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Ekeland's Variational Principle and its Applications to Equilibrium Problems

A. Hosseinpour[†], A. Farajzadeh[‡] and S. Plubtieng[†,1]

[†]Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand.
[†]Center of Excellence in Nonlinear Analysis and Optimization, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand.
e-mail : a.hosseinpour@gmail.com (A. Hosseinpour)
somyotp@nu.ac.th (S. Plubtieng)
[‡]Department of Mathematics, Razi University, Kermanshah, 67149, Iran.
e-mail : farajzadehali@gmail.com (A. Farajzadeh)

Abstract : In this paper, we obtain a generalization of the vectorial form of Ekeland's variational principle for set-valued mapping in complete metric spaces. We get some equivalent results to the mentioned variational principle. As an application of our work, we provide some existence results for equilibrium problems in compact and noncompact spaces.

Keywords : Ekeland's variational principle; equilibrium problems; set-valued mappings; scalarization function; complete metric space.
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1 Introduction

Ekeland's variational principle was first expressed by Ekeland [1, 2] which is developed by many authors and researchers. This principle occurs in many problems like optimization, nonlinear equations, dynamical systems (see [3, 4, 5]). In recent years, many authors have made an effort to generalize this problem. For

[1]Corresponding author.

example in [6] and [7], the authors introduced the equilibrium version of Ekeland's variational principle to get some existence results for equilibrium problems in both compact and noncompact domains. Afterwards, Bianchi et al. [8] considered a vector version of Ekeland's principle for equilibrium problems. They studied bifunctions defined on complete metric spaces with values in locally convex spaces ordered by closed convex cones and obtained some existence results for vector equilibria in compact and noncompact domains. Recently, Ansari [9] studied a vectorial form of equilibrium version of Ekeland-type variational principle in the setting of complete quasi-metric spaces with a w-distance. He also survived Caristi Kirk fixed point theorem for multi-valued maps. In [10], an Ekeland's variational principle for set-valued mappings is investigated by Zeng and Li. They obtained some existence results in equilibrium problems and fixed point theory. In our work, we consider a generalization of vectorial form of Ekeland's variational principle for set-valued mapping in complete metric spaces. We first introduce some basic definitions and concepts in section 2 and then in section 3, we state our main results regarding the vectorial form of Ekeland's variational principle for set-valued mapping. Finally, in section 4, we obtain some existence results for equilibrium problems in both compact and noncompact spaces.

2 Main Definitions

Throughout this paper, let (X, d) be a complete metric space, Y a topological vector space and C a pointed convex cone in X. There is a partial order on X given by $x \prec y$ if and only if $y - x \in C$, for $x, y \in X$. Assume that $\text{int}C \neq \emptyset$, that is interior of C, and $e \in \text{int}C$.

Lemma 2.1 ([11])**.** *Let $\{s_n\}$ be a sequence of real numbers. Let E be the set of numbers x (in the extended real number system) such that $s_{n_k} \to x$, for some subsequences $\{s_{n_k}\}$. Then,* $\inf E = \liminf_{n\to\infty} s_n$ *and* $\liminf_{n\to\infty} s_n \in E$.

Definition 2.2 ([12])**.** Let (X, d) be a metric space. Then, a function $w : X \times X \to R^+$ is called the *w-distance* on X if satisfies the following conditions:

1. $w(x, z) \leq w(x, y) + w(y, z), \quad \forall x, y, z \in X,$
2. w is lower semicontinuous in its second variable,
3. for each $\varepsilon > 0$, there exists $\delta > 0$ such that $w(z, x) \leq \delta$ and $w(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$.

Definition 2.3 ([13])**.** Let X and Y be two topological spaces. A set-valued mapping $F : X \to 2^Y$ is called *lower semicontinuous* (l.s.c) at $x \in X$ if for each open set V with $F(x) \cap V \neq \emptyset$, there is an open set U containing x such that for each $t \in U$, $F(t) \cap V \neq \emptyset$. We say F is l.s.c. on X if it is l.s.c. at all $x \in X$.

