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A Generalized Mittag-Leffler Type Function with Four Parameters

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Abstract : In the present paper, we introduce and study a generalization of Mittag-Leffler type functions. We obtain several results which include integral representations, recurrence relations, differential formula, fractional derivative and integral, Mellin Barnes integral representation and images of this function under the Laplace and Mellin transforms. We also introduce and investigate fractional calculus integral operator involving this generalized Mittag-Leffler type function.

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1 Introduction and Preliminaries

The Mittag-Leffler function has gained importance and popularity due to its applications in solutions of fractional-order differential, integral, integro-differential

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and difference equations arising in several problems of applied sciences such as physics, chemistry, biology and engineering (See for example [1]. This function was introduced by the Swedish mathematician Mittag-Leffler [2] in terms of the following power series

$$E_\alpha(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n + 1)}, x, \alpha \in \mathbb{C}, Re(\alpha) > 0. \tag{1.1}$$

The Mittag-Leffler function (1.1) reduces to the exponential function when $\alpha = 1$. For $0 < \alpha < 1$ it interpolates between the pure exponential e^x and the geometric function $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$; $(|x| < 1)$.

A generalization of (1.1) was studied by Wiman [3] in the form

$$E_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n + \beta)}, x, \alpha, \beta \in \mathbb{C}, Re(\alpha), Re(\beta) > 0. \tag{1.2}$$

A further generalization of (1.2) was studied by Prabhakar [4] as

$$E_{\alpha,\beta}^{\gamma}(x) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \frac{x^n}{n!} = \frac{1}{\Gamma(\gamma)} {}_1\psi_1 \left[\begin{array}{c} (\gamma, 1) \\ (\beta, \alpha) \end{array} \middle| x \right], x, \alpha, \beta, \gamma \in \mathbb{C}, Re(\alpha) > 0, \tag{1.3}$$

where ${}_p\psi_q$ is the Fox-Wright function defined as [5]

$${}_p\psi_q \left[\begin{array}{c} (a_1, A_1), ..., (a_p, A_p) \\ (b_1, B_1), ..., (b_q, B_q) \end{array} \middle| x \right] = \sum_{n=0}^{\infty} \frac{\Gamma(a_1 + A_1 n) ... \Gamma(a_p + A_p n)}{\Gamma(b_1 + B_1 n) ... \Gamma(b_q + B_q n)} \frac{x^n}{n!}, \tag{1.4}$$

$x, a_j, A_j, b_j, B_j \in \mathbb{C}$, $Re(a_j), Re(A_j) > 0$, $j = 1, ..., p$, $Re(B_j) > 0$, $j = 1, ..., q$ and $1 + Re\left(\sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j\right) \geq 0$.

Kiryakova [6] defined a multi-index Mittag-Leffler type function by means of the following power series

$$E_{\left(\frac{1}{\rho_i}\right),(\mu_i)}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\prod_{j=1}^{m} \Gamma\left(\mu_j + \frac{n}{\rho_j}\right)}, \tag{1.5}$$

where $m > 1$ is an integer, $\rho_1, \rho_2, ..., \rho_m$ and $\mu_1, \mu_2, ..., \mu_m$ are arbitrary and real parameters.

Shukla & Prajapati [7] defined a generalization of Mittag-Leffler type function (1.3) in the form

$$E_{\alpha,\beta}^{\gamma,q}(x) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\alpha n + \beta)} \frac{x^n}{n!} = \frac{1}{\Gamma(\gamma)} {}_1\psi_1 \left[\begin{array}{c} (\gamma, q) \\ (\beta, \alpha) \end{array} \middle| x \right], \tag{1.6}$$

where $\alpha, \beta, \gamma \in \mathbb{C}$, $Re(\alpha), Re(\beta), Re(\gamma) > 0$, $q \in (0, 1) \cup \mathbb{N}$.

A further generalization of the above function (1.6) has been introduced and studied by Srivastava & Tomovski [8] wherein the parameter q is replaced by K with $Re\,(K) > 0$.

Some extensions of Mittag-Leffler type functions (1.5) and (1.6) have been introduced and studied by Saxena & Nishimoto [9] in the form

$$E_{\gamma,K}\left[(\alpha_j,\beta_j)_{1,m};x\right] = E_{\gamma,K}\left[(\alpha_1,\beta_1),....,(\alpha_m,\beta_m);x\right] = \sum_{r=0}^{\infty}\frac{(\gamma)_{rK}x^r}{\prod_{j=1}^{m}\Gamma(\alpha_j r+\beta_j)(r)!}$$

$$= \frac{1}{\Gamma(\gamma)}{}_1\psi_m\left[\begin{array}{c}(\gamma,K)\\(\beta_1,\alpha_1),(\beta_2,\alpha_2),....,(\beta_m,\alpha_m)\end{array}\middle|\,x\right], \tag{1.7}$$

where $x,\gamma,\alpha_j,\beta_j \in \mathbb{C}$, $\sum_{j=1}^{m} Re\,(\alpha_j) > Re\,(K) - 1$, $j = 1,....,m$, $Re\,(K) > 0$ and by Paneva-Konovska [10] as 3m-indices M-L type functions in the form:

$$E^{(\gamma_i),m}_{(\alpha_i),(\beta_i)}(x) = \sum_{k=0}^{\infty}\frac{(\gamma_1)_k\,...(\gamma_m)_k}{\Gamma(\alpha_1 k+\beta_1)...\Gamma(\alpha_m k+\beta_m)}\,\frac{x^k}{(k!)^m}. \tag{1.8}$$

