r\in\mathbb{N}$. Hence, from (3.7) we get (3.2).

Finally, to prove (iii) we define the operators $P_r:\ell_\infty\to\ell_\infty$ $(r\in\mathbb{N})$ as in the proof of part (i) for all $x=(x_k)\in\ell_\infty$. Then, we have

$$AS\subset P_r(AS)+(I-P_r)(AS);\quad (r\in\mathbb{N}).$$

Thus, it follows by the elementary properties of the function χ that

$$\begin{aligned}0 &\le \chi(AS)\le\chi(P_r(AS))+\chi((I-P_r)(AS))\\ &= \chi((I-P_r)(AS))\\ &\le \sup_{x\in S}\|(I-P_r)(Ax)\|_{\ell_\infty}\\ &= \sup_{n>r}\|\tilde{A}_n\|_{\ell_1}\end{aligned}$$

for all $r\in\mathbb{N}$ and hence,

$$0\le\chi(AS)\le\lim_{r\to\infty}\left(\sup_{n>r}\|\tilde{A}_n\|_{\ell_1}\right)=\limsup_{n\to\infty}\|\tilde{A}_n\|_{\ell_1}.$$

This and (3.4) together imply (3.3) and complete the proof. □

Corollary 3.3. *Let X denote any of the spaces $c_0(B)$ and $\ell_\infty(B)$. Then, we have*

(i) If $A\in(X,c_0)$, then

$$L_A \text{ is compact if and only if } \lim_{n\to\infty}\left(\sum_{k=0}^{\infty}|\tilde{a}_{nk}|\right)=0.$$

(ii) If $A \in (X,c)$, then

$$L_A \textit{ is compact if and only if } \lim_{n\to\infty}\left(\sum_{k=0}^{\infty}|\tilde{a}_{nk}-\tilde{\alpha}_k|\right)=0.$$

where $\tilde{\alpha}_k = \lim_{n\to\infty}\tilde{a}_{nk}$ for all $k \in \mathbb{N}$.

(iii) If $A \in (X,\ell_\infty)$, then

$$L_A \textit{ is compact if } \lim_{n\to\infty}\left(\sum_{k=0}^{\infty}|\tilde{a}_{nk}|\right)=0. \tag{3.8}$$

Proof. The result follows from Theorem 3.2 by using (1.6). □

It is worth mentioning that the condition in (3.8) is only a sufficient condition for the operator L_A to be compact, where $A \in (X,\ell_\infty)$ and X is any of the spaces $c_0(B)$ and $\ell_\infty(B)$. More precisely, the following example will show that it is possible for L_A to be compact while $\lim_{n\to\infty}\left(\sum_{k=0}^{\infty}|\tilde{a}_{nk}|\right)\neq 0$.

Example 3.4. Let X be any of the spaces $c_0(B)$ and $\ell_\infty(B)$, and define the matrix $A=(a_{nk})$ by $a_{n0}=1$ and $a_{nk}=0$ for $k\geq 1$ ($n\in\mathbb{N}$). Then, we have $Ax = x_0 e$ for all $x=(x_k)\in X$, hence $A\in(X,\ell_\infty)$. Also, since L_A is of finite rank, L_A is compact. On the other hand, by using (2.5), it can easily be seen that $\tilde{A}=A$. Thus $\tilde{A}_n = e^{(0)}$ and so $\|\tilde{A}_n\|_{\ell_1}=1$ for all $n\in\mathbb{N}$. This implies that

$$\lim_{n\to\infty}\|\tilde{A}_n\|_{\ell_1}=1.$$

Finally, we characterize classes of compact operators given by infinite matrices from $\ell_p(B)$ to c_0, c, ℓ_∞ and ℓ_1. Also, we give the necessary and sufficient conditions for $A\in(\ell_1(B),\ell_p)$ to be compact, where $1\leq p<\infty$.

It is easy to see that the space $\ell_p(B)$ is BK-space with the norm

$$\|x\|_{\ell_p(B)} = \|B(r,s,t)(x)\|_{\ell_p} = \left(\sum_{n=0}^{\infty}|B_n(r,s,t)(x)|^p\right)^{1/p}; \quad (1\leq p<\infty). \tag{3.9}$$

Lemma 3.5. *Let $1\leq p<\infty$. Then, we have $\ell_p^\beta=\ell_q$ and*

$$\|a\|_{\ell_p}^{*}=\|a\|_{\ell_q}$$

for all $a=(a_k)\in\ell_q$.