Definition 2.4 ([13])**.** Let X and Y be two topological spaces. A set-valued mapping $F : X \to 2^Y$ is called *upper semicontinuous* (u.s.c) at $x \in X$ if for each open set V containing $F(x)$, there is an open set U containing x such that for each $t \in U$, $F(t) \subset V$. We say F is u.s.c. on X if it is u.s.c. at all $x \in X$.

Proposition 2.5 ([14]). *Let X and Y be topological spaces, and $F : X \to 2^Y$ be a set-valued mapping. Then, F is l.s.c. at $x_0 \in X$ if and only if for any net $\{x_\alpha\} \subset X$ with $x_\alpha \to x_0$ and any $y_0 \in F(x_0)$, there exists $y_\alpha \in F(x_\alpha)$ such that $y_\alpha \to y_0$.*

Proposition 2.6 ([14]). *Let X and Y be topological spaces, and $F : X \to 2^Y$ be a set-valued mapping. If F has compact values (i.e., $F(x)$ is a compact set for each $x \in X$), then F is u.s.c. at $x_0 \in X$ if and only if for any net $\{x_\alpha\} \subset X$ with $x_\alpha \to x_0$ and for any $y_\alpha \in F(x_\alpha)$, there exists $y_0 \in F(x_0)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \to y_0$.*

Definition 2.7 ([10]). Let X and Y be Banach spaces and C a pointed convex cone in Y. We say that a set-valued mapping $F : X \longrightarrow 2^Y$ is *quasi lower semicontinuous* at $x_0 \in X$ if for every $b \in Y$ and $F(x_0) \nsubseteq b - C$, there exists a neighborhood U of x_0 in X such that $F(x) \nsubseteq b - C$, for each $x \in U$. F is quasi lower semicontinuous if and only if F is quasi lower semicontinuous at each point $x \in X$.

Definition 2.8 ([15]). A set-valued mapping $F : X \times X \to 2^Y$ is *bounded from below* on $X \times X$ if there exists $z \in Y$ such that $F(x, y) \subseteq z + C$, for all $x, y \in X$.

Definition 2.9 ([16]). Given a fixed $e \in \text{int}C$ and $a \in X$. The function $\xi_{e,a} : X \longrightarrow R$ defined by

$$\xi_{e,a}(z) = \min\{t \in R \mid z \in a + te - C\} \tag{2.1}$$

is called a *nonlinear (separating) scalarization function.*

Proposition 2.10 ([16]). *The function $\xi_{e,a}$ is well-defined, that is, the minimum in (2.1) is attained.*

Proposition 2.11 ([16]). *For any fixed $e \in intC$, $z \in X$ and $r \in R$, we have*

$$\xi_{e,0}(z) \le r \Leftrightarrow z \in re - C$$

Sometimes, we denote $\xi_{e,0}$ by ξ_e.

Definition 2.12 ([17]). Let $S : X \longrightarrow 2^X$ be a dynamical system. $x^* \in X$ is called a *critical point* of S if and only if $S(x^*) = \{x^*\}$.

Theorem 2.13 ([18]). *Let (X, d) be a complete metric space and $S : X \longrightarrow 2^X$ be a set-valued mapping satisfying the following conditions:*

(i) $x \in S(x)$, for all $x \in X$,

(ii) $S(x)$ is a closed set, for all $x \in X$,

(iii) $x_2 \in S(x_1) \Longrightarrow S(x_2) \subseteq S(x_1)$, *for all* $x_1, x_2 \in X$,

(iv) *If* $\{x_n\}$ *is any sequence in* X *such that* $x_{n+1} \in S(x_n)$, *for all* $n \in N$, *then* $\lim_{n\longrightarrow\infty} d(x_n, x_{n+1}) = 0$.

Then, the set-valued mapping S *has a critical point* x^* *in* X, *that is* $S(x^*) = \{x^*\}$. *Moreover, for all* $x_0 \in X$, *there is a critical point of* S *in* $S(x_0)$.

3 Main Results

In this section, we state the new version of Ekeland's variational principle which was surveyed by Zeng and Li [10].

