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Subtractive Extension of Ideals in Ternary Semirings

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Abstract : In this paper, we introduce the concept of subtractive extension of an ideal of a ternary semiring. Further, 1) A characterization of subtractive extensions of ideals in the ternary semiring of non-positive integers is investigated. 2) The relation between subtractive extensions of a Q-ideal I in a ternary semiring S and the ideals in the quotient ternary semiring $S/I_{(Q)}$ is obtained. 3) We show that a subtractive extension P of a Q-ideal I in a ternary semiring S is a prime (semiprime) ideal if and only if $P/I_{(Q\cap P)}$ is a prime (semiprime) ideal in the quotient ternary semiring $S/I_{(Q)}$.

Keywords : ternary semiring; subtractive ideal; prime ideal; semiprime ideal; Quotient ternary semiring; subtractive extension of an ideal.
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1 Introduction

Generalizing the notion of ternary ring introduced by Lister [1], Dutta and Kar [2] introduced the notion of ternary semiring. A non-empty set S together with a binary operation called addition $(+)$ and a ternary operation called ternary multiplication $(\cdot)$ is called ternary semiring if it satisfies the following conditions for all $a, b, c, d, e \in S$:

1. $(a + b) + c = a + (b + c)$;
2. $a + b = b + a$;

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3. $(a \cdot b \cdot c) \cdot d \cdot e = a \cdot (b \cdot c \cdot d) \cdot e = a \cdot b \cdot (c \cdot d \cdot e)$;
4. there exists $0 \in S$ such that $a + 0 = a = 0 + a$, $a \cdot b \cdot 0 = a \cdot 0 \cdot b = 0 \cdot a \cdot b = 0$;
5. $(a + b) \cdot c \cdot d = a \cdot c \cdot d + b \cdot c \cdot d$;
6. $a \cdot (b + c) \cdot d = a \cdot b \cdot d + a \cdot c \cdot d$;
7. $a \cdot b \cdot (c + d) = a \cdot b \cdot c + a \cdot b \cdot d$.

Clearly, every semiring is a ternary semiring. Denote the sets of all non-positive, negative, non-negative, and positive integers respectively by $\mathbb{Z}_0^-$, $\mathbb{Z}^-$, $\mathbb{Z}_0^+$, and $\mathbb{N}$. The set $\mathbb{Z}_0^-$ is a ternary semiring under usual addition and ternary multiplication of non-positive integers but it is not a semiring.

If there exists an element e in a ternary semiring S such that $eex = exe = xee = x$ for all $x \in S$, then e is called the identity element of S. A ternary semiring S is said to be commutative if $abc = acb = cab$ for all $a, b, c \in S$. The ternary semiring $(\mathbb{Z}_0^-, +, \cdot)$ is commutative with identity element -1. A non-empty subset I of a ternary semiring S is called an ideal of S if the following conditions are satisfied:

1. $a, b \in I$ implies $a + b \in I$;
2. $a \in I, r, s \in S$ implies $rsa, ras, ars \in I$.

An ideal I of a ternary semiring S is called a subtractive ideal (= k-ideal) if x, $x + y \in I$, $y \in S$, then $y \in I$. If S is a commutative ternary semiring with identity element, then a proper ideal I of S is called i) prime if $abc \in I$, $a, b, c \in S$ implies $a \in I$ or $b \in I$ or $c \in I$; ii) semiprime if $a^3 \in I$, $a \in S$ implies $a \in I$. Clearly, every prime ideal is a semiprime ideal. An ideal I of a ternary semiring S is called a Q-ideal (= partitioning ideal) if there exists a subset Q of S such that

1. $S = \cup\{q + I : q \in Q\}$;
2. if q_1, $q_2 \in Q$, then $(q_1 + I) \cap (q_2 + I) \neq \emptyset \Leftrightarrow q_1 = q_2$.

Let I be a Q-ideal of a ternary semiring S. Then $S/I_{(Q)} = \{q + I : q \in Q\}$ forms a ternary semiring under the following addition "$\oplus$"and ternary multiplication "$\odot$", $(q_1 + I) \oplus (q_2 + I) = q' + I$ where $q' \in Q$ is a unique element such that $q_1 + q_2 + I \subseteq q' + I$ and $(q_1 + I) \odot (q_2 + I) \odot (q_3 + I) = q_4 + I$ where $q_4 \in Q$ is a unique element such that $q_1q_2q_3 + I \subseteq q_4 + I$. This ternary semiring is called a quotient ternary semiring of S by I and denoted by $(S/I_{(Q)}\ , \oplus\ , \odot)$ or just $S/I_{(Q)}$. If $q_0 \in Q$ is a unique element such that $q_0 + I = I$, then $q_0 + I$ is the zero element of $S/I_{(Q)}$ [3, Lemma 2.3].

