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Generalized Stability of Euler-Lagrange Quadratic Functional Equation in Random Normed Spaces under Arbitrary t-Norms

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Abstract : The main goal of this paper is the investigation of the generalized Hyers-Ulam stability theorem of the following Euler-Lagrange type quadratic functional equation

$$f(ax+by)+af(x-by)=(a+1)b^2f(y)+a(a+1)f(x),$$

in random normed spaces under arbitrary t-norms, where a, b are fixed integer numbers such that $a \neq -1, 0, 1$ and $b \neq 0$.

Keywords : generalized Hyers–Ulam stability; quadratic functional equation; random normed space.
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1 Introduction

In 1940, Ulam [1] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms (see also [2, 3, 4]).

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Let E_1 and E_2 be real vector spaces. A function $f : E_1 \longrightarrow E_2$ is called a quadratic function if and only if f is a solution of the quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{1.1}$$

It is well known that a function f between real vector spaces is quadratic if and only if there exists a unique symmetric bi-additive function B such that $f(x) = B(x,x)$ for all x, where the mapping B is given by $B(x,y) = (1/4)(f(x+y) - f(x-y))$. The Hyers-Ulam stability of the quadratic functional equation (1.1) was first proved by Skof [5] for function $f : E_1 \longrightarrow E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [6] demonstrated that Skof's theorem is also valid if E_1 is replaced by an Abelian group G. Assume that a function $f : G \longrightarrow E$ satisfies the inequality

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \delta, \tag{1.2}$$

for some $\delta \geq 0$ and for all $x, y \in G$. Then there exists a unique quadratic function $Q : G \longrightarrow E$ such that

$$\|f(x) - Q(x)\| \leq \frac{\delta}{2}, \tag{1.3}$$

for all $x \in G$. Czerwik [7] proved the Hyers-Ulam stability of quadratic functional equation (1.1).
J.M. Rassias investigated the stability of Ulam for the Euler-Lagrange functional equation

$$f(ax+by) + f(bx-ay) = (a^2+b^2)[f(x)+f(y)], \tag{1.4}$$

in [8]. Gordji and Khodaei investigated the generalized Hyers-Ulam stability of other Euler-Lagrange quadratic functional equations [9]. Jun et al. [10] introduced a new quadratic Euler-Lagrange functional equation

$$f(ax+y) + af(x-y) = (a+1)f(y) + a(a+1)f(x), \tag{1.5}$$

for any fixed $a \in \mathbb{Z}$ with $a \neq 0, -1, 1$, which was a modified and instrumental equation for [11], and solved the generalized stability of (1.5). Now, we improve the functional equation (1.5) to the following functional equations:

$$f(ax+by) + af(x-by) = (a+1)f(by) + a(a+1)f(x), \tag{1.6}$$

and

$$f(ax+by) + af(x-by) = (a+1)b^2f(y) + a(a+1)f(x), \tag{1.7}$$

for any fixed numbers $a, b \in \mathbb{Z}$ with $a \neq 0, -1, 1$ and $b \neq 0$ which are generalized versions of (1.5). In this paper, we prove the generalized Hyers-Ulam stability of (1.7).

2 Preliminaries

In this section, we recall some definitions and results which will be used later on in the article (see [12] for more details).

Throughout this paper, the space of all probability distribution functions is denoted by

$$\begin{aligned}\triangle^+ = \{F : \mathbb{R}\bigcup\{-\infty,+\infty\} \longrightarrow [0,1] : F \ \ is\ \ left-continuous\\ and\ \ nondecreasing\ \ on\ \ \mathbb{R}\ \ and\ \ f(0)=o, f(+\infty)=1\},\end{aligned}$$

and the subset $D^+ \subseteq \triangle^+$ is the set $D^+ = \{F \in \triangle^+ : l^- F(+\infty) = 1\}$, where $l^- f(x)$ denotes the left limit of the function f at the point x. The space $\triangle^+$ is partially ordered by the usual pointwise ordering of functions, that is, $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. The maximal element for $\triangle^+$ in this order is the distribution function given by

$$\varepsilon_0(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ 1 & \text{if } t > 0, \end{cases}$$

Definition 2.1. A function $T : [0,1] \times [0,1] \longrightarrow [0,1]$ is a *continuous triangular norm* (briefly, a t-norm) if T satisfies the following conditions:
(a) T is commutative and associative;
(b) T is continuous;
(c) $T(a,b) \leq T(c,d)$ whenever $a \leq c$ and $b \leq d$ for all $a,b,c,d \in [0,1]$.
Three typical examples of continuous t-norms are $T(a,b) = ab$, $T(a,b) = \max(a+b-1,0)$ and $T(a,b) = \min(a,b)$.