Theorem 3.1. *Assume that* (X, d) *is a complete metric space, and the set-valued mapping* $F : X \times X \longrightarrow 2^Y$ *is quasi lower semicontinuous in the second variable and bounded from below on* $X \times X$. *Moreover, suppose that* F *meets the following condition:*

$$F(x, y) \subseteq F(x, z) + F(z, y) - C, \quad \forall x, y, z \in X. \tag{3.1}$$

Then, for all $\varepsilon > 0$ *and* $x_0 \in X$, *there exists* $\overline{x} \in X$ *such that*

$$\begin{cases} F(x_0, \overline{x}) + \varepsilon w(x_0, \overline{x})e \subseteq -C, \\ F(\overline{x}, x) + \varepsilon w(\overline{x}, x)e \nsubseteq -C, \quad \forall x \in X, x \neq \overline{x}. \end{cases}$$

Proof. We choose $\varepsilon = 1$ and define the set-valued mapping $S : X \longrightarrow 2^X$ by

$$S(x) = \{y \in X \mid F(x, y) + w(x, y)e \subseteq -C\} \cup \{x\}$$

$S(x)$ is nonempty, because $x \in S(x)$, for all $x \in X$. We prove that $S(x)$ is closed, for all $x \in X$. Consider any sequence $\{y_n\}$ in $S(x)$ such that $y_n \to y_0$. We must show that $F(x, y_0) + w(x, y_0)e \subseteq -C$. Suppose by contradiction that $F(x, y_0) + w(x, y_0)e \nsubseteq -C$, hence $F(x, y_0) \nsubseteq -w(x, y_0)e - C$. In regard to the quasi lower semicontinuous of F at $y_0 \in X$, there exists a neighborhood U of y_0 in X such that $F(x, y) \nsubseteq -w(x, y)e - C$, for all $y \in U$. Since $y_n \to y_0$, then there exists $N_0 \in N$ such that $F(x, y_n) \nsubseteq -w(x, y_n)e - C$, for all $n > N_0$. Thus, $F(x, y_n) + w(x, y_n)e \nsubseteq -C$, for all $n > N_0$ which is a contradiction. It is easy to prove that if $y \in S(x)$, then $S(y) \subseteq S(x)$. In this regard, let $z \in S(y)$, then

$$F(y, z) + w(y, z)e \subseteq -C. \tag{3.2}$$

On the other hand, since $y \in S(x)$, we have

$$F(x, y) + w(x, y)e \subseteq -C. \tag{3.3}$$

By (3.1), (3.2), (3.3) and condition 1. of Definition 2.2, we obtain $F(x, z) + w(x, z)e \subseteq -C$ implying that $z \in S(x)$. It just reminds to verify (iv) of Theorem 2.13. On this subject, let $\{x_n\}_{n\in N}$ be any sequence in X such that $x_{n+1} \in S(x_n)$, for all $n \in N$ with any $x_1 \in X$. Thus, we have

$$F(x_n, x_{n+1}) + w(x_n, x_{n+1})e \subseteq -C.$$

Since C is a convex cone, then

$$\sum_{i=1}^{n} F(x_i, x_{i+1}) + \sum_{i=1}^{n} w(x_i, x_{i+1})e \subseteq -C. \tag{3.4}$$

Regarding to relations (3.1) and (3.4), we obtain

$$F(x_1, x_{n+1}) + \sum_{i=1}^{n} w(x_i, x_{i+1})e \subseteq \sum_{i=1}^{n} F(x_i, x_{i+1}) + \sum_{i=1}^{n} w(x_i, x_{i+1})e - C \subseteq -C.$$

Hence,

$$\sum_{i=1}^{n} w(x_i, x_{i+1})e \in -F(x_1, x_{n+1}) - C. \tag{3.5}$$

Since F is below bounded on $X \times X$, there exists $z \in Y$ such that

$$F(x_1, x_{n+1}) \subseteq z + C. \tag{3.6}$$

Relations (3.5) and (3.6) imply that $\sum_{i=1}^{n} w(x_i, x_{i+1})e \in -z - C$. Consequently, $z \in -\sum_{i=1}^{n} w(x_i, x_{i+1})e - C$. With reference to Proposition 2.11, we have $\xi_e(z) \leq -\sum_{i=1}^{n} w(x_i, x_{i+1})$. Thus,

