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Regularity of Semihypergroups of Infinite Matrices

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Abstract : A semigroup S is a *regular semigroup* if for every $x \in S, x = xyx$ for some $y \in S$, and a semihypergroup $(H, \circ)$ is a *regular semihypergroup* if for every $x \in H, x \in x \circ y \circ x$ for some $y \in H$. If S is a semigroup and P is a nonempty subset of S, we let (S, P) denote the semihypergroup $(S, \circ)$ where $x \circ y = xPy$ for all $x, y \in S$. Let $\mathbf{BM}(F)$ be the multiplicative semigroup of all bounded $\mathbb{N} \times \mathbb{N}$ matrices over a field F where $\mathbb{N}$ is the set of natural numbers. It is known that $\mathbf{BM}(F)$ is a regular semigroup. Our purpose is to provide necessary and sufficient conditions for a nonempty subset $\mathbf{P}$ of $\mathbf{BM}(F)$ so that $(\mathbf{BM}(F), \mathbf{P})$ is a regular semihypergroup.

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1 Introduction

A semigroup S is called a *regular semigroup* if for every $x \in S, x = xyx$ for some $y \in S$.

By a *hyperoperation* on a nonempty set H is a function $\circ$ from $H \times H$ into $P(H) \smallsetminus \{\emptyset\}$ where $P(H)$ is the power set of H, and $(H, \circ)$ is called a *hypergroupoid.* For $A, B \subseteq H$, let $A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b$. A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if $x \circ (y \circ z) = (x \circ y) \circ z$ for all $x, y, z \in H$. A *hypergroup* is a semihypergroup $(H, \circ)$ satisfying the condition $H \circ x = x \circ H = H$ for all $x \in H$. We call a semihypergroup $(H, \circ)$ a *regular semihypergroup* if for every $x \in H, x \in x \circ y \circ x$ for some $y \in H$. Hence every regular semigroup is a regular semihypergroup. Notice that if $(H, \circ)$ is a hypergroup, then for every $x \in H$, $x \circ H \circ x = H$. This implies that every hypergroup is a regular semihypergroup.

Let S be a semigroup, P a nonempty subset of S and $\circ$ the hyperoperation on S defined by $x \circ y = xPy$ for all $x, y \in S$. Then $(S, \circ)$ is a semihypergroup ([2], page 11) and $(S, \circ)$ will be denoted by (S, P). We note here that if S is a group, then (S, P) is a hypergroup, so it is a regular semihypergroup.

Let $\mathbb{N}$ be the set of natural numbers (positive integers) and F a field. For

$n \in \mathbb{N}$, let $\mathbf{M}_n(F)$ be the multiplicative semigroup of all $n \times n$ matrices over F. It is well-known that $\mathbf{M}_n(F)$ is a regular semigroup ([3], page 114) with identity I_n where I_n is the identity $n \times n$ matrix over F.

By an $\mathbb{N} \times \mathbb{N}$ *matrix* over F we mean an infinite matrix over F of the form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots \\ a_{21} & a_{22} & a_{23} & \dots \\ a_{31} & a_{32} & a_{33} & \dots \\ & \vdots & & \end{bmatrix}$$

and let $\mathbf{M}(F)$ be the set of all $\mathbb{N} \times \mathbb{N}$ matrices over F. We give a remark that associativity $(AB)C = A(BC)$ can fail for $A, B, C \in \mathbf{M}(F)$ even when all products concerned make sense. The following example was given in [1]. Define $A, B, C \in \mathbf{M}(F)$ by

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & \dots \\ 0 & 1 & 1 & 1 & \dots \\ 0 & 0 & 1 & 1 & \dots \\ & & \vdots & & \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & \dots \\ 0 & 1 & -1 & 0 & 0 & \dots \\ 0 & 0 & 1 & -1 & 0 & \dots \\ & & \vdots & & & \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ -1 & 0 & 0 & 0 & \dots \\ -1 & -1 & 0 & 0 & \dots \\ -1 & -1 & -1 & 0 & \dots \\ & & \vdots & & \end{bmatrix}.$$

Then $AB = BC = I$, the identity $\mathbb{N} \times \mathbb{N}$ matrix over F. Thus $(AB)C = C \neq A = A(BC)$.

For a matrix A in $\mathbf{M}_n(F)$ or $\mathbf{M}(F)$, the entry of A in the $i^{\underline{th}}$ row and the $j^{\underline{th}}$ column will be denoted by A_{ij}. A matrix $A \in \mathbf{M}(F)$ is called *bounded* if there is a positive integer N such that $A_{ij} = 0$ for $i > N$ or $j > N$(see [4]). Denote by $\mathbf{BM}(F)$ the set of all bounded matrices in $\mathbf{M}(F)$. Then $\mathbf{BM}(F)$ is a semigroup under matrix multiplication. For each $k \in \mathbb{N}$, let $I(k) \in \mathbf{BM}(F)$ be such that