A t-norm T can be extended (by associativity) in a unique way to an n-array operation taking for $(x_1,\ldots,x_n) \in [0,1]^n$ the value $T(x_1,\ldots,x_n)$ defined by

$$\mathrm{T}_{i=1}^0 x_i = 1, \mathrm{T}_{i=1}^n x_i = T(\mathrm{T}_{i=1}^{n-1} x_i, x_n) = T(x_1,\ldots,x_n).$$

T can also be extended to a countable operation taking for any sequence $(x_n)_{n\in N}$ in $[0,1]$ the value

$$\mathrm{T}_{i=1}^\infty x_i = \lim_{n\to\infty} \mathrm{T}_{i=1}^n x_i. \tag{2.1}$$

The limit on the right side of (2.1) exists since the sequence $\{\mathrm{T}_{i=1}^n x_i\}_{n\in\mathbb{N}}$ is non-increasing and bounded from below.

Definition 2.2. A *random normed space* (briefly, RN-space) is a triple (X,μ,T), where X is a vector space, T is a continuous t-norm and μ is a mapping from X into D^+ such that the following conditions hold:
(PN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;
(PN2) $\mu_{\alpha x}(t) = \mu_x(t/|\alpha|)$ for all x in X , $\alpha \neq 0$ and $t \geq 0$;
(PN3) $\mu_{x+y}(t+s) \geq T(\mu_x(t),\mu_y(s))$ for all $x,y \in X$ and $t,s \geq 0$.

Definition 2.3. Let (X, μ, T) be an RN-space.

(1) A sequence $\{x_n\}$ in X is said to be *convergent* to x in X if, for every $\epsilon > 0$ and $t > 0$, there exists a positive integer N such that $\mu_{x_n - x}(t) > 1 - \epsilon$ whenever $n \geq N$.

(2) A sequence $\{x_n\}$ in X is called *Cauchy* if, for every $\epsilon > 0$ and $t > 0$, there exists a positive integer N such that $\mu_{x_n - x_m}(t) > 1 - \epsilon$ whenever $n \geq m \geq N$.

(3) An RN-space (X, μ, T) is said to be *complete* if every Cauchy sequence in X is convergent to a point in X.

3 Stability of (1.7)

For convenience, we use the following abbreviation, for any fixed integer numbers a and b with $a \neq -1, 0, 1$ and $b \neq 0$,

$$Df(x, y) := f(ax + by) + af(x - by) - (a + 1)b^2 f(y) - a(a + 1)f(x), \qquad (3.1)$$

for all $x, y \in X$.

Theorem 3.1. *Let X be a linear space and (Y, μ, T) be a complete RN-space and $(\mathbb{R}, \mu', T)$ be an RN-space. Let $\varphi : X \times X \longrightarrow [0, \infty)$ be a function such that*

$$\lim_{n \longrightarrow \infty} \mu'_{\varphi(a^n x, a^n y)}(|a|^{2n} t) = 1, \qquad (3.2)$$

for all $x, y \in X$ and $t > 0$. Suppose that a function $f : X \longrightarrow Y$ with $f(0) = 0$ satisfies

$$\mu_{Df(x,y)}(t) \geq \mu'_{\varphi(x,y)}(t), \qquad (3.3)$$

for all $x, y \in X$ and $t > 0$. Then there exists a unique quadratic function $Q : X \longrightarrow Y$ satisfying

$$\mu_{f(x) - Q(x)}(t) \geq T_{i=1}^{\infty}(\mu'_{\varphi(a^{i-1}x, 0)}(|a|^i t)), \qquad (3.4)$$

for all $x \in X$ and $t > 0$. The function Q is given by

$$Q(x) = \lim_{k \longrightarrow \infty} \frac{1}{a^{2n}} f(a^n x), \qquad (3.5)$$

for all $x \in X$ and

$$\lim_{n \longrightarrow \infty} T_{i=1}^{\infty}(\mu'_{\varphi(a^{n+i-1}x)}(|a|^{2n+i} t)) = 1. \qquad (3.6)$$

