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Regular Weakly Continuous Functions in Ideal Topological Spaces

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Abstract : In this paper, we study the concepts of I_{rw}-continuity in ideal topological spaces, and obtain several characterizations and some properties of these functions. Also, we investigate their relationship with other types of functions.

Keywords : I_{rw}-continuous, I_{rw}-irresolute; quasi regular semiclosed map; almost I_{rw}-continuous; $\wp_I$-rw-continuous; I^*_{rw}-continuous; I^*_{rw}-closed map.
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1 Introduction

A nonempty collection I of subsets of a set X is said to be an ideal on X, if it satisfies the following two properties: (1) $A \in I$ and $B \subseteq A$ imply $B \in I$. (2) $A \in I$ and $B \in I$ imply $A \cup B \in I$. A topological space (X, τ) with an ideal I on X is called an ideal topological space (an ideal space) and is denoted by (X, τ, I). For an ideal space (X, τ, I) and a subset $A \subseteq X$, $A^*(I) = \{x \in X : U \cap A \notin I$ for every $U \in \tau(x)\}$, is local function [1] of A with respect to I and τ. It is well known that $Cl^*(A) = A \cup A^*$ defines a Kuratowski closure operator for a topology τ^* finer than τ [2] and $Int^*(A)$ will denote the interior of A in (X, τ^*). In this paper, we introduce and investigate the notions of I_{rw}-continuity, almost I_{rw}-continuity and quasi $*$-rs-normal spaces in ideal topological spaces. Also, we characterize quasi $*$-rs-normal spaces in terms of I_{rw}-open sets.

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By a space, we always mean a topological space (X, τ) on which no separation axioms are assumed unless explicitly stated. In a space (X, τ), the closure and the interior of a subset A of X will be denoted by $Cl(A)$ and $Int(A)$, respectively. A subset A of a space (X, τ) is said to be regular-open [3] (resp. regular-closed, α-open [4]) if $A = Int(Cl(A))$ (resp. $A = Cl(Int(A))$, $A \subseteq Int(Cl(Int(A)))$). The family of all regular-open (resp. regular-closed) subsets of a space (X, τ) is denoted by $RO(X)$ (resp. $RC(X)$). Finite union of regular open sets in (X, τ) is π-open [5] in (X, τ). The complement of a π-open set in (X, τ) is π-closed in (X, τ). The family of all α-open sets in (X, τ) denoted by τ^α is a topology on X finer than τ. The closure of A in (X, τ^α) is denoted by $Cl_\alpha(A)$. A subset A of a space (X, τ) is said to be regular semiopen [6] if there is a regular open set U such that $U \subseteq A \subseteq Cl(U)$. A subset A of a space (X, τ) is said to be rw-closed [7] if $Cl(A) \subset U$ whenever $A \subseteq U$ and U is regular semiopen. A is said to be rw-open (resp. g-open) if $X - A$ is rw-closed (resp. g-closed). A function $f : (X, \tau) \to (Y, \sigma)$ is R-map [8] (resp. completely continuous [9]) if $f^{-1}(V)$ is regular closed in (X, τ) for every regular closed (resp. closed) set V of Y. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be rw-continuous [7] if $f^{-1}(A)$ is rw-closed in (X, τ) for every $A \in \sigma$. A space (X, τ) is said to be quasi-normal [5] if for disjoint π-closed sets F_1 and F_2 there exist disjoint open sets U_1, U_2 such that $F_1 \subseteq U_1$ and $F_2 \subseteq U_2$. A subset A of an ideal space (X, τ, I) is $*$-perfect [10] (resp. $*$-closed [11]) if $A = A^*$ (resp. $A^* \subseteq A$). A subset A of an ideal space (X, τ, I) is I_g-closed [12] if $A^* \subseteq U$ whenever U is open in X and $A \subseteq U$. An ideal I is said to be completely codense [17] if $PO(X) \cap I = \{\phi\}$, where $PO(X)$ is the family of all pre-open sets in (X, τ). $\mathcal{N}$ denotes the ideal of all nowhere dense subset in (X, τ). A function $f : (X, \tau, I) \to (Y, \sigma)$ is $*$-continuous [15] (resp. I_g-continuous [15]) if $f^{-1}(A)$ is $*$-closed (resp. I_g-closed) in X for every closed set A of Y. A space (X, τ, I) is said to be $*$-normal if for any two disjoint closed sets A and B in (X, τ), there exist disjoint $*$-open sets U, V such that $A \subseteq U$ and $B \subseteq V$. A subset A of an ideal space (X, τ, I) is said to be a I_{rw}-closed [16] if $A^* \subseteq U$ whenever $A \subseteq U$ and U is regular semiopen. The complement of I_{rw}-closed is said to be I_{rw}-open. The family of all I_{rw}-closed (resp. I_{rw}-open) subsets of a space (X, τ, I) is denoted by $IRWC(X)$ (resp. $IRWO(X)$).

