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A Common Fixed Point Theorem in Fuzzy Metric Space Using the Property (CLRg)

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Abstract : In this paper, we generalize the results of Kumar and Fisher [S. Kumar, B. Fisher, A common fixed point theorem in fuzzy metric space using property (E.A.) and implicit relation, Thai J. Math. 8 (3) (2010) 439–446.] using weakly compatible mappings along with property (CLRg). We also provide an example in support our result.

Keywords : fuzzy metric space; common fixed point; weakly compatible maps; implicit relation; property (CLRg).
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1 Introduction

It proved a turning point in the development of mathematics when the notion of fuzzy set was introduced by Zadeh [1]. This notion laid the foundation of fuzzy mathematics. Kramosil and Michalek [2] introduced the notion of a fuzzy metric space by generalizing the concept of the probabilistic metric space to the fuzzy situation. George and Veeramani [3] modified the concept of fuzzy metric spaces introduced by Kramosil and Michalek [2]. There are many view points of the notion of the metric space in fuzzy topology for instance one can refer to Kaleva and Seikkala [4], Kramosil and Michalek [2] and George and Veeramani [3]. This proved a milestone in fixed point point theory of fuzzy metric space and afterwards

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a flood of papers appeared for fixed point theorems in fuzzy metric space.

Mishra et al. [5] introduced the concept of compatible maps in FM-spaces which was further generalised by Singh and Jain [6] by introducing the notion of weak compatibility in FM-spaces. In 2002, Aamri and Moutawakil [7] introduced property (E.A.), which is a true generalization of non-compatible maps in metric spaces. Common fixed points for a pair of maps under the notion of property (E.A.) and non-compatible maps were studied by Pant and Pant [8]. Recently, Sintunavarat and Kumam [9] introduced a new concept of property (CLRg). The importance of property (CLRg) ensures that one does not require the closeness of range subspaces and hence, now a days, authors are giving much attention to this property for generalizing the results present in the literature. Works noted in the references [10–14] are some examples.

Popa [15, 16] introduced the idea of implicit function to prove a common fixed point theorem in metric spaces. Jain [17] further extended the result of Popa [15, 16] in fuzzy metric spaces. Afterwards, implicit relations are used as a tool for finding common fixed point of contraction maps (see, [18–23]). Altun and Turkoglu [24] proved two common fixed point theorems on complete FM-space with an implicit relation. In [24], common fixed point theorems have been proved for continuous compatible maps of type (α) or (β). Kumar and Fisher [25] generalized the results of Altun and Turkoglu [24] by removing the assumption of continuity, relaxing compatibility to weak compatibility and replacing the completeness of the space with a set of four alternative conditions for functions satisfying an implicit relation in FM-space. Our aim is to further generalize the result of Kumar and Fisher [25] by using the property (CLRg) and relaxing many conditions involved.

2 Preliminaries

Before we give our main result we need the following definitions:

Definition 2.1 ([1])**.** A fuzzy set A in X is a function with domain X and values in $[0,1]$.

Definition 2.2 ([26])**.** A binary operation $*:[0,1]\times[0,1]\to[0,1]$ is a continuous t-norm if $([0,1],*)$ is a topological abelian monoid with unit 1 s.t. $a*b\le c*d$ whenever $a\le c$ and $b\le d$ for all $a,b,c,d\in[0,1]$.

Definition 2.3 ([3])**.** The 3-tuple $(X,M,*)$ is called a *fuzzy metric space* if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2\times[0,\infty)$ satisfying the following conditions:

(FM-1) $M(x,y,0)>0$,

(FM-2) $M(x,y,t)=1$ iff $x=y$,

(FM-3) $M(x,y,t)=M(y,x,t)$,

(FM-4) $M(x,y,t)*M(y,z,s)\le M(x,z,t+s)$, (FM-5) $M(x, y, .) : (0, \infty) \to [0, 1]$ is continuous, for all $x, y, z \in X$ and $s, t > 0$.

Throughout this paper, we consider M to be a fuzzy metric space with condition:

(FM-6) $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$.

Definition 2.4 ([3]). Let $(X, M, *)$ be fuzzy metric space. A sequence $\{x_n\}$ in X is said to be

(i) *Convergent to a point* $x \in X$, if $\lim\limits_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$;

(ii) *Cauchy sequence* if $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$, for all $t > 0$ and $p > 0$.

Definition 2.5 ([3]). A fuzzy metric space $(X, M, *)$ is said to be *complete* if and only if every Cauchy sequence in X is convergent.