For $a, b \in (\mathbb{Z}_0^-, +, \cdot)$ and $a \neq 0$, we define $a \mid b$ if and only if $b = \alpha\beta a$ for some $\alpha, \beta \in \mathbb{Z}_0^-$. An ideal I of $(\mathbb{Z}_0^-, +, \cdot)$ is said to be generated by a subset $A = \{a_1, a_2, \cdots, a_n\}$ of $\mathbb{Z}_0^-$ if for every $x \in I$, there exist $\alpha_i, \beta_i \in \mathbb{Z}_0^-$ such that $x = \sum_{i=1}^{n} \alpha_i\beta_i a_i$. If $A = \{a\}$, then $\mathbb{Z}_0^-\mathbb{Z}_0^- a$ is called a principal ideal generated by a. For $a_1, a_2, \cdots, a_k \in \mathbb{Z}_0^-$, we denote i) $\langle a_1, a_2, \cdots, a_k\rangle$ = the ideal generated by $a_1, a_2, \cdots, a_k$ in the ternary semiring $\mathbb{Z}_0^-$; ii) $(a_1, a_2, \cdots, a_k)$ = g.c.d. of $a_1, a_2, \cdots, a_k$. For example, $(-4, -6) = 2$.

The concept of subtractive extension of ideals in semirings is recently introduced by Chaudhari and Bonde [7]. In section 2, we extend this concept of subtractive extension of an ideal to ternary semirings and obtain a characterization of subtractive extension of ideals in the ternary semiring $(\mathbb{Z}_0^-, +, \cdot)$. In section 3, we obtain the relation between subtractive extensions of a Q-ideal I in a ternary semiring S and the ideals in the quotient ternary semiring $S/I_{(Q)}$ and hence prove that a subtractive extension P of a Q-ideal of a ternary semiring S is a prime (semiprime) ideal if and only if $P/I_{(Q\cap P)}$ is a prime (semiprime) ideal of the quotient ternary semiring $S/I_{(Q)}$.

Here we list some results that we need throughout the paper.

Lemma 1.1. [4, Lemma 3.12] *Let $I = \langle a_1, a_2, \cdots, a_n\rangle \subseteq \mathbb{Z}_0^-$. If $(a_1, a_2, \cdots, a_n) = d$, then there exists a largest $r \in \mathbb{Z}_0^-$ such that $(-1)(-d)k \in I$ for all $k \leq r$.*

Theorem 1.2. [5, Theorem 2.7] *Every ideal of $\mathbb{Z}_0^-$ is finitely generated.*

Theorem 1.3. [6, Theorem 5.5] *An ideal I of $\mathbb{Z}_0^-$ is a subtractive ideal if and only if I is a principal ideal.*

Lemma 1.4. [3, Lemma 1.4] *Let I be an ideal of a ternary semiring S and a, $x \in S$ such that $a + I \subseteq x + I$. Then*

1. $a + r + I \subseteq x + r + I$;
2. $rsa + I \subseteq rsx + I$;
3. $ras + I \subseteq rxs + I$;
4. $ars + I \subseteq xrs + I$ *for all* $r, s \in S$.

Lemma 1.5. [3, Lemma 2.2] *Let I be a Q-ideal of a ternary semiring S. If $x \in S$, then there exists a unique $q \in Q$ such that $x + I \subseteq q + I$. Hence $x = q + a$ for some $a \in I$.*

2 Subtractive Extension of Ideals in the Ternary Semiring $\mathbb{Z}_0^-$

In this section, we extend the concept of subtractive extension of an ideal for semirings to ternary semirings and obtain a characterization of subtractive extension of ideals in the ternary semiring $(\mathbb{Z}_0^-, +, \cdot)$.

Definition 2.1. Let I be an ideal of a ternary semiring S. An ideal A of S with $I \subseteq A$ is said to be a *subtractive extension* of I if $x \in I$, $x + y \in A$, $y \in S$, then $y \in A$.

Clearly, every subtractive ideal of a ternary semiring S containing an ideal I of S is a subtractive extension of I. Also every ideal of a ternary semiring S is a subtractive extension of $\{0\}$. Let $(\mathbb{Z}^- \cup \{-\infty\}, \max, \cdot)$, the commutative ternary semiring with zero element $-\infty$ and identity -1. For $n \in \mathbb{Z}^-$, we denote $I_n = \{r \in \mathbb{Z}^- : r \leq n\} \cup \{-\infty\}$. Clearly, I_n is an ideal in the ternary semiring $(\mathbb{Z}^- \cup \{-\infty\}, \max, \cdot)$. The following lemma can be proved easily.