A multivariate analogue of generalized Mittg-leffler type function is defined by Saxena et al. [11] in the form

$E^{(\gamma_j)}_{(\rho_j),\lambda}(z_1,...,z_m) = E^{(\gamma_1,...,\gamma_m)}_{(\rho_1,...,\rho_m),\lambda}(z_1,...,z_m) = \sum_{k_1,...,k_m=0}^{\infty}\frac{(\gamma_1)_{k_1}\cdots(\gamma_k)_{k_m}}{\Gamma\left(\lambda+\sum_{j=1}^{m}\rho_i k_i\right)}\frac{z_1^{k_1}...z_m^{k_m}}{(k_1)!...(k_m)!}$,
where $\lambda,\gamma_j,\rho_j,z_j \in \mathbb{C}, Re\,(\rho_j) > 0, j = 1,2,...,m$.

A generalized multiparameter function of Mittag-Leffler type is defined by Kalla *et al.* [12] in the form

$$HE^{\lambda_1,\lambda_2,...,\lambda_r}_{\mu_1,\mu_2,...,\mu_r}(z) \equiv HE^{\lambda}_{\mu}(z) = \sum_{k=0}^{\infty}\frac{(-1)^k}{\prod_{i=1}^{r}\Gamma(1+\mu_i+\lambda_i k)}\left(\frac{z}{\Lambda}\right)^{\Lambda k+M}, \tag{1.9}$$

where $\mu_i \in \mathbb{C}, \lambda_i > 0, i = 1,2,...,r, \sum_{i=1}^{r}\mu_i = M$ and $\sum_{i=1}^{r}\lambda_i = \Lambda$.

Recently, Garg *et al.* [13] studied a Mittag-Leffler type function of two variables in the form

$$E_1(x,y) = E_1\left(\begin{array}{c}\gamma_1,\alpha_1;\gamma_2,\beta_1\\\delta_1,\alpha_2,\beta_2;\delta_2,\alpha_3;\delta_3,\beta_3\end{array}\middle|\begin{array}{c}x\\y\end{array}\right)$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\gamma_1)_{\alpha_1 m}(\gamma_2)_{\beta_1 n}}{\Gamma(\delta_1+\alpha_2 m+\beta_2 n)}\frac{x^m}{\Gamma(\delta_2+\alpha_3 m)}\frac{y^n}{\Gamma(\delta_3+\beta_3 n)}. \tag{1.10}$$

2 A New Mittag-Leffler Type Function

Let us note that special functions mentioned in Section 1, such as the classical Mittag-Leffler function (1.1), (1.2) and its generalizations as (1.3), (1.5), (1.8) can be presented as particular cases of the Fox-Wright function (1.4).

Avoiding increasing number of variables and parameters, in the present paper, we investigate the following Mittag-Leffler type function with four parameters $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ as

$$ {}_{\gamma,\delta}E_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{(\gamma)_{\delta n}}{\Gamma(\alpha n+\beta)} x^n, x \in \mathbb{C}, Re(\alpha) > Re(\delta) > 0. \tag{2.1} $$

Remark 2.1. *Particularly for $\delta = 1$ in (2.1), ${}_{\gamma,\delta}E_{\alpha,\beta}(x)$ reduces to another new generalized Mittag-Leffler type function as*

$$ {}_{\gamma,1}E_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{(\gamma)_n x^n}{\Gamma(\alpha n+\beta)}. \tag{2.2} $$

Remark 2.2. *On taking $\delta \to 0$ in (2.1), ${}_{\gamma,\delta}E_{\alpha,\beta}(x)$ reduces to the Mittag-Leffler function $E_{\alpha,\beta}(x)$ as*

$$ {}_{\gamma,0}E_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n+\beta)} = E_{\alpha,\beta}(x). \tag{2.3} $$

Remark 2.3. *On taking $\delta \to 0$, $\beta = 1$ in (2.1), ${}_{\gamma,\delta}E_{\alpha,\beta}(x)$ reduces to the Mittag-Leffler function $E_\alpha(x)$ as*

$$ {}_{\gamma,0}E_{\alpha,1}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n+1)} = E_\alpha(x). \tag{2.4} $$

Theorem 2.4. *For $Re(\alpha) > Re(\delta) > 0$, the Mittag-Leffler type function ${}_{\gamma,\delta}E_{\alpha,\beta}(x)$ defined by (2.1) is an entire function in the complex plane. The order ρ and type σ of ${}_{\gamma,\delta}E_{\alpha,\beta}(x)$ are given by*

$$ \frac{1}{\rho} = Re(\alpha - \delta) \tag{2.5} $$

and

$$ \sigma = \frac{1}{\rho}\left(\frac{\{Re(\delta)\}^{Re(\delta)}}{\{Re(\alpha)\}^{Re(\alpha)}}\right)^{\rho}. \tag{2.6} $$

Further when $Re(\alpha) = Re(\delta) > 0$, the power series in (2.1) converges absolutely for $|x| < 1$.