Lemma 2.6 ([27]). *$M(x, y, \cdot)$ is non-decreasing for all $x, y \in X$.*

Lemma 2.7 ([27]). *Let $x_n \to x$ and $y_n \to y$, then*

(i) *$\lim_{n\to\infty} M(x_n, y_n, t) \geq M(x, y, t)$, for all $t > 0$,*

(ii) *$\lim_{n\to\infty}(x_n, y_n, t) = M(x, y, t)$, for all $t > 0$, if $M(x, y, t)$ is continuous.*

Lemma 2.8 ([5]). *If for all $x, y \in X$, $t > 0$ and for a number $k \in (0, 1)$;*

$$M(x, y, kt) \geq M(x, y, t), \quad \text{then } x = y.$$

Definition 2.9 ([5]). Let A and B be maps from a FM-space $(X, M, \cdot)$ into itself. The maps A and B are said to be *compatible* (or *asymptotically commuting*), if for all t,

$$\lim_{n\to\infty} M(AB_{xn}, BA_{xn}, t) = 1,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z \quad \text{for some } z \in X.$$

From the above definition it is inferred that A and B are non-compatible maps from a FM-space $(X, M, \cdot)$ into itself if $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty}$ for some $z \in X$, but either $\lim_{n\to\infty} M(AB_{xn}, BAx_n, t) \neq 1$ or the limit does not exist.

Definition 2.10 ([6]). Let A and B be maps from a FM-space $(X, M, \cdot)$ into itself. The maps are said to be *weakly compatible* if they commute at their coincidence points. Note that compatible mappings are weakly compatible but converse is not true in general.

Definition 2.11 ([8]). Let A and B be two self-maps of a FM-space $(X, M, \cdot)$. We say that A and B satisfy the property (E.A.) if there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z \quad \text{for some } z \in X.$$

Note that weakly compatible and property (E.A.) are independent to each other (see [15], Example 2.2).

Definition 2.12 ([9])**.** Let (X, d) be a metric space. Two mappings $f : X \to X$ and $g : X \to X$ are said to satisfy property (CLRg) if there exists sequences $\{x_n\}$ in X such that

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = g(p), \quad \text{for some } p \text{ in } X.$$

Similarly, we can have the property (CLR_T) and the property (CLR_S) if in the Definition 2.12, the mapping $g : X \to X$ has been replaced by the mapping $T : X \to X$ and $S : X \to X$ respectively.

Our result deal with the following implicit relation used by Altun and Turkoglu [24].

Definition 2.13 ([24])**.** Let $I = [0, 1]$, $*$ be a continuous t-norm and $\mathcal{F}$ be the set of all real continuous functions $F : I^6 \to R$ satisfying the following conditions:

(F-1) F is non-increasing in the fifth and sixth variables,

(F-2) if for some constant $k \in (0, 1)$ we have

(F-a) $F\left(u(kt) \cdot v(t), v(t), u(t), 1, u\left(\frac{t}{2}\right) * v\left(\frac{t}{2}\right)\right) \geq 1,$

or

(F-b) $F\left(u(kt), v(t), u(t), v(t), u\left(\frac{t}{2}\right) * v\left(\frac{t}{2}\right), 1\right) \geq 1,$

for any fixed $t > 0$ and any non-decreasing functions $u, v : (0, \infty) \to I$, then there exists $h \in (0, 1)$ with $u(ht) \geq v(t) * u(t)$,

(F-3) if for some constant $k \in (0, 1)$, we have $F(u(kt), u(t), 1, 1, u(t), u(t)) \geq 1$ for any fixed $t > 0$ and any non-decreasing function $u : (0, \infty) \to I$ then $u(kt) \geq u(t)$.

3 Main Results

In [24], Altun and Turkoglu proved the following result:

Theorem 3.1. *Let $(X, M, *)$ be a complete fuzzy metric space with $a * b = \min\{a, b\}$. Let A, B, S, T be maps from X into itself satisfying the following conditions:*

(3.1) *$A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$;*

(3.2) *one of the maps A, B, S, T is continuous;*

(3.3) *the pairs (A, S) and (B, T) are compatible of type (α);*

(3.4) *there exists* $k \in (0,1)$ *and* $F \in \mathcal{F}$ *such that*

$$F\{M(Ax,By,kt), M(Sx,Ty,t), M(Ax,Sx,t), M(By,Ty,t), M(Ax,Ty,t), M(By,Sx,t)\} \geq 1$$

for all $x, y \in X$ *and* $t > 0$.

Then A, B, S, T *have a unique common fixed point in* X.

In [25], Kumar and Fisher generalized Theorem 3.1 (which is Theorem 1 in [24]) as follows:

Theorem 3.2. *Let* $(X, M, *)$ *be a fuzzy metric space with* $a * b = \min\{a, b\}$. *Further, let* (A, S) *and* (B, T) *be weakly compatible pairs of self-maps of* X *satisfying* (3.1), (3.4) *with the following condition:*

(3.5) *one of the pairs* (A, S) *or* (B, T) *satisfies property (E.A.).*

If the range of one of the maps A, B, S *or* T *is a complete subspace of* X*, then* A, B, S, T *have a unique common fixed point in* X.