Lemma 2.2. *Let $I \subseteq A$ be non-zero ideals of the ternary semiring $S = (\mathbb{Z}^- \cup \{-\infty\}, max, \cdot)$. Then*

1) I is a subtractive ideal of S if and only if $I = I_n$ for some $n \in \mathbb{Z}^-$;

2) A is a subtractive extension of I if and only if $I \subseteq I_n \subseteq A$ for some $n \in \mathbb{Z}^-$.

Example 2.3. Let $I = \{-6\} \cup I_{-8}$, $A = \{-4\} \cup I_{-6}$ be ideals in the ternary semiring $S = (\mathbb{Z}^- \cup \{-\infty\}, \max, \cdot)$. Then by Lemma 2.2, A is a subtractive extension of I but not a subtractive ideal.

By Theorem 1.2, every ideal of $(\mathbb{Z}_0^-, +, \cdot)$ is finitely generated. Now the following theorem gives a characterization of subtractive extensions of non-zero ideals in the ternary semiring $(\mathbb{Z}_0^-, +, \cdot)$:

Theorem 2.4. *Let $I = \langle b_1, b_2, ..., b_m\rangle$ be a non-zero ideal of $S = (\mathbb{Z}_0^-, +, \cdot)$ and $d = (b_1, b_2, ..., b_m)$. Then an ideal A of S is a subtractive extension of I if and only if $A = \langle a\rangle$ where $a \mid -d$.*

Proof. Let A be a subtractive extension of I. Suppose that A is not a principal ideal. Then by Theorem 1.2, $A = \langle a_1, a_2, ..., a_n\rangle$ where $a_n < a_{n-1} < ... < a_1 < -1$, $a_i \nmid a_j$ for all $i < j$, $j = 2, 3, ..., n$, $n \geq 2$. Let $d' = (a_1, a_2, ..., a_n)$. By Lemma 1.1, there exist $r_1, r_2 \in \mathbb{Z}^-$ such that $(-1)(-d)k \in I$ and $(-1)(-d')s \in A$ for all $k \leq r_1$, $s \leq r_2$. Hence $(-1)(-d)k \in I$ and $(-1)(-d')k \in A$ for all $k \leq r$ where $r = \min\{r_1, r_2\}$ ...(1). Since $-d, -d' < 0$, $(-1)(-d')r, (-1)(-d)r + (-1) \leq r$. So by (1), $(-d)(-d')r = (-1)(-d)(-1)(-d')r \in I$ and $(-d)(-d')r + (-d') = (-1)(-d')((-1)(-d)r + (-1)) \in A$. Since A is a subtractive extension of I, $-d' \in A$. Hence $-d' = a_1$. So $a_1 \mid a_2$, a contradiction. Now $A = \langle a\rangle$ for some $a \in \mathbb{Z}_0^-$. Since $I \subseteq A$, $a \mid -d$. Conversely, suppose that $A = \langle a\rangle$ where $a \mid -d$. Clearly, $I \subseteq A$. By Theorem 1.3, A is a subtractive ideal of S and hence A is a subtractive extension of I. □

The following example shows that the sum (union) of two subtractive extensions of I need not be a subtractive extension of I.

Example 2.5. Let $I = \langle -12, -18\rangle$, $A = \langle -2\rangle$ and $B = \langle -3\rangle$ be ideals in the ternary semiring $\mathbb{Z}_0^-$. By Theorem 2.4, A, B are subtractive extensions of I, but $A + B = \{0, -2, -3, -4, -5, -6, ...\} = \langle -2, -3\rangle$, is not a subtractive extension of I. An inspection will show that $A \cup B$ is not an ideal of $\mathbb{Z}_0^-$ and hence $A \cup B$ is not a subtractive extension of I.

Denote $\mathbb{Z}_0^+ \times \mathbb{Z}_0^+ = (\mathbb{Z}_0^+, +, \cdot) \times (\mathbb{Z}_0^+, +, \cdot)$, the semiring with pointwise addition and pointwise multiplication. Also denote $\mathbb{Z}_0^- \times \mathbb{Z}_0^- = (\mathbb{Z}_0^-, +, \cdot) \times (\mathbb{Z}_0^-, +, \cdot)$, the ternary semiring with pointwise addition and pointwise ternary multiplication. For a subset A of $\mathbb{Z}_0^- \times \mathbb{Z}_0^-$, we denote $A^* = \{(n, m) \in \mathbb{Z}_0^+ \times \mathbb{Z}_0^+ : (-n, -m) \in A\}$. The following lemma can be proved easily.