Proof. Here we follow the classical techniques used by Kriyakova [6], and Paneva-Konovska [10] to find the order and type of Mittag-Leffler type functions. The radius of convergence R of the power series ${}_{\gamma,\delta}E_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{(\gamma)_{\delta n} x^n}{\Gamma(\alpha n+\beta)} = \sum_{n=0}^{\infty} \phi_n x^n$ is given by

$$ R = \limsup_{n\to\infty} \left|\frac{\phi_n}{\phi_{n+1}}\right|. \tag{2.7} $$

Using the asymptotic formula [14]

$$ \frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\left[1 + \frac{1}{2z}(a-b)(a+b-1) + O\left(\frac{1}{z^2}\right)\right], \tag{2.8} $$

where a and b are fixed arbitrary complex numbers and $-\pi < \arg z < \pi$, we get from (2.1)

$$R = \limsup_{n\to\infty}\left|\frac{\phi_n}{\phi_{n+1}}\right| = \limsup_{n\to\infty}\left|\frac{\Gamma(\gamma+\delta n)}{\Gamma(\gamma+\delta+\delta n)}.\frac{\Gamma(\alpha n+\alpha+\beta)}{\Gamma(\alpha n+\beta)}\right| \sim \frac{\{Re(\alpha)\}^{Re(\alpha)}}{\{Re(\delta)\}^{Re(\delta)}}.n^{Re(\alpha-\delta)}$$

$$= \begin{cases} \infty, & \text{when } Re\,(\alpha) > Re\,(\delta) > 0 \\ 1, & \text{when } Re\,(\alpha) = Re\,(\delta) > 0 \end{cases}, \tag{2.9}$$

which proves first part of the theorem.

The order ρ of an entire function ${}_{\gamma,\delta}E_{\alpha,\beta}\,(x) = \sum_{n=0}^{\infty}\phi_n x^n$ is given by the formula

$$\rho = \limsup_{n\to\infty}\frac{n\ln n}{\ln\left[1/\left|\phi_n\right|\right]}. \tag{2.10}$$

The Stirling formula in the form [14]

$$\Gamma\,(z+a) \sim \sqrt{2\pi}z^{z+a-\frac{1}{2}}.e^{-z}, |z| \to \infty, \tag{2.11}$$

gives

$$\frac{1}{|\phi_n|} = \left|\frac{\Gamma(\gamma)\Gamma(\alpha n+\beta)}{\Gamma(\gamma+\delta n)}\right| \sim \left|\Gamma\,(\gamma)\right| e^{Re\left[\left(\alpha n+\beta-\frac{1}{2}\right)\ln(\alpha n)-\left(\gamma+\delta n-\frac{1}{2}\right)\ln(\delta n)-(\alpha-\delta)n\right]}, n \to \infty. \tag{2.12}$$

Hence by definition (2.10), we have

$$\frac{1}{\rho} = \limsup_{n\to\infty}\frac{\ln\left[1/\left|\phi_n\right|\right]}{n\ln n}$$

$$= \lim_{n\to\infty}\frac{\ln|\Gamma(\gamma)|+Re\left[\left(\alpha n+\beta-\frac{1}{2}\right)\ln(\alpha n)-\left(\gamma+\delta n-\frac{1}{2}\right)\ln(\delta n)-(\alpha-\delta)n\right]}{n\ln n}$$
$$= Re\,(\alpha-\delta)\,,$$

which is the required result (2.5).

Further, the type σ of the function ${}_{\gamma,\delta}E_{\alpha,\beta}\,(x) = \sum_{n=0}^{\infty}\phi_n x^n$ of order ρ is determined by the relation

$$\sigma e\rho = \limsup_{n\to\infty}\left[n\left|\phi_n\right|^{\rho/n}\right]. \tag{2.13}$$

Proceeding as above for finding ρ, we can easily obtain

$$\sigma = \frac{1}{\rho}\left(\frac{\{Re\,(\delta)\}^{Re(\delta)}}{\{Re\,(\alpha)\}^{Re(\alpha)}}\right)^{\rho}, \tag{2.14}$$

which is as given by (2.6). □

3 Some Results for ${}_{\gamma,\delta}E_{\alpha,\beta}\left(x\right)$

Integral Representations

Result 1(a). For $x, \alpha, \beta, \gamma, \delta \in \mathbb{C}$, $Re\left(\alpha\right) > Re\left(\delta\right) > 0$, $Re\left(\gamma\right) > 0$, the following integral representation of ${}_{\gamma,\delta}E_{\alpha,\beta}\left(x\right)$ holds

$$ {}_{\gamma,\delta}E_{\alpha,\beta}\left(x\right) = \frac{1}{\Gamma\left(\gamma\right)} \int_0^{\infty} e^{-t} t^{\gamma-1} E_{\alpha,\beta}\left(xt^{\delta}\right) dt. \tag{3.1} $$