$$\sum_{i=1}^{n} w(x_i, x_{i+1}) \leq -\xi_e(z)$$

Set $T_n = \sum_{i=1}^{n} w(x_i, x_{i+1})$. Obviously, $\{T_n\}$ is an increasing and bounded from above sequence which implies that $\{T_n\}$ is a convergent sequence. Hence, $w(x_i, x_{i+1}) \to 0$. As we see, all assumptions of Theorem 2.13 hold. Therefore, for all $x_0 \in X$, there exists $\overline{x} \in S(x_0)$ such that $S(\overline{x}) = \{\overline{x}\}$, that is

$$F(x_0, \overline{x}) + w(x_0, \overline{x})e \subseteq -C,$$

and

$$F(\overline{x}, x) + w(\overline{x}, x)e \nsubseteq -C, \quad \forall x \in X, x \neq \overline{x}.$$

This completes the proof. □

Remark 3.2. *Suppose that $F(x, x) = 0_Y$, for all $x \in X$. Then, from Theorem 3.1, we conclude that*

$$F(\overline{x}, x) + w(\overline{x}, x)e \nsubseteq -intC, \quad \forall x \in X.$$

Indeed, Theorem 3.1 implies that

$$F(\overline{x}, x) + w(\overline{x}, x)e \nsubseteq -intC, \quad \forall x \in X, x \neq \overline{x}$$

Assume that $F(\overline{x}, \overline{x}) + w(\overline{x}, \overline{x})e \subseteq -intC$. Since $w(\overline{x}, \overline{x}) \geq 0$ and $e \in intC$, convexity of C imply that $w(\overline{x}, \overline{x})e \in C$. Therefore, $F(\overline{x}, \overline{x}) + w(\overline{x}, \overline{x})e \subseteq C$ which is a contradiction.

4 Existence Results for Equilibrium Problems

Let (X, d) be a complete metric space and C a proper closed convex cone of X with $\text{int}C \neq \emptyset$. Let $F : X \times X \to 2^Y$ be a set-valued mapping. An equilibrium problem is finding $\overline{x} \in X$ such that $F(\overline{x}, y) \nsubseteq -\text{int}C$, for all $y \in X$. We may abbreviate this problem with EP from now on. In this section, we intend to provide sufficient conditions to solve the mentioned EP using the new version of Ekeland's variational principle.

Theorem 4.1. *Assume that all the assumptions of Theorem 3.1 hold. Moreover, suppose that X is a compact space, F is upper semicontinuous in the first variable and $F(x, x) = 0_Y$, for all $x \in X$. Then, EP has at least one solution.*

Proof. Consider $\varepsilon = 1/n$. By Theorem 3.1 and Remark 3.2, for all $n \in N$, there exists $x_n \in X$ such that

$$F(x_n, y) + \frac{1}{n} w(x_n, y)e \nsubseteq -\text{int}C, \quad \forall y \in X. \tag{4.1}$$

By the compactness of X, $\{x_n\}$ has a convergent subsequence in X, say $\{x_{n_k}\}$. Hence, there exists $\overline{x} \in X$ such that $x_{n_k} \to \overline{x}$ as $k \to \infty$. We claim that $\overline{x}$ is a solution of EP. Suppose not, therefore $F(\overline{x}, y) \subseteq -\text{int}C$, for some $y \in X$. From (4.1), we get

$$F(x_{n_k}, y) + \frac{1}{n_k} w(x_{n_k}, y)e \nsubseteq -\text{int}C, \quad \forall y \in X.$$

Thus,

$$\forall k \in N \quad \exists z_k \in F(x_{n_k}, y) + \frac{1}{n_k} w(x_{n_k}, y)e \quad \text{s.t} \quad z_k \notin -\text{int}C \tag{4.2}$$

Hence,

$$\forall k \in N \quad \exists y_k \in F(x_{n_k}, y) \quad \text{s.t} \quad z_k = y_k + \frac{1}{n_k} w(x_{n_k}, y)e \tag{4.3}$$