Lemma 2.6. *Let I be a subset of $\mathbb{Z}_0^- \times \mathbb{Z}_0^-$. Then*

1) I is an ideal if and only if I^* is an ideal of $\mathbb{Z}_0^+ \times \mathbb{Z}_0^+$.

2) I is a Q-ideal if and only if I^* is a Q^*-ideal of $\mathbb{Z}_0^+ \times \mathbb{Z}_0^+$.

3) I is a subtractive ideal if and only if I^* is a subtractive ideal of $\mathbb{Z}_0^+ \times \mathbb{Z}_0^+$.

4) If $I \subseteq A$ are ideals of $\mathbb{Z}_0^- \times \mathbb{Z}_0^-$, then A is a subtractive extension of I if and only if A^* is a subtractive extension of I^* in $\mathbb{Z}_0^+ \times \mathbb{Z}_0^+$.

Theorem 2.7. *Let I be a subset of $\mathbb{Z}_0^- \times \mathbb{Z}_0^-$. Then*

1) I is an ideal if and only if $I = J_1 \times J_2$ where J_1, J_2 are ideals of the ternary semiring $\mathbb{Z}_0^-$.

2) I is a principal ideal if and only if $I = J_1 \times J_2$ where J_1, J_2 are principal ideals in the ternary semiring $\mathbb{Z}_0^-$.

3) I is a Q-ideal if and only if $I = J_1 \times J_2$ where J_1, J_2 are Q_1, Q_2-ideals respectively in the ternary semiring $\mathbb{Z}_0^-$ with $Q = Q_1 \times Q_2$.

4) I is a subtractive ideal if and only if $I = J_1 \times J_2$ where J_1, J_2 are subtractive ideals in the ternary semiring $\mathbb{Z}_0^-$.

Proof. Follows from Lemma 2.6, [7, Lemma 2.1 and Lemma 2.2] and [5, Lemma 2.3 and Lemma 2.4]. □

Theorem 2.8. *Let I be an ideal of $\mathbb{Z}_0^- \times \mathbb{Z}_0^-$. Then the following statements are equivalent:*

1) I is a principal ideal;

2) I is a Q-ideal;

3) I is a subtractive ideal.

Proof. Follows from Theorem 2.7 and [5, Theorem 2.6]. □

Theorem 2.9. *Let $I \subseteq A$ be ideals of the ternary semiring $\mathbb{Z}_0^- \times \mathbb{Z}_0^-$. Then A is a subtractive extension of I if and only if $A = A_1 \times A_2$ where an ideal A_i is a subtractive extension of an ideal J_i (i = 1, 2) in the ternary semiring $\mathbb{Z}_0^-$ with $I = J_1 \times J_2$.*

Proof. Let A be a subtractive extension of I. By Theorem 2.7, $A = A_1 \times A_2$ and $I = J_1 \times J_2$ where A_1, A_2, J_1, J_2 are ideals of the ternary semiring $\mathbb{Z}_0^-$. Let $x \in J_1$, $x + y \in A_1$, $y \in \mathbb{Z}_0^-$. Then $(x, 0) \in I$ and $(x, 0) + (y, 0) = (x + y, 0) \in A_1 \times A_2 = A$. Since A is a subtractive extension of I, $(y, 0) \in A$ and hence $y \in A_1$. Now A_1 is a subtractive extension of J_1. Similarly, A_2 is a subtractive extension of J_2. Conversely, suppose that $A = A_1 \times A_2$ where A_i is a subtractive extension of J_i $(i = 1, 2)$ with $I = J_1 \times J_2$. Since A_i is a subtractive extensions of J_i $(i = 1, 2)$, $A_1 \times A_2 = A$ is a subtractive extension of $J_1 \times J_2 = I$. □

Corollary 2.10. *Let $I \subseteq A$ be ideals of the semiring $\mathbb{Z}_0^+ \times \mathbb{Z}_0^+$. Then A is a subtractive extension of I if and only if $A = A_1 \times A_2$ where an ideal A_i is a subtractive extension of an ideal J_i (i = 1, 2) with $I = J_1 \times J_2$.*

Proof. Follows from Lemma 2.6 (4) and Theorem 2.9. □

3 Ideal Theory in Quotient Ternary Semirings

Let $I \subseteq A$ be ideals of a ternary semiring S. Then we denote

1) $\overline{A_I} = \{x \in S : x + i \in A \text{ for some } i \in I\}$, and will be called the closure of A with respect to I;

2) $\widetilde{A} = \{x \in S : \text{ there exists } q + I \in S/I_{(Q)} \text{ such that} x \in q + I \text{ and } (q + I) \cap A \neq \emptyset\}$, and will be called the closure of A with respect to $I_{(Q)}$ where I is a Q-ideal of S;

3) $\overline{A} = \{x \in S : x + y \in A \text{ for some } y \in A\}$, is called the k-closure of A [6].