(b) For $x, \alpha, \beta, \gamma, \delta \in \mathbb{C}$, $Re\left(\alpha\right) > Re\left(\delta\right) > 0$, $Re\left(\beta\right) > Re\left(\gamma\right) > 0$, we obtain an integral representation of ${}_{\gamma,\alpha}E_{\alpha,\beta}\left(x\right)$, as follows:

$$ {}_{\gamma,\alpha}E_{\alpha,\beta}\left(x\right) = \frac{1}{\Gamma\left(\gamma\right)\Gamma\left(\beta-\gamma\right)} \int_0^{1} t^{\gamma-1}\left(1-t\right)^{\beta-\gamma-1}\left(1-xt^{\alpha}\right)^{-1} dt. \tag{3.2} $$

Proof. (a) The gamma function is defined by

$$ \Gamma\left(z\right) = \int_0^{\infty} e^{-t} t^{z-1} dt, Re\left(z\right) > 0, \tag{3.3} $$

we obtain the integral representation (3.1) by making use of this definition in the power series given by (2.1) with $Re\left(\gamma\right) > 0$ and changing the order of summation and integration, permissible under the conditions stated with the result, evaluating the integral and expressing the power series thus obtained, in terms of the Mittag-Leffler function using definition (1.2).

(b) For the integral representation (3.2) of ${}_{\gamma,\alpha}E_{\alpha,\beta}\left(x\right)$, we write

$$ {}_{\gamma,\alpha}E_{\alpha,\beta}\left(x\right) = \sum_{n=0}^{\infty} \frac{\left(\gamma\right)_{\alpha n}}{\Gamma\left(\alpha n+\beta\right)} x^n = \frac{1}{\Gamma\left(\gamma\right)\Gamma\left(\beta-\gamma\right)} \sum_{n=0}^{\infty} \frac{\Gamma\left(\gamma+\alpha n\right)\Gamma\left(\beta-\gamma\right)}{\Gamma\left(\alpha n+\beta\right)} x^n. \tag{3.4} $$

Now using the definition of beta function in (3.4), changing the order of summation and integration and evaluating the integral thus obtained, we arrive at the integral representation (3.2). □

Recurrence Relations

Result 2. For $x, \alpha, \beta, \gamma, \delta \in \mathbb{C}$, $Re\left(\alpha\right) > Re\left(\delta\right) > 0$, we have

(a)

$$ {}_{\gamma,\delta}E_{\alpha,\beta}\left(x\right) = \beta\,{}_{\gamma,\delta}E_{\alpha,\beta+1}\left(x\right) + \alpha x\,{}_{\gamma,\delta}E'_{\alpha,\beta+1}\left(x\right), \tag{3.5} $$

(b) $\beta\left(\beta+1\right){}_{\gamma,\delta}E_{\alpha,\beta+3}\left(x\right) + \alpha x\left\{2+\alpha+2\beta\right\}{}_{\gamma,\delta}E'_{\alpha,\beta+3}\left(x\right) + \alpha^2 x^2\,{}_{\gamma,\delta}E''_{\alpha,\beta+3}\left(x\right)$

$$ = {}_{\gamma,\delta}E_{\alpha,\beta+1}\left(x\right) - {}_{\gamma,\delta}E_{\alpha,\beta+2}\left(x\right), \tag{3.6} $$

where ${}_{\gamma,\delta}E'_{\alpha,\beta}\left(x\right)$ denotes differentiation of ${}_{\gamma,\delta}E_{\alpha,\beta}\left(x\right)$ with respect to x.

Proof. (a) The recurrence relation (3.5) can easily be obtained by considering its right-hand side, performing differentiation and then writing the two series in one to form ${}_{\gamma,\delta}E_{\alpha,\beta}(x)$.

(b) To prove recurrence relation (3.6), we begin with ${}_{\gamma,\delta}E_{\alpha,\beta+2}(x)$ and write it as

$$ {}_{\gamma,\delta}E_{\alpha,\beta+2}(x) = {}_{\gamma,\delta}E_{\alpha,\beta+1}(x) - S, \tag{3.7}$$

where

$$S = \sum_{n=0}^{\infty} \frac{(\gamma)_{\delta n}}{(\alpha n+\beta+1)\,\Gamma(\alpha n+\beta)} x^n. \tag{3.8}$$

Using simple identity $\frac{1}{u} = \frac{1}{u(u+1)} + \frac{1}{u+1}$ and the result $\Gamma(n+1) = n\Gamma(n)$, we can write

$$S = \sum_{n=0}^{\infty} \frac{(\gamma)_{\delta n}(\alpha n+\beta)}{\Gamma(\alpha n+\beta+3)} x^n + \sum_{n=0}^{\infty} \frac{(\gamma)_{\delta n}(\alpha n+\beta+1)(\alpha n+\beta)}{\Gamma(\alpha n+\beta+3)} x^n$$

$$= p\sum_{n=0}^{\infty} \frac{(\gamma)_{\delta n}\,x^n}{\Gamma(\alpha n+\beta+3)} + q\sum_{n=0}^{\infty} \frac{(\gamma)_{\delta n}\,n}{\Gamma(\alpha n+\beta+3)} x^n + r\sum_{n=0}^{\infty} \frac{(\gamma)_{\delta n}\,n^2}{\Gamma(\alpha n+\beta+3)} x^n, \tag{3.9}$$

where $p = \beta(\beta+2)$, $q = 2\alpha(\beta+1)$ and $r = \alpha^2$.

