Thai Journal of Mathematics
Volume 14 (2016) Number 2 : 505–516
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

On Best Approximation and Best Coapproximation

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Abstract : In this paper, the concept of best coapproximation is employed to prove various characterizations of proximinal, coproximinal, Chebyshev and co-Chebyshev subspaces. The continuity properties of metric projection, metric co-projection and related maps have also been discussed. The underlying spaces are normed linear spaces and metric linear spaces.

Keywords : best approximation; best coapproximation; Chebyshev set; co-Chebyshev set; metric projection; metric co-projection.
2010 Mathematics Subject Classification : 41A50; 41A52; 41A65.

1 Introduction

A kind of approximation, called best coapproximation by Papini and Singer [1], was introduced in 1972 by Franchetti and Furi to characterize real Hilbert spaces among real reflexive Banach spaces. As in the case of best approximation, the theory of best coapproximation has been developed to a large extent in normed linear spaces and in Hilbert spaces by C. Franchetti and M. Furi, L. Hetzelt, H. Mazaheri, T.D. Narang, P.L. Papini and I. Singer, Geetha S. Rao and her students, and by many others (see e.g. [1–9] and references cited therein). The situation in case of metric linear spaces and metric spaces is somewhat different. Although, some attempts have been made to develop the theory of best coapproximation in such spaces (see e.g. [10, 11] but this theory is very less developed as compared to

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theory of best approximation. The present paper is a step in this direction. This paper deals with some characterizations of proximinal, coproximinal, Chebyshev and co-Chebyshev subspaces. We also discuss continuity properties of the metric projection, metric coprojection and related maps in this paper.

2 Preliminaries

In this section, we give some definitions and basic facts concerning best approximation and best coappproximation.

Given a subset G of a metric space (X, d), an element $g_0 \in G$ is called a *best approximation (best coapproximation)* to $x \in X$ if

$$d(x, g_0) \leq d(x, g) \ (d(g_0, g) \leq d(x, g))$$

for all $g \in G$. The set of all such $g_0 \in G$ is denoted by $P_G(x)(R_G(x))$. The set G is called *proximinal (coproximinal)* if $P_G(x)$ $(R_G(x))$ contains at least one element for every $x \in X$. If for each $x \in X$, $P_G(x)(R_G(x))$ has exactly one element, then the set G is called *Chebyshev (co-Chebyshev)*.

We shall denote the set $\{x \in X : P_G(x) \neq \phi\}$ $(\{x \in X : R_G(x) \neq \phi\})$ by $D(P_G)$ $(D(R_G))$ and the set $\{x \in X : g_0 \in P_G(x)\}$ $(\{x \in X : g_0 \in R_G(x)\})$ by $P_G^{-1}(g_0)$ $(R_G^{-1}(g_0))$.

For a proximinal (coproximinal) subset G of X, a mapping $P_G(R_G) : X \to 2^G$($\equiv$ the collection of all subsets of G) defined by $P_G(x)(R_G(x)) = \{g_0 \in G : d(x, g_0) \leq d(x, g) \ for\ every\ g \in G\}$ $(\{g_0 \in G : d(g_0, g) \leq d(x, g) \ for\ every\ g \in G\})$ is called *metric projection (metric coprojection)*.

Remarks 2.1.

1. *A proximinal subset of a metric space need not be coproximinal:* *Let $X = \mathbb{R}^2$ and $G = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, then G is a compact subset of $\mathbb{R}^2$ and hence proximinal. However, G is not coproximinal as $(0, 0) \in \mathbb{R}^2$ does not have any best coapproximation in G.*

2. *A coproximinal subset of a metric space need not be proximinal:* *Let $X = \mathbb{R} - \{1\}$ and $M = (1, 2]$, then M is a coproximinal subset of X but is not proximinal.*

3. *A Chebyshev subset of a metric space need not be co-Chebyshev:* *Let $X = \mathbb{R}$ and $G = [1, 2]$, then G is Chebyshev but not co-Chebyshev.*

4. *A co-Chebyshev subset of a metric space need not be Chebyshev:* *Let $X = R^2$ with the metric $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$ and $G = \{(x, y) \in R^2 : x = y\}$. Then, X is a real Banach space and G is a proximinal subspace of X. We have $P_G(x, y) = \{\alpha(x, x) + (1 - \alpha)(y, y) : 0 \leq \alpha \leq 1\}$ i.e. G is not Chebyshev, but $R_G(x, y) = \{(\frac{x+y}{2}, \frac{x+y}{2})\}$ i.e. G is co-Chebyshev.*

5. *The set $P_G(x)$ $(R_G(x))$ is closed if G is closed.*

6. *Every proximinal (coproximinal) set is closed.*
7. *The set* $P_G^{-1}(g_0)(R_G^{-1}(g_0))$ *is a closed set for every* $g_0 \in G$.
8. *If* G *is subspace of a metric linear space* (X,d)*, then* $g_0 \in P_G(x)$ $(g_0 \in R_G(x))$ *if and only if* $x - g_0 \in P_G^{-1}(0)$ $(R_G^{-1}(0))$ *and* $P_G(x+g) = P_G(x) + g$ $(R_G(x+g) = R_G(x) + g)$ *for every* $g \in G$.
9. *If* G *is subspace of a metric linear space* (X,d)*, then* $d(g,0) = d(g, R_G^{-1}(0))$ *for every* $g \in G$.