We can easily show that i) $I \subseteq \overline{I} \subseteq \overline{A_I} \subseteq \overline{A}$; ii) $\overline{A_A} = \overline{A}$ where $I \subseteq A$ are ideals of S.

Example 3.1. Let $I = \{-8\} \cup I_{-10}$, $A = \{-6\} \cup I_{-8}$ be ideals in the ternary semiring $S = (\mathbb{Z}^- \cup \{-\infty\}, \max, \cdot)$. An inspection will show that $\overline{A_I} = \{-6\} \cup I_{-8}$, $\overline{I} = I_{-8}$ and $\overline{A} = I_{-6}$. Now $I \subsetneq \overline{I} \subsetneq \overline{A_I} \subsetneq \overline{A}$.

Theorem 3.2. *Let $I \subseteq A$ be ideals of a ternary semiring S. Then $\overline{A_I}$ is the smallest subtractive extension of I containing A.*

Proof. 1) Let a_1, $a_2 \in \overline{A_I}$ and $r, s \in S$. Then there exist $i_1, i_2 \in I$ such that $a_1 + i_1$, $a_2 + i_2 \in A$. Hence $(a_1 + a_2) + (i_1 + i_2) = a_1 + i_1 + a_2 + i_2 \in A$ where $i_1 + i_2 \in I$. So $a_1 + a_2 \in \overline{A_I}$. Similarly, rsa_1, $ra_1s, a_1rs \in \overline{A_I}$. Hence $\overline{A_I}$ is an ideal of S.
2) Clearly, $A \subseteq \overline{A_I}$.
3) Let $i \in I$, $a + i \in \overline{A_I}$, $a \in S$. Then there exists $i^{'} \in I$ such that $a + i + i^{'} \in A$. Now $i + i^{'} \in I$ implies $a \in \overline{A_I}$. Hence $\overline{A_I}$ is a subtractive extension of I.
4) Let J be a subtractive extension of I containing A and let $x \in \overline{A_I}$. Then there exists $i \in I$ such that $x + i \in A \subseteq J$. Since J is a subtractive extension of I, $x \in J$. Hence $\overline{A_I} \subseteq J$. □

Corollary 3.3. *Let $I \subseteq A$ be ideals of a ternary semiring S. Then*

$$\overline{A_I} = \cap\{J : J \text{ is a subtractive extension of } I \text{ containing } A\}.$$

Now we have the following:

Theorem 3.4. *Let I, A, B be ideals of a ternary semiring S such that $I \subseteq A, B$. Then*

1) A is a subtractive extension of $I \Leftrightarrow \overline{A_I} = A$.

2) $\overline{(\overline{A_I})_I} = \overline{A_I}$.

3) $A \subseteq B \Rightarrow \overline{A_I} \subseteq \overline{B_I}$.

4) $\overline{(A \cap B)_I} = \overline{A_I} \cap \overline{B_I}$.

5) If A, B are subtractive extensions of I, then $A \cap B$ is a subtractive extension of I.

6) If J is an ideal of S such that $I \subseteq J \subseteq A$, then $\overline{A_I} \subseteq \overline{A_J}$.

Now by Theorem 3.4, we have i) A is a subtractive ideal of $S \Leftrightarrow \overline{A} = A$. ii) $\overline{\overline{A}} = \overline{A}$. iii) $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$.

Corollary 3.5. *Let A, B be ideals of a ternary semiring S. Then $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.*

Proof. By Theorem 3.4 (3), $\overline{A \cap B} = \overline{(A \cap B)_{(A \cap B)}} \subseteq \overline{A_{(A \cap B)}}$. Since $A \cap B \subseteq A$, by Theorem 3.4 (6), $\overline{A_{(A \cap B)}} \subseteq \overline{A_A} = \overline{A}$. Hence $\overline{A \cap B} \subseteq \overline{A}$. Similarly, $\overline{A \cap B} \subseteq \overline{B}$. Now $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$. □

The following example shows that equality in Corollary 3.5 may not hold.

Example 3.6. Let $A = \{-4\} \cup I_{-8}$, $B = \{-6, -7\} \cup I_{-10}$ be ideals in the ternary semiring $S = (\mathbb{Z}^- \cup \{-\infty\}, \max, \cdot)$. Then $A \cap B = I_{-10}$. By Lemma 2.2, $A \cap B$ is a subtractive ideal of S and hence $\overline{A \cap B} = I_{-10}$. An inspection will show that $\overline{A} = I_{-4}$ and $\overline{B} = I_{-6}$. Hence $\overline{A} \cap \overline{B} = I_{-6}$. Now $\overline{A \cap B} \subsetneq \overline{A} \cap \overline{B}$.