Proof Letting y by 0 in (3.3), we get

$$\mu_{f(ax) - a^2 f(x)}(t) \geq \mu'_{\varphi(x,0)}(t), \qquad (3.7)$$

and then

$$\mu_{\frac{1}{a^2} f(ax) - f(x)}(t) \geq \mu'_{\frac{1}{a^2}\varphi(x,0)}(t), \qquad (3.8)$$

for all $x \in X$ and $t > 0$. Replacing x by $a^n x$ in (3.8), we get

$$\mu_{\frac{1}{|a|^{2(n+1)}} f(a^{n+1}x) - \frac{1}{|a|^{2n}} f(a^n x)} \left(\frac{t}{|a|^{n+1}} \right) \geq \mu'_{\varphi(a^n x, 0)}(|a|^{(n+1)} t). \qquad (3.9)$$

Now, since, $\frac{1}{|a|^{2n}} f(a^n x) - f(x) = \sum_{k=0}^{n-1} \frac{1}{|a|^{2(k+1)}} f(a^{k+1}x) - \frac{1}{|a|^{2k}} f(a^k x)$ and $t \geq \sum_{k=0}^{n-1} \frac{t}{|a|^{k+1}}$ we have

$$\begin{aligned} \mu_{\frac{1}{|a|^{2n}} f(a^n x) - f(x)}(t) &\geq T_{k=0}^{n-1} \left(\mu_{\frac{1}{|a|^{2(k+1)}} f(a^{k+1}x) - \frac{1}{|a|^{2k}} f(a^k x)} \left(\frac{t}{|a|^{k+1}} \right) \right) \qquad (3.10) \\ &\geq T_{k=0}^{n-1} \mu'_{\varphi(a^k x, 0)}(|a|^{k+1} t) \\ &\geq T_{i=1}^{n} \mu'_{\varphi(a^{i-1} x, 0)}(|a|^i t). \end{aligned}$$

Now, we prove that, the sequence $\{(1/a^{2n}) f(a^n x)\}$ is convergence. Replacing x by $a^m x$ in (3.10), we get

$$\mu_{\frac{f(a^(n+m)x)}{|a|^{2(n+m)}} - \frac{f(a^m x)}{a^{2m}}}(t) \geq T_{i=1}^{n} \mu'_{\varphi(a^{i+m-1} x, 0)}(|a|^{2m+i} t). \qquad (3.11)$$

Since the right hand side of the inequality tends to 1 as m, n tend to ∞, then $\{(1/a^{2n}) f(a^n x)\}$ is a Cauchy sequence in Y. Since Y is a complete RN-space, this sequence converges for all $x \in X$. Therefore, we can define a mapping $Q : X \longrightarrow Y$ by

$$Q(x) = \lim_{k \longrightarrow \infty} \frac{1}{a^{2k}} f(a^k x), \qquad (3.12)$$

for all $x \in X$. Replacing x, y with $2^n x$ and $2^n y$, respectively, in (3.3) and by (3.2), it follows that

$$\begin{aligned} \mu_{DQ(x,y)}(t) &= \lim_{n \longrightarrow \infty} \mu_{\frac{1}{|a|^{2n}} Df(a^n x, a^n y)}(t) \\ &\geq \lim_{n \longrightarrow \infty} \mu_{\frac{1}{|a|^{2n}} \varphi(a^n x, a^n y)}(t) \\ &= 1, \end{aligned}$$

for all $x, y \in X$ and $t > 0$, therefore Q satisfies (3.1), and so the function Q is quadratic. Finally, to prove the uniqueness of the quadratic function Q, let us assume that there exists a quadratic function $Q' : X \longrightarrow Y$ which satisfying the inequality (3.4). Since $Q'(a^n x) = |a|^{2n} Q'(x)$ and $Q(a^n x) = |a|^{2n} Q(x)$. Then we have

$$\begin{aligned} \mu_{Q(x) - Q'(x)}(2t) &:= \mu_{Q(a^n x) - Q'(a^n x))}(|a|^{2n} t) \\ &\geq T\Big(\mu_{Q(a^n x) - f(a^n x)}(|a|^{2n} t), \mu_{Q'(a^n x) - f(a^n x)}(|a|^{2n} t)\Big) \\ &\geq T\left(T_{i=1}^{\infty} \mu'_{\varphi(a^{i+n-1} x, 0)}(|a|^{2n+i} t), T_{i=1}^{n} \mu'_{\varphi(a^{i+n-1} x, 0)}(|a|^{2n+i} t))\right), \end{aligned}$$

for all $x \in X$, $t > 0$ and $n \in \mathbb{N}$. Therefore, $Q(x) - Q'(x) = 0$ for all $x \in X$ when $n \longrightarrow \infty$.

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