Lemma 1.1. [14] If (X, τ, I) be an ideal space and I is completely codense, then $\tau^* \subseteq \tau^\alpha$.

2 I_{rw}-Continuous Functions

Definition 2.1. A function $f : (X, \tau, I) \to (Y, \sigma)$ is said to be I_{rw}-*continuous* if $f^{-1}(A)$ is I_{rw}-closed in (X, τ, I) for every closed set A of Y.

Theorem 2.2. *For a function $f : (X, \tau, I) \to (Y, \sigma)$, the following hold.*
1. If f is rw-continuous, then f is I_{rw}-continuous.
2. If f is I_g-continuous, then f is I_{rw}-continuous.

Example 2.3. Let $X = Y = \{a, b, c, d, e\}$, $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}, \{b, c, d, e\}, X\}$, $I = \{\phi, \{b\}, \{d\}, \{b, d\}\}$ and $\sigma = \{\phi, \{a, c, d, e\}, Y\}$. Then the identity function $f : (X, \tau, I) \to (Y, \sigma)$ is I_{rw}-continuous but not rw-continuous.

Example 2.4. Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$, $I = \{\phi, \{c\}\}$ and $\sigma = \{\phi, \{b\}, X\}$. Then the identity function $g : (X, \tau, I) \to (Y, \sigma)$ is I_{rw}-continuous but not I_g-continuous.

Definition 2.5. A function $f : (X, \tau, I) \to (Y, \sigma, I)$ is said to be *I_{rw}-irresolute* if $f^{-1}(A)$ is I_{rw}-closed in (X, τ, I) for every I_{rw}-closed set A of (Y, σ, I).

Example 2.6. Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$, $I = \{\phi, \{c\}\}$ and $\sigma = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. Let $f : (X, \tau, I) \to (Y, \sigma, I)$ be the identity map. Then f is I_{rw}-irresolute.

Definition 2.7. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *quasi regular semiclosed* if $f(A)$ is regular semiclosed in (Y, σ) for every regular semiclosed set A in (X, τ).

Example 2.8. Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{c\}, \{a, c\}, X\}$ and $\sigma = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity map. Then f is quasi regular semiclosed.

Theorem 2.9. *If $f : (X, \tau, I) \to (Y, \sigma, I)$ is I_{rw}-continuous and quasi regular semiclosed, then f is I_{rw}-irresolute.*

Proof. Assume that A is I_{rw}-closed in Y. Let $f^{-1}(A) \subseteq U$, where U is regular semiopen in X. Then $(X - U) \subseteq f^{-1}(Y - A)$ and hence $f(X - U) \subseteq Y - A$. Since f is quasi regular semiclosed, $f(X - U)$ is regular semiclosed. Then, since $Y - A$ is I_{rw}-open. By Theorem 2.9 in [16], $f(X - U) \subseteq Int^*(Y - A) = Y - Cl^*(A)$. Thus, $f^{-1}(Cl^*(A)) \subseteq U$. Since f is I_{rw}-continuous, $f^{-1}(Cl^*(A))$ is I_{rw}-closed. Therefore, $Cl^*(f^{-1}(Cl^*(A))) \subseteq U$ and hence $Cl^*(f^{-1}(A)) \subseteq Cl^*(f^{-1}(Cl^*(A))) \subseteq U$ which proves that $f^{-1}(A)$ is I_{rw}-closed and therefore f is I_{rw}-irresolute. □

Definition 2.10. A function $f : (X, \tau, I) \to (Y, \sigma)$ is said to be *almost I_{rw}-continuous* if $f^{-1}(A)$ is I_{rw}-closed in (X, τ, I) for every $A \in RC(Y)$.