Next, we observe that

$$\frac{d}{dx}\left[x\,{}_{\gamma,\delta}E_{\alpha,\beta+3}(x)\right] = \sum_{n=0}^{\infty} \frac{(\gamma)_{\delta n}(n+1)}{\Gamma(\alpha n+\beta+3)} x^n$$

and

$$\frac{d^2}{dx^2}\left[x^2\,{}_{\gamma,\delta}E_{\alpha,\beta+3}(x)\right] = \sum_{n=0}^{\infty} \frac{(\gamma)_{\delta n}\left(n^2+3n+2\right)}{\Gamma(\alpha n+\beta+3)} x^n,$$

which gives

$$\sum_{n=0}^{\infty} \frac{(\gamma)_{\delta n}\,n}{\Gamma(\alpha n+\beta+3)} x^n = x\,{}_{\gamma,\delta}E'_{\alpha,\beta+3}(x) \tag{3.10}$$

and

$$\sum_{n=0}^{\infty} \frac{(\gamma)_{\delta n}\,n^2}{\Gamma(\alpha n+\beta+3)} x^n = x\,{}_{\gamma,\delta}E'_{\alpha,\beta+3}(x) + x^2\,{}_{\gamma,\delta}E''_{\alpha,\beta+3}(x). \tag{3.11}$$

Using (3.10) and (3.11) in (3.9), we obtain the value of S which being substituted in (3.7), provides the recurrence relation (3.6). □

Differential Formula

Result 3. For $w, x, \alpha, \beta, \gamma, \delta \in \mathbb{C}$, $Re(\alpha) > Re(\delta) > 0$ and $m \in \mathbb{N}$, we have

$$\left(\frac{d}{dx}\right)^m \left[x^{\beta-1}\,{}_{\gamma,\delta}E_{\alpha,\beta}(wx^\alpha)\right] = x^{\beta-m-1}\,{}_{\gamma,\delta}E_{\alpha,\beta-m}(wx^\alpha). \tag{3.12}$$

Proof. Using the definition of ${}_{\gamma,\delta}E_{\alpha,\beta}\left(wx^{\alpha}\right)$ given by (2.1), in the left hand side of (3.12), then changing the order of differentiation and summation and writing the result in terms of ${}_{\gamma,\delta}E_{\alpha,\beta-m}\left(wx^{\alpha}\right)$ we arrive at the required result (3.12). □

Fractional Calculus Operators

Result 4. Let $a \in \mathbb{R}^{+}$ and $\omega, \lambda, \alpha, \beta, \gamma, \delta \in \mathbb{C}$, $Re\left(\alpha\right) > Re\left(\delta\right) > 0$ then for $x > a$ there hold the relations

$$I_{a+}^{\lambda}\left(\left(t-a\right)^{\beta-1}{}_{\gamma,\delta}E_{\alpha,\beta}\left[\omega\left(t-a\right)^{\alpha}\right]\right)\left(x\right) = \left(x-a\right)^{\beta+\lambda-1}{}_{\gamma,\delta}E_{\alpha,\beta+\lambda}\left[\omega\left(x-a\right)^{\alpha}\right], \tag{3.13}$$

$$D_{a+}^{\lambda}\left(\left(t-a\right)^{\beta-1}{}_{\gamma,\delta}E_{\alpha,\beta}\left[\omega\left(t-a\right)^{\alpha}\right]\right)\left(x\right) = \left(x-a\right)^{\beta-\lambda-1}{}_{\gamma,\delta}E_{\alpha,\beta-\lambda}\left[\omega\left(x-a\right)^{\alpha}\right]. \tag{3.14}$$

Proof. The Riemann-Liouville fractional integral operator of order $\lambda \in \mathbb{C}$, $Re\left(\lambda\right) > 0$ is defined as (Miller & Ross [16], Samko *et al.* [15])

$$I_{a+}^{\lambda}\left(f\left(t\right)\right)\left(x\right) = \frac{1}{\Gamma\left(\lambda\right)}\int_{a}^{x}\left(x-t\right)^{\lambda-1}f\left(t\right)dt, x > a. \tag{3.15}$$

The Riemann-Liouville fractional derivative of order $\lambda \in \mathbb{C}$, $n-1 < Re\left(\lambda\right) \leq n$, $n \in \mathbb{N}$ is defined as (Miller & Ross [16])

$$D_{a+}^{\lambda}\left(f\left(t\right)\right)\left(x\right) = D^{n}I_{a+}^{n-\lambda}f\left(x\right) = \frac{1}{\Gamma\left(\lambda\right)}D^{n}\int_{a}^{x}\left(x-t\right)^{n-\lambda-1}f\left(t\right)dt, x > a. \tag{3.16}$$

Using definitions (3.15) and (2.1) and applying the known result (Samko *et al.* [15])

$$I_{a+}^{\lambda}\left(\left(x-a\right)^{\beta-1}\right) = \frac{\Gamma\left(\beta\right)}{\Gamma\left(\lambda+\beta\right)}\left(x-a\right)^{\beta+\lambda-1}, \lambda, \delta \in \mathbb{C}, Re\left(\lambda\right), Re\left(\beta\right) > 0, \tag{3.17}$$

we obtain (3.13). □

Next, using (2.1), (3.16) and (3.13) we arrive at (3).