We now generalize Theorem 3.2 as follows:

Theorem 3.3. *Let* $(X, M, *)$ *be a fuzzy metric space with* $a * b = \min\{a, b\}$. *Let* A, B, S, T *be maps from* X *into itself satisfying (3.4) with the following conditions:*

(3.6) $B(X) \subseteq S(X)$ *and the pair* (B, T) *satisfies property* (CLR_T),

or

$A(X) \subseteq T(X)$ *and the pair* (A, S) *satisfies property* (CLR_S);

(3.7) *the pairs* (A, S) *and* (B, T) *are weakly compatible.*

Then A, B, S, T *have a unique common fixed point in* X.

Proof. Without loss of generality, assume that $B(X) \subseteq S(X)$ and the pair (B, T) satisfies property (CLR_T), then there exists a sequence $\{x_n\}$ in X such that Bx_n and Tx_n converges to Tx, for some x in X as $n \to \infty$. Since $B(X) \subseteq S(X)$, so there exists a sequence $\{y_n\}$ in X such that $Bx_n = Sy_n$, hence $Sy_n \to Tx$ as $n \to \infty$.

We shall show that $\lim_{n\to\infty} Ay_n = Tx$. Let $\lim_{n\to\infty} Ay_n = z$. Taking $x = y_n$, $y = x_n$ in (3.4),

$$F\{M(Ay_n, Bx_n, kt), M(Sy_n, Tx_n, t), M(Ay_n, Sy_n, t), M(Bx_n, Tx_n, t), M(Ay_n, Tx_n, t), M(Bx_n, Sy_n, t),\} \geq 1.$$

Letting $n \to \infty$, we have

$$F\{M(z, Tx, kt), 1, M(z, Tx, t), 1, M(z, Tx, t), 1\} \geq 1.$$

On the other hand, since

$$M(z,Tx,t) \geq M\left(z,Tx,\frac{t}{2}\right) = M\left(z,Tx,\frac{t}{2}\right) * 1,$$

and F is non-increasing in the fifth variable, we have, for any $t > 0$

$$F\left\{M(z,Tx,kt),1,M(z,Tx,t),1,M\left(z,Tx,\frac{t}{2}\right),1\right\} \\ \geq F\{M(z,Tx,kt),1,M(z,Tx,t),1,M(z,Tx,t)\} \geq 1,$$

which implies by (F-2), that $z = Tx$. Subsequently, we have Bx_n, Tx_n, Sy_n, Ay_n converges to z. We shall show that $Bx = z$.

Taking $x = y_n, y = x$ in (3.4),

$$F\{M(Ay_n,Bx,kt),M(Sy_n,Tx,t),M(Ay_n,Sy_n,t),M(Bx,Tx,t), \\ M(Ay_n,Tx,t),M(Bx,Sy_n,t)\} \geq 1.$$

Letting $n \to \infty$, we have

$$F\{M(z,Bx,kt),1,1,M(z,Bx,t),1,M(z,Bx,t)\} \geq 1.$$

On the other hand, since

$$M(z,Bx,t) \geq N\left(z,Bx,\frac{t}{2}\right) = M\left(z,Bx,\frac{t}{2}\right) * 1,$$

and F is non-increasing in the sixth variable, we have, for any $t > 0$

$$F\left\{M(z,Bx,kt),1,1,M(z,Bx,t),1,M\left(z,Bx,\frac{t}{2}\right) * 1\right\} \\ \geq F\{M(z,Bx,kt),1,1,M(z,Bx,t),1,M(z,Bx,t)\} \geq 1,$$

which implies by (F-2) that $z = Bx = Tx$. Since, the pair (B,T) is weak compatible, it follows that $Bz = Tz$.

Also, since $B(X) \subseteq S(X)$, there exists some y in X such that $Bx = Sy(= z)$.

We next show that $Sy = Ay(= z)$. Taking $y = x_n, x = y$ in (3.4),

$$F\{M(Ay,Bx_n,kt),M(Sy,Tx_n,t),M(Ay,Sy,t),M(Bx_n,Tx_n,t), \\ M(Ay,Tx_n,t),M(Bx_n,Sy,t)\} \geq 1.$$

Letting $n \to \infty$, we have

$$F\{M(Ay,z,kt),1,M(Ay,z,t),1,M(Ay,z,t),1\} \geq 1.$$

Other the other hand, since

$$M(Ay,z,t) \geq M\left(Ay,z,\frac{t}{2}\right) = M\left(Ay,z,\frac{t}{2}\right) * 1,$$

and F is non-increasing in the fifth variable, we have, for any $t > 0$

$$F\left\{M(Ay, z, kt), 1, M(Ay, z, t), 1, M\left(Ay, z, \frac{t}{2}\right) * 1, 1\right\}$$
$$\geq F\{M(Ay, z, kt), 1, M(Ay, z, t), 1, M(Ay, z, t), 1\} \geq 1,$$

which implies by (F-2) that $Ay = z = Sy$. But the pair (A, S) is weakly compatible, it follows that $Az = Sz$.