In the next lemma we give the relation between $\overline{A_I}$ and $\widetilde{A}$.

Lemma 3.7. *Let I be a Q-ideal of a ternary semiring S and A be an ideal of S with $I \subseteq A$. Then $\overline{A_I} = \widetilde{A}$.*

Proof. Let $x \in \overline{A_I}$. Then there exists $i_1 \in I$ such that $x + i_1 \in A$...(1). By Lemma 1.5, there exists $q \in Q$ such that $x \in q + I$. Then $x = q + i_2$ for some $i_2 \in I$. So $x + i_i = q + i_2 + i_1 \in q + I$. Thus $(q + I) \cap A \neq \emptyset$. Hence $x \in \widetilde{A}$. Now $\overline{A_I} \subseteq \widetilde{A}$. For other inclusion, let $z \in \widetilde{A}$. Then there exists $q + I \in S/I_{(Q)}$ such that $z \in q + I$ and $(q + I) \cap A \neq \emptyset$. So $z = q + i'$ for some $i' \in I$. Let $y = q + i'' \in (q + I) \cap A$ where $i'' \in I$. Since $i' \in I \subseteq A$ and A is an ideal of S, $z + i'' = q + i'' + i' \in A$. Hence $z \in \overline{A_I}$. Now $\widetilde{A} \subseteq \overline{A_I}$. □

Theorem 3.8. *Let I be a Q-ideal of a ternary semiring S and A be an ideal of S with $I \subseteq A$. Then $\widetilde{A}$ is the smallest subtractive extension of I containing A.*

Proof. Follows from Lemma 3.7 and Theorem 3.2. □

Corollary 3.9. [7, Proposition 2.16] *Let I be a Q-ideal of a semiring S and A be an ideal of S with $I \subseteq A$. Then $\widetilde{A}$ is the smallest subtractive extension of I containing A.*

Now we extend results of Chaudhari and Bonde [7, Lemma 2.6 and Theorems 2.7, 2.10, 2.12] for semirings to ternary semirings. For the sake of completeness we give the proof of the following lemma which is exactly similar to the proof of [7, Lemma 2.6].

Lemma 3.10. *Let I be a Q-ideal of a ternary semiring S and A be an ideal of S with $I \subseteq A$. Then A is a subtractive extension of I if and only if I is a $Q \cap A$-ideal of A.*

Proof. Let A be a subtractive extension of I. Let $a \in A$. Then there exists a unique $q \in Q$ such that $a \in q + I$. So $a = q + i$ for some $i \in I$. Since A is a subtractive extension of I, $q \in A$. Hence $q \in Q \cap A$. If $(q_1 + I) \cap (q_2 + I) \neq \emptyset$ for some $q_1, q_2 \in Q \cap A$, then $q_1 = q_2$ because I is a Q-ideal of S. Thus I is a $Q \cap A$-ideal of A. Conversely, suppose that I is a $Q \cap A$-ideal of A and $x \in I$, $x + y \in A$, $y \in S$. Since I is a $Q \cap A$-ideal of A, there exists a unique $q_1 \in Q \cap A$ such that $x + y + I \subseteq q_1 + I$. Also since I is a Q-ideal of S, there exists a unique $q_2 \in Q$ such that $y + I \subseteq q_2 + I$. By using Lemma 1.4, $x + y + I \subseteq x + q_2 + I \subseteq q_2 + I$ as $x \in I$. So $(q_1 + I) \cap (q_2 + I) \neq \emptyset$. Since I is a Q-ideal of S, $q_2 = q_1 \in A$. Now $y \in q_2 + I \subseteq A$. Hence A is a subtractive extension of I. □

Theorem 3.11. *Let $I \subseteq A$ be ideals of a ternary semiring S and I a Q-ideal of S. Then following statements are equivalent:*

1) A is a subtractive extension of I;

2) I is a $Q \cap A$-ideal of A;

3) $A/I_{(Q \cap A)}$ is an ideal of a ternary semiring $S/I_{(Q)}$;

4) $A/I_{(Q \cap A)} \subseteq S/I_{(Q)}$.