Mellin-Barnes Integral Representation

Result 5. For $\min\left(\delta,\alpha\right) > 0, \alpha < \delta + 2$ and $\left|argx\right| < \frac{\pi}{2}\left(\delta-\alpha+2\right)$ we have

$${}_{\gamma,\delta}E_{\alpha,\beta}\left(x\right) = \frac{1}{2\pi i\Gamma\left(\gamma\right)}\int_{L}\frac{\Gamma\left(-s\right)\Gamma\left(1+s\right)\Gamma\left(\gamma+\delta s\right)}{\Gamma\left(\beta+\alpha s\right)}x^{s}ds, \tag{3.18}$$

where for details of contour L and convergence of (3.18), we refer the book by Srivastava *et al.* [17].

Proof. Making use of relation between ${}_{\gamma,\delta}E_{\alpha,\beta}(x)$ and the H-function by comparing their series representations, we obtain the Mellin-Barnes integral representation as follows

$$ {}_{\gamma,\delta}E_{\alpha,\beta}(x) = H^{1,2}_{2,2}\left[x\left|\begin{array}{c}(1-\gamma,\delta),(0,1)\\(0,1),(1-\beta,\alpha)\end{array}\right.\right] = \frac{1}{2\pi i\Gamma(\gamma)}\int_L \tfrac{\Gamma(-s)\Gamma(1+s)\Gamma(\gamma+\delta s)}{\Gamma(\beta+\alpha s)} x^s ds. \tag{3.19}$$

□

Integral Transforms

Result 6. Euler (Beta) transform

For $a,b,\alpha,\beta,\gamma,\delta,\sigma\in\mathbb{C}$, $Re(a), Re(b), Re(\sigma)>0$, $Re(\alpha)>Re(\delta)>0$, we have

$$\int_0^1 u^{a-1}(1-u)^{b-1}\,{}_{\gamma,\delta}E_{\alpha,\beta}(xu^\sigma)\,du = \frac{\Gamma(b)}{\Gamma(\gamma)}{}_3\psi_2\left[\begin{array}{c}(\gamma,\delta),(a,\sigma),(1,1);\\(\beta,\alpha)(a+b,\sigma);\end{array}x\right]. \tag{3.20}$$

Particularly for $a=\beta$, $b=\mu$, $\sigma=\alpha$, (3.20) reduces to the integral

$$\int_0^1 u^{\beta-1}(1-u)^{\mu-1}\,{}_{\gamma,\delta}E_{\alpha,\beta}(xu^\alpha)\,du = \Gamma(\mu)\,{}_{\gamma,\delta}E_{\alpha,\beta+\mu}(x). \tag{3.21}$$

Proof. Using the definition of ${}_{\gamma,\delta}E_{\alpha,\beta}(x)$ given by (2.1), changing the order of differentiation and integration, which is permissible under the given conditions, using the definition of the beta function and writing the power series thus obtained in terms of the Fox-Wright function ${}_p\psi_q$ defined by (1.4), we arrive at the required result (3.20).

Particularly for $a=\beta$, $b=\mu$, $\sigma=\alpha$, we have

$$\tfrac{\Gamma(\mu)}{\Gamma(\gamma)}{}_3\psi_2\left[\begin{array}{c}(\gamma,\delta),(a,\sigma),(1,1);\\(\beta,\alpha)(a+b,\sigma);\end{array}x\right] = \frac{\Gamma(\mu)}{\Gamma(\gamma)}\sum_{n=0}^{\infty}\tfrac{\Gamma(\gamma+\delta n)}{\Gamma(\beta+\mu+\alpha n)}x^n = \Gamma(\mu)\,{}_{\gamma,\delta}E_{\alpha,\beta+\mu}(x),$$

which proves (3.21). □

Result 7. Laplace transform

(a) For $a,\alpha,\beta,\gamma,\delta,\sigma,s\in\mathbb{C}$, $Re(\alpha)>Re(\delta+\sigma)$, $Re(\delta), Re(\sigma), Re(s)>0$, we have

$$L\left\{u^{a-1}{}_{\gamma,\delta}E_{\alpha,\beta}(xu^\sigma);s\right\}$$

$$=\int_0^\infty u^{a-1}e^{-su}{}_{\gamma,\delta}E_{\alpha,\beta}(xu^\sigma)\,du = \frac{s^{-a}}{\Gamma(\gamma)}{}_3\psi_1\left[\begin{array}{c}(\gamma,\delta),(a,\sigma),(1,1);\\(\beta,\alpha);\end{array}\frac{x}{s^\sigma}\right], \tag{3.22}$$

(b) $L^{-1}\left\{s^{-1}{}_{\gamma,\delta}E_{\alpha,\beta}\left(s^{-1}\right);x\right\} = E^{\gamma,\delta}_{\alpha,\beta}(x)$.