According to Proposition 2.6, since F is upper semicontinuous in its first variable there exist $\overline{y} \in F(\overline{x}, y)$ and a subsequence $\{y_{k_m}\}$ of $\{y_k\}$ such that $y_{k_m} \to \overline{y}$. From (4.3) and boundedness of w, we obtain

$$z_{k_m} = y_{k_m} + \frac{1}{n_{k_m}} w(x_{n_{k_m}}, y)e \to \overline{y} \in F(\overline{x}, y) \subseteq -\text{int}C.$$

Therefore, there exists $M_0 \in N$ such that for all $m > M_0$, $z_{k_m} \in -\text{int}C$ which contradicts with (4.2). This completes the proof. □

Theorem 4.2. *Assume that all the assumptions of Theorem 3.1 hold. Let F be upper semicontinuous in the first variable, $F(x, x) = 0_Y$ and $F(x, x) \subseteq -C$, for all $x \in X$. In addition, suppose that for any fixed point $x_0 \in X$ there is a compact set $K \subseteq X$ such that*

$$\forall x \in X - K \quad \exists y \in X \quad \textit{with} \quad w(y, x_0) < w(x, x_0) \quad \textit{and} \quad F(x, y) \subseteq -C. \tag{4.4}$$

Moreover, let the set $\{y \in X \mid w(y, x_0) \leq w(x, x_0),\ F(x, y) \subseteq -C\}$ is compact, for all $x \in X$. Then, EP has at least one solution.

Proof. We define $S : X \to 2^Y$ by

$$S(x) = \big\{y \in X \mid w(y, x_0) \leq w(x, x_0),\ F(x, y) \subseteq -C\big\}. \tag{4.5}$$

Obviously, $S(x) \neq \emptyset$, for all $x \in X$. We first show that for any $x, y \in X$ if $y \in S(x)$, then $S(y) \subseteq S(x)$. Let $t \in S(y)$, then $w(t, x_0) \leq w(y, x_0)$ and $F(y, t) \subseteq -C$. Since $y \in S(x)$, then $w(y, x_0) \leq w(x, x_0)$ and $F(x, y) \subseteq -C$. Thus, we have

$$w(t, x_0) \leq w(y, x_0) \leq w(x, x_0) \tag{4.6}$$

On the other hand, from (3.1), we have

$$F(x, t) \subseteq F(x, y) + F(y, t) - C \subseteq -C \tag{4.7}$$

Thus, from (4.6) and (4.7), we get $t \in S(x)$ and hence $S(y) \subseteq S(x)$. Since K is a compact set, then Theorem 4.1 grantees the existence of a point $x_k \in K$ such that

$$F(x_k, y) \nsubseteq -C, \quad \forall y \in K. \tag{4.8}$$

We claim that x_k is a solution of EP on X. Assume that this assertion is not true, therefore

$$\exists\, \overline{x} \in X \quad \text{such that} \quad F(x_k, \overline{x}) \subseteq -C \tag{4.9}$$

set

$$a = \min_{y \in S(\overline{x})} w(y, x_0). \tag{4.10}$$

Based on the assumption of the theorem, the minimum is achieved. In this way, there exists $x_1 \in S(\overline{x})$ such that $w(x_1, x_0) = a$. It is clear that $x_1 \notin K$. Because otherwise, since $x_1 \in S(\overline{x})$, we have $F(\overline{x}, x_1) \subseteq -C$ and according to (4.9), we obtain

$$F(x_k, x_1) \subseteq F(x_k, \overline{x}) + F(\overline{x}, x_1) - C \subseteq -C$$

which contradicts (4.8). Now by (4.4), there exists $y_1 \in X$ such that $w(y_1, x_0) < w(x_1, x_0)$ and $F(x_1, y_1) \subseteq -C$. Therefore, $y_1 \in S(x_1) \subseteq S(\overline{x})$. So, $w(y_1, x_0) \leq w(x_1, x_0) = a$ contradicting (4.10) since $y_1 \in S(\overline{x})$. □

5 Conclusion

In the present paper, we study the vectorial form of Ekeland's variational principle for set-valued mapping in complete metric space. We obtained some existence results for equilibrium problems in compact and noncompact spaces.

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