Proof. (1) ⇒ (2) Follows from Lemma 3.10.
(2) ⇒ (3) As A is an ideal of S, $A/I_{(Q \cap A)}$ is an ideal of ternary semiring $S/I_{(Q)}$.
(3) ⇒ (4) Trivial.
(4) ⇒ (1) Let $x \in I$, $x + y \in A$, $y \in S$. Then $x \in I = q_0 + I$ where $q_0 + I$ is the zero element of $S/I_{(Q)}$. Now by definition of quotient ternary semiring, there exists a unique $q_1 + I \in A/I_{(Q \cap A)} \subseteq S/I_{(Q)}$ and a unique $q_2 + I \in S/I_{(Q)}$ such that $x + y \in q_1 + I$ and $y \in q_2 + I$. Here $x + y \in (q_0 + I) \oplus (q_2 + I) = q_2 + I$. So $(q_1 + I) \cap (q_2 + I) \neq \emptyset$. Hence $q_2 = q_1 \in A$. Now $y \in q_2 + I \subseteq A$. □

Corollary 3.12. [3, Lemma 3.4] *Let I a Q-ideal of a ternary semiring S and A a subtractive ideal of S with $I \subseteq A$. Then I is a $Q \cap A$-ideal of A.*

The following example shows that the converse of Corollary 3.12 is not true.

Example 3.13. Let $I = \langle -4 \rangle \times \{0\}$, $A = \langle -2 \rangle \times \langle -2, -3 \rangle$ be ideals in the ternary semiring $S = \mathbb{Z}_0^- \times \mathbb{Z}_0^-$. By [8, Example 6], [5, Lemma 2.4] and Theorem 2.7, I is a Q-ideal of S where $Q = \{0, -1, -2, -3\} \times \mathbb{Z}_0^-$. Clearly, $I \subseteq A$. By Theorem 2.4, $\langle -2 \rangle$ is a subtractive extension of $\langle -4 \rangle$. Also $\langle -2, -3 \rangle$ is a subtractive extension of $\{0\}$. Hence by Theorem 2.9, A is a subtractive extension of I. By Lemma 3.10, I is a $Q \cap A$-ideal of A. But by Theorem 1.3 and Theorem 2.7(4), A is not a subtractive ideal of S.

Theorem 3.14. *Let I be a Q-ideal of a ternary semiring S. Then L is an ideal of $S/I_{(Q)}$ if and only if there exists an ideal A of S such that A is a subtractive extension of I and $A/I_{(Q\cap A)} = L$.*

Proof. Let L be an ideal of a ternary semiring $S/I_{(Q)}$. Denote $A = \{x \in S :$ there exist a unique $q \in Q$ such that $x + I \subseteq q + I \in L\}$.
(1) Let $a \in I$. Then $a + I \subseteq I = q_0 + I \in L$, so $a \in A$. Now $I \subseteq A$.
(2) Let $x, y \in A$. Then there exist unique $q_1, q_2 \in Q$ such that $x + I \subseteq q_1 + I \in L$ and $y+I \subseteq q_2+I \in L$. Again there exists unique $q_3 \in Q$ such that $(q_1+I)\oplus(q_2+I)$ $= q_3 + I \in L$ where $q_1 + q_2 + I \subseteq q_3 + I$. By Lemma 1.4, $x + I \subseteq q_1 + I$ and $y+I \subseteq q_2+I \Rightarrow x+y+I \subseteq q_1+y+I \subseteq q_1+q_2+I \subseteq q_3+I \in L$. Now $x+y \in A$. Similarly, if $x \in A$ and $r, s \in S$, then $rsx, rxs, xrs \in A$. Hence A is an ideal of S.
(3) Let $x \in I$, $x + y \in A$, $y \in S$. So $x + y \in q + I \in L$. Since I is a Q-ideal of S, there exists a unique $q^{'} \in Q$ such that $y \in q^{'} + I$. Since $x \in I$, $x + y \in q^{'} + I$. So $(q + I) \cap (q^{'} + I) \neq \emptyset$ implies $q = q^{'}$. Now $y \in q^{'} + I = q + I \in L$. Thus $y \in A$. Hence A is a subtractive extension of I.
(4) Clearly, $A/I_{(Q\cap A)} \subseteq L$. Now if $q+I \in L$, then $q \in A$. So $L \subseteq A/I_{(Q\cap A)}$. Thus $A/I_{(Q\cap A)} = L$.
Conversely, suppose that A is a subtractive extension of I and $A/I_{(Q\cap A)} = L$. Then by Theorem 3.11, L is an ideal of $S/I_{(Q)}$. □

Theorem 3.15. *Let S be a ternary semiring, I a Q-ideal of S and P a subtractive extension of I. Then P is a prime ideal of S if and only if $P/I_{(Q\cap P)}$ is a prime ideal of $S/I_{(Q)}$.*