$$(I(k))_{ij} = \begin{cases} 1 & \text{if } i = j \in \{1, \dots, k\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then for all $k, l \in \mathbb{N}$, $I(k)I(l) = I(l)I(k) = I(k)$ if $k \leq l$. It is clear that if $A \in \mathbf{BM}(F)$ and $N \in \mathbb{N}$ are such that $A_{ij} = 0$ for $i > N$ or $j > N$, then $I(k)A = AI(k) = A$ for every $k \geq N$. It follows that $\mathbf{BM}(F)$ is a semigroup without identity. We have from [4] that $\mathbf{BM}(F)$ is also a regular semigroup. Hence $\mathbf{BM}(F)$ is a regular semigroup without identity. For $A \in \mathbf{BM}(F)$ and $k \in \mathbb{N}$, A is called *k-right* [*k-left*] *invertible* if $AB = I(k)$ [$BA = I(k)$] for some $B \in \mathbf{BM}(F)$, and B is called a *k-right* [*k-left*] *inverse* of A in $\mathbf{BM}(F)$. We observe that if $A \in \mathbf{BM}(F)$ has a

k-right [k-left] inverse in $\mathbf{BM}(F)$, then A has an infinitely many k-right [k-left] inverses in $\mathbf{BM}(F)$. Let B be a k-right [k-left] inverse of A and $N \in \mathbb{N}$ such that $A_{ij} = 0 = B_{ij}$ for $i > N$ or $j > N$. Then $B_{ii} = 0$ for all $i > N$. For $t > N$, define $B^{(t)} \in \mathbf{BM}(F)$ by

$$B^{(t)}_{ij} = \begin{cases} 1 & \text{if } i = j = t, \\ B_{ij} & \text{otherwise.} \end{cases}$$

It is clear that $B^{(t)} \neq B^{(r)}$ for all distinct t, r greater than N and $AB^{(t)} = AB = I(k)$ [$B^{(t)}A = BA = I(k)$] for all $t > N$.

Our purpose is to provide the following facts.

(1) For $\emptyset \neq \mathbf{P} \subseteq \mathbf{M}_n(F)$, the semihypergroup $(\mathbf{M}_n(F), \mathbf{P})$ is regular if and only if $\mathbf{P}$ contains an invertible matrix in $\mathbf{M}_n(F)$.

(2) For $\emptyset \neq \mathbf{P} \subseteq \mathbf{BM}(F)$, the semihypergroup $(\mathbf{BM}(F), \mathbf{P})$ is regular if and only if for every $k \in \mathbb{N}$, there are elements $A, B \in \mathbf{P}$ such that $I(k)A$ and $BI(k)$ are k-right invertible and k-left invertible in $\mathbf{BM}(F)$, respectively.

2 Main Results

In the remainder, F denotes any field. Recall that for $n \in \mathbb{N}$, $A, B \in \mathbf{M}_n(F)$, $AB = I_n$ implies that $BA = I_n$.

Theorem 2.1. *Let $n \in \mathbb{N}$ and $\mathbf{P}$ a nonempty subset of $\mathbf{M}_n(F)$. Then the semihypergroup $(\mathbf{M}_n(F), \mathbf{P})$ is regular if and only if $\mathbf{P}$ contains an invertible matrix in $\mathbf{M}_n(F)$.*

Proof. Assume that $(\mathbf{M}_n(F), \mathbf{P})$ is a regular semihypergroup. Then $I_n \in I_n\mathbf{P}C\mathbf{P}I_n$ for some $C \in \mathbf{M}_n(F)$. Thus $I_n \in \mathbf{P}C\mathbf{P}$ which implies that $I_n = ACB$ for some $A, B \in \mathbf{P}$, that is, $A(CB) = I_n$. Hence A is invertible in $\mathbf{M}_n(F)$.

For the converse, let $\mathbf{P}$ has an invertible matrix in $\mathbf{M}_n(F)$, say A. To show that $(\mathbf{M}_n(F), \mathbf{P})$ is a regular semihypergroup, let $B \in \mathbf{M}_n(F)$. Since $\mathbf{M}_n(F)$ is a regular semigroup, $B = BCB$ for some $C \in \mathbf{M}_n(F)$. Consequently,

$$B = BAA^{-1}CA^{-1}AB = BA(A^{-1}CA^{-1})AB \in B\mathbf{P}(A^{-1}CA^{-1})\mathbf{P}B.$$

□

Theorem 2.2. *For $\emptyset \neq \mathbf{P} \subseteq \mathbf{BM}(F)$, the semihypergroup $(\mathbf{BM}(F), \mathbf{P})$ is regular if and only if for every $k \in \mathbb{N}$, there are $A, B \in \mathbf{P}$ such that $I(k)A$ is k-right invertible and $BI(k)$ is k-left invertible in $\mathbf{BM}(F)$.*

Proof. First, assume that $(\mathbf{BM}(F), \mathbf{P})$ is a regular semihypergroup and let $k \in \mathbb{N}$. Then $I(k) \in I(k)\mathbf{P}C\mathbf{P}I(k)$ for some $C \in \mathbf{BM}(F)$. Thus $I(k) = I(k)ACBI(k)$ for some $A, B \in \mathbf{P}$. This implies that $I(k)A$ is k-right invertible and $BI(k)$ is k-left invertible.