Lemma 3.6. *If $a=(a_k)\in\{\ell_p(B)\}^\beta$, then $\tilde{a}=(\tilde{a}_k)\in\ell_q$ and the equality*

$$\sum_{k=0}^{\infty}a_k x_k = \sum_{k=0}^{\infty}\tilde{a}_k y_k \tag{3.10}$$

holds for every $x=(x_k)\in\{\ell_p(B)\}^\beta$, where $\tilde{a}=(\tilde{a}_k)$ is as in Lemma 2.1.

Lemma 3.7. *Let $1 \leq p < \infty$ and $\tilde{a} = (\tilde{a}_k)$ be defined as in Lemma 2.1. Then, we have*

$$\|a\|^*_{l_p(B)} = \begin{cases} \left(\sum_{k=0}^{\infty} |\tilde{a}_k|^q\right)^{1/q}, & (1 < p < \infty) \\ \sup_k |\tilde{a}_k|, & (p = 1) \end{cases}$$

for all $a = (a_k) \in \{\ell_p(B)\}^\beta$.

Proof. Let $a = (a_k) \in \{\ell_p(B)\}^\beta$. Then we have from Lemma 3.6 that $\tilde{a} = (\tilde{a}_k) \in \ell_q$ and the equality (3.10) holds for all $x = (x_k) \in \{\ell_p(B)\}^\beta$ and $y = (y_k) \in \ell_p$, which are connected by the relation (2.2). Also, we can write by (3.9) that $x \in S_{\ell_p(B)}$ if and only if $y \in S_{\ell_p}$. Thus, we have from (3.10) that

$$\begin{aligned} \|a\|^*_{\ell_p(B)} &= \sup_{x \in S_{\ell_p(B)}} \left|\sum_{k=0}^{\infty} a_k x_k\right| \\ &= \sup_{y \in \ell_p} \left|\sum_{k=0}^{\infty} \tilde{a}_k y_k\right|. \end{aligned} \tag{3.11}$$

Further, since $\tilde{a} \in \ell_q$, we get by Lemma 3.5 and (3.11) that

$$\|a\|^*_{l_p(B)} = \|\tilde{a}\|^*_{\ell_p} = \|\tilde{a}\|_{\ell_q} < \infty$$

which concludes the proof. □

Lemma 3.8. *Let $X \supset \phi$ be a BK-space. If $A \in (X, \ell_1)$, then*

$$\lim_{r\to\infty}\left(\sup_{N \in \mathcal{F}_r} \left\|\sum_{n \in N} A_n\right\|^*_X\right) \leq \|L_A\|_\chi \leq 4 \cdot \lim_{r\to\infty}\left(\sup_{N \in \mathcal{F}_r} \left\|\sum_{n \in N} A_n\right\|^*_X\right).$$

Also, L_A is compact if and only if

$$\lim_{r\to\infty}\left(\sup_{N \in \mathcal{F}_r} \left\|\sum_{n \in N} A_n\right\|^*_X\right) = 0.$$

Theorem 3.9. *Let $1 \leq p < \infty$. If $A \in (\ell_1(B), \ell_p)$, then*

$$\|L_A\|_\chi = \lim_{r\to\infty}\left(\sup_k \sum_{n=r}^{\infty} |\tilde{a}_{nk}|^p\right)^{1/p}. \tag{3.12}$$

Proof. Let $S = S_{\ell_1(B)}$. Then, we have by Lemma 1.2 that $L_A(S) = AS \in \ell_p$. Thus, from (1.5) and Lemma 1.5 we can write that

$$\|L_A\|_\chi = \chi(AS) = \lim_{r\to\infty}\left(\sup_{x \in S} \|(I - P_r)(Ax)\|_{\ell_p}\right), \tag{3.13}$$

where $P_r : \ell_p \to \ell_p$ $(r \in \mathbb{N})$ is the operator defined by $P_r(x) = (x_0, x_1, ..., x_r, 0, 0, ...)$ for all $x = (x_k) \in \ell_p$.