Next, we claim that $Az = Bz$. Taking $x = z, y = z$ in (3.4),

$$F\{M(Az, Bz, kt), M(Az, Bz, t), 1, 1, M(Az, Bz, t), M(Az, Bz, t)\} \geq 1,$$

which implies by (F-3) that $Az = Bz$. Hence, $Az = Bz = Sz = Tz$.

We now show that $z = Az$. Taking $x = z, y = x$ in (3.4),

$$F\{M(Az, Bx, kt), M(Sz, Tx, t), M(Az, Sz, t), M(Bx, Tx, t),$$
$$M(Az, Tx, t), M(Bx, Sz, t)\} \geq 1,$$

that is,

$$F\{M(Az, z, kt), M(Az, z, t), 1, 1, M(Az, z, t), M(Az, z, t)\} \geq 1.$$

Therefore, $z = Az = Bz = Sz = Tz$, that is z is the common fixed point of the maps A, B, S, T. Uniqueness of z follows immediately from (F-3) and (3.4). □

Example 3.4. Let $(X, M, *)$ be a fuzzy metric space with $X = [0, 1]$, a t-norm $*$ be defined by $a * b = \min\{a, b\}$ for all a, b in [0,1] and M be a fuzzy set on $X^2 \times (0, \infty)$ defined by

$$M(x, y, t) = \left[\exp\left(\frac{|x-y|}{t}\right)\right]^{-1}$$

for all x, y in X and $t > 0$.
Let $F : I^6 \to R$ be defiend by $F(u_1, u_2, u_3, u_4, u_5, u_6) = \dfrac{u_1}{\min\{u_2, u_3, u_4, u_5, u_6\}}$.
Let $t > 0$, $0 < u(t)$, $v(t) \leq 1$, $k \in (0, \frac{1}{2})$, where $u, v : [0, \infty) \to I$ are non-decreasing functions. Suppose that

$$F\left(u(kt), v(t), v(t)u(t), 1, u\left(\frac{t}{2}\right) * v\left(\frac{t}{2}\right)\right) \geq 1,$$

that is,

$$F\left(u(kt), v(t), v(t), u(t), 1, u\left(\frac{t}{2}\right) * v\left(\frac{t}{2}\right)\right) = \frac{u(kt)}{\min\left\{v(t), v(t), u(t), 1, u(\frac{t}{2}) * v\frac{t}{2}\right\}}$$
$$\geq 1.$$

Thus $u(ht) \geq v(t) * u(t)$ if $h = 2k \in (0,1)$. A similar argument works if (F_b) is assumed. Finally, suppose that $t > 0$ is fixed, $u : [0,\infty) \to I$ is a non-decreasing function and

$$F(u(kt), u(t), 1, 1u(t), u(t)) = \frac{u(kt)}{u(t)} \geq 1,$$

for some $k \in (0,1)$. Then we have $u(kt) \geq u(t)$ and thus $F \in \mathcal{F}$.

Define the mappings $A, B, S, T : X \to X$ by

$$Ax = \frac{x}{27}, \quad Bx = \frac{x}{9}, \quad Sx = \frac{x}{3}, \quad Tx = x,$$

respectively. Then, for some $k \in [\frac{1}{9}, 1)$, we have

$$\begin{aligned} M(Ax, By, kt) &= \left[\exp\left(\frac{|\frac{x}{27} - \frac{y}{9}|}{kt}\right)\right]^{-1} \\ &\geq \left[\exp\left(\frac{|\frac{x}{3} - y|}{t}\right)\right]^{-1} \\ &= M(Sx, Ty, t) \\ &\geq \min\{M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), M(Ax, Ty, t), \\ &\qquad M(By, Sx, t)\}. \end{aligned}$$

Thus, the condition (3.4) of Theorem 3.3 is satisfied.

Further, the pairs (A, S) and (B, T) are weakly compatible. Also, $B(X) = [0, \frac{1}{9}] \subseteq [0, \frac{1}{3}] = S(X)$. Considering the sequence $\{x_n\} = \{\frac{1}{n}\}$ so that $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = 0 = T(0)$, hence the pair (B, T) satisfies property (CLR_T).

Therefore, all the conditions of Theorem 3.3 are satisfied. Indeed 0 is the unique common fixed point of the mappings A, B, S, T.

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