For the converse, assume that for every $k \in \mathbb{N}$, there are $A, B \in \mathbf{P}$ such that $I(k)A$ and $BI(k)$ are k-right invertible and k-left invertible in $\mathbf{BM}(F)$, respectively. To prove that $(\mathbf{BM}(F), \mathbf{P})$ is a regular semihypergroup, let $C \in \mathbf{BM}(F)$. Since $\mathbf{BM}(F)$ is a regular semigroup, $C = CDC$ for some $D \in \mathbf{BM}(F)$. Let $N \in \mathbb{N}$ be such that $C_{ij} = 0$ if $i > N$ or $j > N$. Then $I(N)C = CI(N) = C$. It follows that $C = CI(N)DI(N)C$. By assumption, there are $E, F \in \mathbf{P}$ such that $I(N)E$ is N-right invertible and $FI(N)$ is N-left invertible in $\mathbf{BM}(F)$. Then $I(N)EK = I(N)$ and $LFI(N) = I(N)$ for some $K, L \in \mathbf{BM}(F)$. Consequently,

$$\begin{aligned} C = CI(N)DI(N)C &= CI(N)EKDLFI(N)C \\ &= CE(KDL)FC \ \in \ C\mathbf{P}(KDL)\mathbf{P}C. \end{aligned}$$

Hence the theorem is proved. □

Remark 2.3. For $k \in \{1, \ldots, n\}$, let $I_k \in \mathbf{M}_n(F)$ be such that

$$(I_k)_{ij} = \begin{cases} 1 & \text{if } i = j \leq k, \\ 0 & \text{otherwise.} \end{cases}$$

Then $I_kI_n = I_nI_k = I_k$ for all $k \in \{1, \ldots, n\}$. For $k \in \{1, \ldots, n\}$, k-right [k-left] invertible matrices in $\mathbf{M}_n(F)$ are defined analogously as in $\mathbf{BM}(F)$. If $A, B \in \mathbf{M}_n(F)$ are such that $AB = BA = I_n$, then for every $k \in \{1, \ldots, n\}$, $I_k = (I_kA)B = B(AI_k)$. Hence to be analogous to Theorem 2.2, Theorem 2.1 can be restated as follows : For $\emptyset \neq \mathbf{P} \subseteq \mathbf{M}_n(F)$, the semihypergroup $(\mathbf{M}_n(F), \mathbf{P})$ is regular if and only if for every $k \in \{1, \ldots, n\}$, there is a matrix $A \in \mathbf{P}$ such that I_kA is k-right invertible and AI_k is k-left invertible in $\mathbf{M}_n(F)$.

Example 2.4. It is clear that if $\mathbf{P} = \{\ I(k) \mid k \in \mathbb{N}\}$, then by Theorem 2.2, $(\mathbf{BM}(F), \mathbf{P})$ is a regular semihypergroup.

Next, define $A_k, B_k \in \mathbf{M}(F)$ for $k \in \mathbb{N}$, by

$$A_1 = B_1 = I(1),$$

$$A_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ & & \vdots & & & \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ & & \vdots & & & \end{bmatrix}, \quad \ldots$$

$$B_2 = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ & & \vdots & & & \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & -1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ & & \vdots & & & \end{bmatrix}, \quad \ldots$$

Then $A_1B_1 = I(1) = B_1A_1, A_2B_2 = I(2) = B_2A_2,\ A_3B_3 = I(3) = B_3A_3, \ldots$. It follows that $(I(k)A_k)(B_kI(k))$ $=(I(k)B_k)(A_kI(k))$ $=I(k)$ for all $k \in \mathbb{N}$. Hence by Theorem 2.2, if

$$\mathbf{P}_1 = \{A_k \mid k \in \mathbb{N}\} \text{ and } \mathbf{P}_2 = \{B_k \mid k \in \mathbb{N}\},$$

then both $(\mathbf{BM}(F), \mathbf{P}_1)$ and $(\mathbf{BM}(F), \mathbf{P}_2)$ are regular semihypergroups.

Remark 2.5. For $\emptyset \neq \mathbf{P} \subseteq \mathbf{BM}(F)$, if $(\mathbf{BM}(F), \mathbf{P})$ is a regular semihypergroup, then $\mathbf{P}$ must be an infinite set. To show this, suppose that $\mathbf{P}$ is finite. Then there is a positive integer N such that $A_{ij} = 0$ for all $A \in \mathbf{P}$ and $i > N$ or $j > N$. Let $k > N$ and $B \in \mathbf{BM}(F)$ be such that $I(k) \in I(k)\mathbf{P}B\mathbf{P}I(k)$. Then $I(k) = I(k)CBDI(k)$ for some $C, D \in \mathbf{P}$. By the property of N, $I(k)C = C$ and $DI(k)= D$. Thus $I(k) = CBD$. Since $D_{ik} = 0$ for all $i \in \mathbb{N}$, it follows that $(CBD)_{kk} = \sum_{i \in \mathbb{N}} (CB)_{ki}D_{ik} = 0$. It is a contradiction since $(I(k))_{kk} = 1$.

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