Proof. Let P be a prime ideal of S. Suppose that $q_1 + I, q_2 + I, q_3 + I \in S/I_{(Q)}$ and $(q_1 + I) \odot (q_2 + I) \odot (q_3 + I) = q_4 + I \in P/I_{(Q\cap P)}$ where $q_4 \in Q \cap P$ is a unique element such that $q_1q_2q_3 + I \subseteq q_4 + I$. So $q_1q_2q_3 = q_4 + i$ for some $i \in I$. Now $q_1q_2q_3 \in P$ implies $q_1 \in P$ or $q_2 \in P$ or $q_3 \in P$. Hence $q_1 + I \in P/I_{(Q\cap P)}$ or $q_2 + I \in P/I_{(Q\cap P)}$ or $q_3 + I \in P/I_{(Q\cap P)}$. Conversely, suppose that $P/I_{(Q\cap P)}$ is a prime ideal of $S/I_{(Q)}$. Let $abc \in P$ where $a, b, c \in S$. Since I is a Q-ideal of S, there exist unique $q_1, q_2, q_3, q_4 \in Q$ such that $a \in q_1 + I$, $b \in q_2 + I$, $c \in q_3 + I$ and $abc \in (q_1 + I) \odot (q_2 + I) \odot (q_3 + I) = q_4 + I$ where $q_1q_2q_3 + I \subseteq q_4 + I$. So $abc = q_4 + i^{'}$ for some $i^{'} \in I$. Since P is a subtractive extension of I, $q_4 \in P$. So $(q_1 + I) \odot (q_2 + I) \odot (q_3 + I) = q_4 + I \in P/I_{(Q\cap P)}$. Since $P/I_{(Q\cap P)}$ is a prime ideal, we may assume $q_1 + I \in P/I_{(Q\cap P)}$. Now $a \in q_1 + I \Rightarrow a = q_1 + i^{''}$ for some $i^{''} \in I \Rightarrow a \in P$ as $q_1 \in Q \cap P \subseteq P$. □

Theorem 3.16. *Let S be a ternary semiring, I a Q-ideal of S and P a subtractive extension of I. Then P is a semiprime ideal of S if and only if $P/I_{(Q\cap P)}$ is a semiprime ideal of $S/I_{(Q)}$.*

Proof. Let P be a semiprime ideal of S. Suppose that $q + I \in S/I_{(Q)}$ and $(q + I) \odot (q + I) \odot (q + I) = q^{'} + I \in P/I_{(Q\cap P)}$ where $q^{'} \in Q \cap P$ is a unique element such that $q^3 + I \subseteq q^{'} + I$. So $q^3 = q^{'} + i$ for some $i \in I$. Now $q^3 \in P$ implies $q \in P$. Hence $q + I \in P/I_{(Q\cap P)}$. Conversely, suppose that $P/I_{(Q\cap P)}$ is a semiprime ideal of $S/I_{(Q)}$. Let $a^3 \in P$ where $a \in S$. Since I is a Q-ideal of S, there exist unique $q, q^{'} \in Q$ such that $a \in q + I$ and $a^3 \in (q + I) \odot (q + I) \odot (q + I) = q^{'} + I$. So $a^3 = q^{'} + i^{'}$ for some $i^{'} \in I$. Since P is a subtractive extension of I, $q^{'} \in P$ where $q^3 + I \subseteq q^{'} + I$. So $(q+I) \odot (q+I) \odot (q+I) = q^{'} + I \in P/I_{(Q\cap P)}$. Since $P/I_{(Q\cap P)}$ is a semiprime ideal, $q + I \in P/I_{(Q\cap P)}$. Now $a \in q + I \Rightarrow a = q + i^{''}$ for some $i^{''} \in I \Rightarrow a \in P$ as $q \in Q \cap P \subseteq P$. □

Chaudhari and Ingale [5, Theorem 3.4], proved that if I, J are subtractive ideals of a ternary semiring S, then $I \cup J$ is an ideal of S if and only if $I \subseteq J$ or $J \subseteq I$. For subtractive extensions we have:

Example 3.17. Let $I = \langle -4\rangle \times \{0\}$, $A = \langle -2\rangle \times \langle -2\rangle$, $B = \langle -4\rangle \times T$ be ideals in the ternary semiring $S = \mathbb{Z}_0^- \times \mathbb{Z}_0^-$ where $T = \{n \in \mathbb{Z}_0^- : n \leq -3\} \cup \{0\}$. By Theorem 2.4, $\langle -2\rangle$ and $\langle -4\rangle$ are subtractive extensions of $\langle -4\rangle$. Also $\langle -2\rangle$, T are subtractive extensions of $\{0\}$. Hence by Theorem 2.9, A, B are subtractive extensions of I. By Example 3.13, $A \cup B = \langle -2\rangle \times \langle -2, -3\rangle$ is a subtractive extension of I. But $A \nsubseteq B$ and $B \nsubseteq A$.

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