Example 2.11. Let $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$, $I = \{\phi, \{c\}\}$ and $\sigma = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. Let $f : (X, \tau, I) \to (Y, \sigma)$ be the identity map. Then f is almost I_{rw}-continuous.

Theorem 2.12. *For a function $f : (X, \tau, I) \to (Y, \sigma)$, the following are equivalent.*
1. f is almost I_{rw}-continuous.
2. $f^{-1}(A) \in IRWO(X)$ for every $A \in RO(Y)$.
3. $f^{-1}(Int(Cl(A))) \in IRWO(X)$ for every $A \in \sigma$.
4. $f^{-1}(Cl(Int(A))) \in IRWC(X)$ for every closed set A of Y.

Proof. (1) $\Leftrightarrow$ (2) Obvious. (2) $\Leftrightarrow$ (3) Suppose $A \in RO(Y)$, we have $A = Int(Cl(A))$ and $f^{-1}(Int(Cl(A))) \in IRWO(X)$. Conversely, suppose $A \in \sigma$, we have $Int(Cl(A)) \in RO(Y)$ and $f^{-1}(Int(Cl(A))) \in IRWO(X)$. (3) $\Leftrightarrow$ (4) Let A be a closed set in Y. Then $Y - A \in \sigma$. We have $f^{-1}(Int(Cl(Y - A))) = f^{-1}(Y - (Cl(Int(A)))) = X - f^{-1}(Cl(Int(A))) \in IRWO(X)$. Hence, $f^{-1}(Int(Cl(A))) \in IRWC(X)$. Converse can be obtained similarly. □

Theorem 2.13. *The following hold for the functions $f : (X, \tau, I) \to (Y, \sigma, J)$ and $g : (Y, \sigma, J) \to (Z, \mu)$. Then*

1. *gof is I_{rw}-continuous, if f is almost I_{rw}-continuous and g is completely continuous.*
2. *gof is I_{rw}-continuous, if f is I_{rw}-continuous and g is continuous.*
3. *gof is I_{rw}-continuous, if f is I_{rw}-irresolute and g is I_{rw}-continuous.*
4. *gof is almost I_{rw}-continuous, if f is almost I_{rw}-continuous and g is R-map.*
5. *gof is almost I_{rw}-continuous, if f is I_{rw}-irresolute and g is almost I_{rw}-continuous.*

Definition 2.14. A function $f : (X, \tau, I) \to (Y, \sigma)$ is said to be *$\wp_I$-rw-continuous* if $f^{-1}(A)$ is $\wp_I$-rw-set in (X, τ, I) for every closed set A of Y.

Theorem 2.15. *A function $f : (X, \tau, I) \to (Y, \sigma)$ is $*$-continuous if and only if it is $\wp_I$-rw-continuous and I_{rw}-continuous.*

Proof. This is an immediate consequence of Theorem 2.14. in [16]. □

Remark 2.16. *The notions of $\wp_I$-rw-continuity and I_{rw}-continuity are independent as shown in the following example.*

Example 2.17. Let $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$, $\sigma = \{\phi, \{c\}, \{a, b\}, X\}$, $\gamma = \{\phi, \{b\}\}$ and $I = \{\phi, \{c\}\}$. Then

1. The identity function $f : (X, \tau, I) \to (X, \gamma)$ is $\wp_I$-rw-continuous but not I_{rw}-continuous.
2. The identity function $g : (X, \tau, I) \to (X, \sigma)$ is I_{rw}-continuous but not $\wp_I$-rw-continuous.

3 Quasi $*$-rs-Normal Spaces

Definition 3.1. A space (X, τ, I) is said to be *quasi $*$-rs-normal* if for any two disjoint regular semiclosed sets A and B in (X, τ), there exist disjoint $*$-open sets U, V such that $A \subseteq U$ and $B \subseteq V$.