Now, let $x = (x_k) \in \ell_1(B)$. Since $A \in (\ell_1(B), \ell_p)$, we obtain that $\tilde{A} \in (\ell_1, \ell_p)$ and $Ax = \tilde{A}y$, where $y = (y_k) \in \ell_1$ is the associated sequence defined by (2.2). Therefore, we have that

$$\begin{aligned} \|(I-P_r)(Ax)\|_{\ell_p} &= \|(I-P_r)(\tilde{A}y)\|_{\ell_p} \\ &= \left(\sum_{n=r+1}^{\infty} \left|\tilde{A}_n(y)\right|^p\right)^{1/p} \\ &= \left(\sum_{n=r+1}^{\infty} \left|\sum_{k=0}^{\infty} \tilde{a}_{nk} y_k\right|^p\right)^{1/p} \\ &\le \sum_{k=0}^{\infty} \left(\sum_{n=r+1}^{\infty} |\tilde{a}_{nk} y_k|^p\right)^{1/p} \\ &\le \|y\|_{\ell_1} \left(\sup_k \left(\sum_{n=r+1}^{\infty} |\tilde{a}_{nk}|^p\right)^{1/p}\right) \\ &= \|x\|_{\ell_1(B)} \left(\sup_k \left(\sum_{n=r+1}^{\infty} |\tilde{a}_{nk}|^p\right)^{1/p}\right) \end{aligned}$$

for every $n \in \mathbb{N}$. This yields that

$$\sup_{x \in S} \|(I-P_r)(Ax)\|_{\ell_p} \le \sup_k \left(\sum_{n=r+1}^{\infty} |\tilde{a}_{nk}|^p\right)^{1/p}$$

for every $n \in \mathbb{N}$. Hence, from (3.13) we have that

$$\|L_A\|_\chi \le \lim_{r\to\infty} \left\{ \sup_k \left(\sum_{n=r+1}^{\infty} |\tilde{a}_{nk}|^p\right)^{1/p} \right\}. \tag{3.14}$$

Conversely, let $e_B^{(k)} \in \ell_1(B)$ such that $B(r,s,t)(e_B^{(k)}) = e^{(k)}$ $(k \in \mathbb{N})$, that is, $e^{(k)}$ is the associated sequence of $e_B^{(k)}$ for each $k \in \mathbb{N}$. Then, we have that $Ae_B^{(k)} = \tilde{A}e^{(k)} = (\tilde{a}_{nk})_{n=0}^{\infty}$ for every $k \in \mathbb{N}$. Now, let $E = \{e_B^{(k)} : k \in \mathbb{N}\}$. Then, $E \subset S$ and $AE \subset AS$ which implies that

$$\chi(AE) \le \chi(AS) = \|L_A\|_\chi. \tag{3.15}$$

Moreover, we can write from Lemma 1.5 and (3.15) that

$$\begin{aligned} \chi(AE) &= \lim_{r\to\infty} \left\{ \sup_k \left(\sum_{n=r+1}^{\infty} \left|A_n(e_B^{(k)})\right|^p\right)^{1/p} \right\} \\ &= \lim_{r\to\infty} \left\{ \sup_k \left(\sum_{n=r+1}^{\infty} \left|\tilde{a}_{nk}\right|^p\right)^{1/p} \right\} \\ &\le \|L_A\|_\chi. \end{aligned}$$

Thus, we get (3.12) from (3.14) and (3.15). □

Corollary 3.10. *Let $1 \leq p < \infty$. If $A \in (\ell_1(B), \ell_p)$, then L_A is compact if and only if*

$$\lim_{r\to\infty}\left(\sup_k \sum_{n=r}^{\infty} |\tilde{a}_{nk}|^p\right)^{1/p} = 0.$$

Proof. This is an immediate consequence of (1.6) and Theorem 3.9. □

Theorem 3.11. *Let $1 < p < \infty$ and $q = p/(p-1)$. If $A \in (\ell_p(B), \ell_1)$, then*

$$\lim_{r\to\infty}\left(\sup_{N\in\mathcal{F}_r}\left(\sum_{k=0}^{\infty}\left|\sum_{n\in N}\tilde{a}_{nk}\right|^q\right)^{1/q}\right)\leq\|L_A\|_\chi\leq 4\cdot\lim_{r\to\infty}\left(\sup_{N\in\mathcal{F}_r}\left(\sum_{k=0}^{\infty}\left|\sum_{n\in N}\tilde{a}_{nk}\right|^q\right)^{1/q}\right) \tag{3.16}$$

and

$$L_A \text{ is compact if and only if } \lim_{r\to\infty}\left(\sup_{N\in\mathcal{F}_r}\left(\sum_{k=0}^{\infty}\left|\sum_{n\in N}\tilde{a}_{nk}\right|^q\right)^{1/q}\right) = 0. \tag{3.17}$$