Proof. (a) Using the definition of the generalized Mittag-Leffler type function ${}_{\gamma,\delta}E_{\alpha,\beta}\left(x\right)$ given by (2.1), changing the order of summation and integration, evaluating the integral and writing the resulting series in terms of the Fox-Wright function ${}_p\psi_q$ defined by (1.4), we arrive at the result (3.22).
(b) By taking the Laplace transform of the generalized Mittag-Leffler type function $E_{\alpha,\beta}^{\gamma,\delta}\left(x\right)$, we arrive at the required result. □

Result 8. Mellin transform
For $\alpha > \delta > 0, 0 < s < \frac{\gamma}{\delta}$we have

$$M\left\{{}_{\gamma,\delta}E_{\alpha,\beta}\left(wx\right);s\right\} = \int_0^\infty x^{s-1}{}_{\gamma,\delta}E_{\alpha,\beta}\left(wx\right)dx = \frac{\Gamma\left(s\right)\Gamma\left(1-s\right)\Gamma\left(\gamma-\delta s\right)}{w^s\Gamma\left(\gamma\right)\Gamma\left(\beta-\alpha s\right)}. \tag{3.23}$$

Proof. Let $M\left\{{}_{\gamma,\delta}E_{\alpha,\beta}\left(wx\right);s\right\} = f\left(s\right)$ then by the Mellin inversion theorem, we have

$$M^{-1}\left\{f\left(s\right);x\right\} = \frac{1}{2\pi i}\int_L f\left(s\right)x^{-s}ds. \tag{3.24}$$

Using the Mellin Barnes integral representation of ${}_{\gamma,\delta}E_{\alpha,\beta}\left(x\right)$, given by (3.18) and the Mellin inversion theorem (3.24), we obtain

$$f\left(s\right) = \frac{\Gamma\left(-s\right)\Gamma\left(1-s\right)\Gamma\left(\gamma-\delta s\right)}{w^s\Gamma\left(\gamma\right)\Gamma\left(\beta-\alpha s\right)},$$

which proves the required result (3.23). □

4 An Integral Operator Involving the Generalized Mittag-Leffler Type Function and Its Properties

We consider the following integral operator

$$\left(\Im_{a+;\alpha,\beta}^{\omega,\gamma,\delta}\phi\right)\left(x\right) = \int_a^x \left(x-t\right)^{\beta-1}{}_{\gamma,\delta}E_{\alpha,\beta}\left[\omega\left(x-t\right)^{\alpha}\right]\phi\left(t\right)dt, x > a,$$

where $\gamma,\omega \in \mathbb{C}, Re\left(\alpha\right) > Re\left(\delta\right) > 0, Re\left(\beta\right) > 0$.

Particularly for $\omega = 0$, $\Im_{a+;\alpha,\beta}^{\omega,\gamma,\delta}$ corresponds to the right handed Riemann-Liouville fractional integral operator.

Theorem 4.1. *Under the various parametric constraints stated with the definition* (2.1)*, let the function* ϕ *be in the space* $L\left(a,b\right)$ *of Lebesgue measurable functions given by*

$$L\left(a,b\right) = \left\{f : \left\|f\right\|_1 = \int_a^b \left|f\left(x\right)\right|dx < \infty\right\}, \tag{4.1}$$

then the integral operator $\Im^{\omega,\gamma,\delta}_{a+;\alpha,\beta}\phi$ *is bounded on* $L(a,b)$ *and*

$$\left\|\Im^{\omega,\gamma,\delta}_{a+;\alpha,\beta}\phi\right\|_1 \leq B\left\|\phi\right\|_1, \tag{4.2}$$

where

$$B = (b-a)^{Re(\beta)} \sum_{n=0}^{\infty} \frac{|(\gamma)_{\delta n}| \left|\omega (b-a)^{Re(\alpha)}\right|^n}{\{Re(\alpha)n + Re(\beta)\}|\Gamma(\alpha n+\beta)|}. \tag{4.3}$$

Proof. First of all, we observe that if ϕ_n denote the n^{th} term of the series in (4.3) then

$$\frac{\phi_{n+1}}{\phi_n} = \frac{|\Gamma(\gamma+\delta+\delta n)|}{|\Gamma(\gamma+\delta n)|}\cdot\frac{\{Re(\alpha)n+Re(\beta)\}}{\{Re(\alpha)(n+1)+Re(\beta)\}}\cdot\frac{|\Gamma(\alpha n+\beta)|}{|\Gamma(\alpha n+\alpha+\beta)|}|\omega|(b-a)^{Re(\alpha)}$$

$$\sim \frac{|\delta|^{Re(\delta)}|\omega|(b-a)^{Re(\alpha)}}{|\alpha|^{Re(\alpha)}} n^{Re(\delta)-Re(\alpha)}$$

$$\to 0,\ \text{as}\ n\to\infty, when Re(\alpha) > Re(\delta) > 0 \tag{4.4}$$

Hence the series in right hand side of (4.3) is bounded and the constant B is finite.