Example 3.2. Let $X = \{a, b, c\}$, $\tau = \{\phi, \{c\}, \{a, b\}, X\}$ and $I = \{\phi, \{c\}\}$. Then regular semiopen sets are $\phi, \{c\}, \{a, b\}, X$ and $*$-open sets are $\phi, \{c\}, \{a, b\}, X$. Let $A = \{c\}$ and $B = \{a, b\}$. Clearly, the space (X, τ, I) is quasi $*$-rs-normal.

Theorem 3.3. *Let (X, τ, I) be an ideal space. Then the following are equivalent.*

1. *(X, τ, I) is quasi $*$-rs-normal.*
2. *For every pair of disjoint regular semiclosed sets A and B, there exist disjoint I_g-open sets U, V such that $A \subseteq U$ and $B \subseteq V$.*
3. *For every pair of disjoint regular semiclosed sets A and B, there exist disjoint I_{rw}-open sets U, V such that $A \subseteq U$ and $B \subseteq V$.*
4. *For each regular semiclosed set A and for each regular semiopen set V containing A, there exists an I_{rw}-open set U such that $A \subseteq U \subseteq Cl^*(U) \subseteq V$.*
5. *For each rw-closed set A and for each rw-open set V containing A, there exists an $*$-open set U such that $A \subseteq U \subseteq Cl^*(U) \subseteq V$.*

Proof. It is obvious that(1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3). (3) $\Rightarrow$ (4) : Suppose that A is regular semiclosed and V is a regular semiopen set containing A. Then $A \cap V^c = \phi$. By assumption, there exist I_{rw}-open sets U and W such that $A \subseteq U$, $V^c \subseteq W$. Since V^c is regular semiclosed and W is I_{rw}-open, by Theorem 2.9 in [16], $V^c \subseteq Int^*(W)$ and so $(Int^*(W))^c \subseteq V$. Again, $U \cap W = \phi$ implies that $U \cap Int^*(W) = \phi$ and so $Cl^*(U) \subseteq (Int^*(W))^c \subseteq V$. Hence, U is the required I_{rw}-open set such that $A \subseteq U \subseteq Cl^*(U) \subseteq V$.

(4) $\Rightarrow$ (5) : Let A be a regular semiclosed set and V be a regular semiopen set such that $A \subseteq V$. By hypothesis, there exist I_{rw}-open set W such that $A \subseteq W \subseteq Cl^*(W) \subseteq V$. By Theorem 2.9 in [16], $A \subseteq Int^*(W)$. If $U = Int^*(W)$, then U is a $*$-open set and $A \subseteq U \subseteq Cl^*(U) \subseteq Cl^*(W) \subseteq V$. Therefore, $A \subseteq U \subseteq Cl^*(U) \subseteq V$.

(5) $\Rightarrow$ (1) : Let A and B be disjoint regular semiclosed sets. Then B^c is a regular semiopen set containing A. By assumption, there exists an $*$-open set U such that $A \subseteq U \subseteq Cl^*(U) \subseteq B^c$. If $V = (Cl^*(U))^c$, then U and V are disjoint $*$-open sets such that $A \subseteq U$ and $B \subseteq V$. □

Definition 3.4. A function $f : (X, \tau, I) \to (Y, \sigma, I)$ is said to be I^*_{rw}-*continuous* if $f^{-1}(A)$ is I_{rw}-closed in (X, τ, I) for every $*$-closed set A of (Y, σ, I).

Example 3.5. Let $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$ and $I = \{\phi, \{c\}\}$. Let $f : (X, \tau, I) \to (X, \tau, I)$ be the identity map. Then f is I^*_{rw}-continuous.

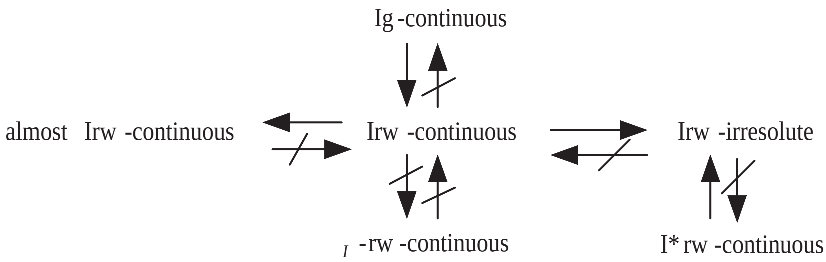
Theorem 3.6. *Let $f : (X, \tau, I) \to (Y, \sigma, I)$ be a I^*_{rw}-continuous quasi regular semiclosed injection and Y is quasi $*$-rs-normal, then X is quasi $*$-rs-normal.*