Proof. Let $A \in (\ell_p(B), \ell_1)$. Since $A_n \in \{\ell_p(B)\}^\beta$ for all $n \in \mathbb{N}$, we derive from Lemma 3.7 that

$$\left\|\sum_{n\in N} A_n\right\|^*_{\ell_p(B)} = \left\|\sum_{n\in N}\tilde{A}_n\right\|^*_{\ell_q}. \tag{3.18}$$

Thus, we get (3.16) and (3.17) from Lemma 3.8 and (3.18). □

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References

[1] F. Başarır, Summability Theory and Its Appliactions, Bentham Science Publishers, ISBN:978-1-60805-252-3, 2011.

[2] E. Malkowsky, V. Rakocevic, An introduction into the theory of sequence spaces and measure of noncompactness, in: Zb. Rad. (Beogr) 9 (17) (2000), Matematicki institut SANU, Belgrade (2000), 143-234.

[3] V. Rakocevic, Funkcionalna analiza, Naucna knjiga, Belgrad, 1994.

[4] E. Malkowsky, Compact matrix operators between some BK-spaces, Modern Methods of Analysis its Applications, Anamaya Publ., New Delhi. (2010) 86-120.

[5] B. de Malafosse, V. Rakočević, Applications of measure of noncompactness in operators on the spaces $s_\alpha, s_\alpha^0, s_\alpha^c, \ell_\alpha^p$, J. Math. Anal. Appl. 323 (1) (2006) 131-145.

[6] A. Sönmez, Some new sequence spaces derived by the domain of the triple band matrix, Comput. Math. Appl. 62 (2011) 641-650.

[7] M. Mursaleen, A.K. Noman, On some new difference sequence spaces of non-absolute type, Math. Comput. Modelling. 52 (2010) 603-617.

[8] M. Mursaleen, V. Karakaya, H. Polat, N. Şimşek, Meausre of noncompactness of matrix operators on some difference sequence spaces of weighted means, Comput. Math. Appl. 62 (2011) 814-820.

[9] E. Evren Kara, M. Baarr, On compact operators and some Euler $B^{(m)}$-difference sequence spaces, J. Math. Anal. Appl. 379 (2011) 499-511.

[10] M. Başarır, E. Evren Kara, On the B-difference sequence space derived by generalized weighted mean and compact operators, J. Math. Anal. Appl. 391 (2012) 67-81.

[11] M. Başarır, E. Evren Kara, On compact operators on the Riesz B^m-difference sequence space, IJST (2011) 279-285.

[12] Başarır, E. Evren Kara, On some difference sequence spaces of weighted means and compact operators, Annals of Func. Anal. 2 (2) 116-131.

[13] M. Mursaleen, Abdullah K. Noman, Compactness of matrix operators on some new difference sequence spaces, Linear Algebra and its Applications 436 (1) 41-52.

[14] M. Mursaleen, S.A. Mohiuddine, Applications of measures of noncompactness to the infinite system of differential equations in spaces, Nonlinear Analysis: Theory, Methods Applications 75 (4) 2111-2115.

[15] M. Mursaleen, Abdullah K. Noman, Compactness by the Hausdorff measure of noncompactness, Nonlinear Analysis: Theory, Methods Applications 73 (8) 2541-2557.

[16] V. Rakocevic, Measures of noncompactness and some applications, Filomat 12 (1998) 87-120.

[17] A. Wilansky, Summability through Functional Analysis, North-Holland Mathematics Studies, Amsterdam 85 (1984).

[18] C. Mongkolkeha, P. Kumam "Some geometric properties of Lacunary sequence spaces related to fixed point property," Abst. Appl. Anal. (2011) Article ID 903736.

[19] C. Mongkolkeha, P. Kumam, On H-properties and uniform Opial property of generalized Cesàro sequence spaces, Journal of Inequalities and Applications (2012) 2012:76.

[20] M. Mursaleen, A.K. Noman, Applications of the Hausdorff measure of noncompactness in some sequence spaces of weighted means, Comput. Math. Appl. 60 (2010) 1245-1258.

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