Using the definition of $\|.\|_1$, given in (4.1) and the definition of integral operator, given in (2.1) and then interchanging the order of integration, we have

$$\left\|\Im^{\omega,\gamma,\delta}_{a+;\alpha,\beta}\phi\right\|_1 = \int_a^b \left|\int_a^x (x-t)^{\beta-1}\,{}_{\gamma,\delta}E_{\alpha,\beta}\left[w(x-t)^{\alpha}\right]\phi(t)\,dt\right| dx$$

$$\leq \int_a^b \left[\int_t^b (x-t)^{Re(\beta)-1}\left|{}_{\gamma,\delta}E_{\alpha,\beta}\left[w(x-t)^{\alpha}\right]\right| dx\right]|\phi(t)|\,dt$$
$$= \int_a^b \left[\int_0^{b-t} u^{Re(\beta)-1}\left|{}_{\gamma,\delta}E_{\alpha,\beta}\left[wu^{\alpha}\right]\right| du\right]|\phi(t)|\,dt$$

$$\leq \int_a^b \left[\int_0^{b-a} u^{Re(\beta)-1}\left|{}_{\gamma,\delta}E_{\alpha,\beta}\left[wu^{\alpha}\right]\right| du\right]|\phi(t)|\,dt.$$

For the inner integral, using (2.1), carrying out term by term integration and taking into account (4.3), we obtain

$$\int_0^{b-a} u^{Re(\beta)-1}\left|{}_{\gamma,\delta}E_{\alpha,\beta}\left[wu^{\alpha}\right]\right| du \leq \sum_{n=0}^{\infty}\frac{|(\gamma)_{\delta n}||\omega|^n}{|\Gamma(\alpha n+\beta)|}\int_0^{b-a} u^{Re(\beta)+Re(\alpha)n-1}du = B. \tag{4.5}$$

Now using (4.5) in, we arrive at (4.2). □

Theorem 4.2. *Let* $\lambda\in\mathbb{C}$, $Re(\lambda)>0$ *and* $\omega,\gamma,\delta,\alpha,\beta\in\mathbb{C}$, *then the relations* $I^{\lambda}_{a+}\Im^{\omega,\gamma,\delta}_{a+;\alpha,\beta}\phi = \Im^{\omega,\gamma,\delta}_{a+;\alpha,\beta+\lambda}\phi = \Im^{\omega,\gamma,\delta}_{a+;\alpha,\beta}I^{\lambda}_{a+}\phi$ *hold for function* $\phi\in L(a,b)$.

Proof. Using (2.1) and (3.15) and interchanging the order of integration, we have for $x > a$

$$\begin{aligned}\left(I_{a+}^{\lambda}\Im_{a+;\alpha,\beta}^{\omega,\gamma,\delta}\phi\right)(x) &= \tfrac{1}{\Gamma(\lambda)}\int_a^x (x-u)^{\lambda-1}\left[\int_a^u (u-t)^{\beta-1}\,{}_{\gamma,\delta}E_{\alpha,\beta}\left[\omega\,(u-t)^{\alpha}\right]\phi\,(t)\,dt\right]du\\ &= \frac{1}{\Gamma(\lambda)}\int_a^x\left[\int_t^x (x-u)^{\lambda-1}(u-t)^{\beta-1}\,{}_{\gamma,\delta}E_{\alpha,\beta}\left[\omega\,(u-t)^{\alpha}\right]du\right]\phi\,(t)\,dt\\ &= \frac{1}{\Gamma(\lambda)}\int_a^x\left[\int_0^{x-t}(x-t-\tau)^{\lambda-1}\tau^{\beta-1}{}_{\gamma,\delta}E_{\alpha,\beta}\left(\omega\tau^{\alpha}\right)d\tau\right]\phi\,(t)\,dt\\ &= \frac{1}{\Gamma(\lambda)}\int_a^x\left[I_{0+}^{\lambda}\left[\tau^{\beta-1}{}_{\gamma,\delta}E_{\alpha,\beta}\left(\omega\tau^{\alpha}\right)\right](x-t)\right]\phi\,(t)\,dt. \qquad (4.6)\end{aligned}$$

Using formula (3.13), we obtain

$$\begin{aligned}\left(I_{a+}^{\lambda}\Im_{a+;\alpha,\beta}^{\omega,\gamma,\delta}\phi\right)(x) &= \frac{1}{\Gamma(\lambda)}\int_a^x (x-t)^{\beta+\lambda-1}\,{}_{\gamma,\delta}E_{\alpha,\beta+\lambda}\left(\omega\,(x-t)^{\alpha}\right)\phi\,(t)\,dt\\ &= \left(\Im_{a+;\alpha,\beta+\lambda}^{\omega,\gamma,\delta}\phi\right)(x),\end{aligned}$$

which proves first part of the relation (4.2) and the rest can be proved similarly. □

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