Proof. Let A and B are disjoint regular semiciosed sets of X. Since f is quasi regular semiclosed injection, $f(A)$ and $f(B)$ are disjoint regular semiclosed sets of Y. By the quasi $*$-rs-normality of Y, there exist disjoint $*$-open sets U and V of Y such that $f(A) \subseteq U$ and $f(B) \subseteq V$. Since f is I^*_{rw}-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint I_{rw}-open sets containing A and B respectively. It follows from Theorem 3.3 that X is quasi $*$-rs-normal. □

Definition 3.7. A function $f : (X, \tau, I) \to (Y, \sigma, I)$ is said to be *I^*_{rw}-closed* if $f(A)$ is I_{rw}-closed in (Y, σ, I) for every $*$-closed set A of X.

Example 3.8. Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$, $I = \{\phi, \{c\}\}$ and $\sigma = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$. Let $f : (X, \tau, I) \to (Y, \sigma, I)$ be the identity map. Then f is I^*_{rw}-closed.

Remark 3.9. *From the above definition and some types of continuous functions in ideal topological spaces, we have the following diagram:*



Theorem 3.10. *Let $f : (X, \tau, I) \to (Y, \sigma, I)$ is a regular semicontinuous (resp. continuous) I^*_{rw}-closed surjection and X is a quasi $*$-rs-normal (resp. $*$-normal), then Y is $*$-rs-normal.*

Proof. Let A and B are disjoint closed sets of Y. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint regular semiclosed (resp. closed) sets of X. Since X is quasi $*$-rs-normal (resp. $*$-normal), there exist disjoint $*$-open sets U and V such that $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$. Now, we set $K = Y - f(X - U)$ and $L = Y - f(X - V)$. Then K and L are I_{rw}-open sets of Y such that $A \subseteq K$, $B \subseteq L$, Since A, B are disjoint closed sets and K and L are I_{rw}-open. We have $A \subseteq Int^*(K)$ and $B \subseteq Int^*(L)$ [[13], Theorem 2.10.] and $Int^*(K) \cap Int^*(L) = \phi$. Hence, Y is $*$-rs-normal. □

Theorem 3.11. *Let (X, τ, I) be an ideal space and I is completely codense. Then (X, τ, I) is quasi-normal if and only if it is quasi $*$-rs-normal.*

Proof. Suppose that A and B are disjoint π-closed sets. Since X is quasi-normal, there exist disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$. But every open (resp. π-closed) set is $*$-open (resp. regular semiclosed) set and hence, quasi $*$-rs-normal.

Conversely, suppose that A and B are disjoint π-closed sets of X. Then there exist disjoint $*$-open sets U and V such that $A \subseteq U$ and $B \subseteq V$. Since I is completely codense. By Lemma 1.1, $\tau^* \subseteq \tau^\alpha$ and so U, $V \in \tau^\alpha$. Hence, $A \subseteq U \subseteq Int(Cl(Int(U))) = G$ and $B \subseteq V \subseteq Int(Cl(Int(V))) = H$. Therefore, G and H are disjoint open sets containing A and B respectively. Therefore, X is quasi-normal. □

Corollary 3.12. *Let (X, τ, I) be an ideal space, where I is completely codense. Then the following are equivalent.*

1. *(X, τ, I) is quasi normal.*
2. *For every pair of disjoint regular semiclosed sets A and B, there exist disjoint I_{rw}-open sets U, V such that $A \subseteq U$ and $B \subseteq V$.*
3. *For each regular semiclosed set A and for each regular semiopen set V containing A, there exists an I_{rw}-open set U such that $A \subseteq U \subseteq Cl^*(U) \subseteq V$.*
4. *For each regular semiclosed set A and for each regular semiopen set V containing A, there exists an $*$-open set U such that $A \subseteq U \subseteq Cl^*(U) \subseteq V$.*
5. *For every pair of disjoint regular semiclosed sets A and B, there exist disjoint $*$-open sets U, V such that $A \subseteq U$ and $B \subseteq V$.*

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