For metric spaces X and Y, a mapping $u : X \to 2^Y$ is called

1. *upper (K)-semi-continuous (u.(K)-s.c.)* if $x_n \to x$, $y_n \in u(x_n)$, $y_n \to y$ imply $y \in u(x)$,
2. *lower (K)-semi-continuous (l.(K)-s.c.)* if $x_n \to x$, $y \in u(x)$ imply the existence of a sequence $\{y_n\}$ such that $y_n \in u(x_n)$ and $y_n \to y$,
3. *upper semi-continuous (lower semi-continuous)* if the set

$$H = \{x \in X : u(x) \bigcap N \neq \phi\}$$

is closed(open) for every closed(open) subset $N \subseteq Y$.

For a closed linear subspace G of a metric linear space (X,d), the *canonical mapping* w_G of X onto X/G is defined as $w_G(x) = x + G$, $x \in X$. It is easy to see that the mapping w_G is linear, continuous and open.

3 Characterizations of Proximinal, Coproximinal, Chebyshev and Co-Chebyshev Subspaces

In this section, we discuss characterizations of proximinal, coproximinal, Chebyshev and co-Chebyshev subspaces in metric linear spaces and normed linear spaces. We start with a characterization of Chebyshev subspaces proved in [12] (see also [13]):

Theorem 3.1. *Let* G *be a proximinal subspace of a normed linear space* X*, then the following statements are equivalent:*
(i) P_G *is one-valued and linear.*
(ii) $P_G^{-1}(0)$ *is a closed linear subspace of* X.
(iii) $P_G^{-1}(0)$ *is a convex subset of* X.

Motivated by this result of Holmes and Kripke [12], we prove the following characterization of Chebyshev subspaces in metric linear spaces:

Theorem 3.2. *Let* G *be a proximinal linear subspace of a metric linear space* (X,d)*, then the following are equivalent:*
(i) P_G *is one-valued and additive.*
(ii) $P_G^{-1}(0)$ *is a closed additive subgroup of* X.

Proof. (i)$\Rightarrow$ (ii) It is known that $P_G^{-1}(0)$ is closed. Let $x, y \in P_G^{-1}(0)$, then $P_G(x) = \{0\}$ *and* $P_G(y) = \{0\}$. Since P_G is additive $P_G(x+y) = P_G(x) + P_G(y) = 0$ implies that $x + y \in P_G^{-1}(0)$. Since $x \in P_G^{-1}(0)$, we have $d(x, 0) = \inf_{g \in G} d(x, g) = \inf_{g \in G} d(-g, -x) = d(-x, G)$. This implies that $d(-x, 0) = d(-x, G)$ i.e. $-x \in P_G^{-1}(0)$. Hence $P_G^{-1}(0)$ is additive subgroup of X.
(ii)$\Rightarrow$(i) Let $g_1, g_2 \in P_G(x)$. Then $x - g_1, x - g_2 \in P_G^{-1}(0)$. Since $P_G^{-1}(0)$ is additive subgroup, we have $(x - g_1) - (x - g_2) \in P_G^{-1}(0)$ i.e. $g_2 - g_1 \in P_G^{-1}(0)$. Also $g_2 - g_1 \in G$. Therefore, $g_2 - g_1 \in P_G^{-1}(0) \bigcap G = \{0\}$. This gives $g_1 = g_2$. Hence P_G is single-valued.

Let $x, y \in X$, $g_1 \in P_G(x)$ *and* $g_2 \in P_G(y)$ i.e. $x - g_1 \in P_G^{-1}(0)$ and $y - g_2 \in P_G^{-1}(0)$. Since $P_G^{-1}(0)$ is additive subgroup of X, we have $(x - g_1) + (y - g_2)) \in P_G^{-1}(0)$ i.e. $(x + y) - (g_1 + g_2) \in P_G^{-1}(0)$. Consider, $P_G(x + y) - (g_1 + g_2) = P_G(x + y - (g_1 + g_2)) = 0$, as P_G is single-valued. This gives $P_G(x+y) = g_1 + g_2 = P_G(x) + P_G(y)$. Hence P_G is additive. □

Analogous to Theorem 3.2, we have the following characterization of the co-Chebyshev subspaces:

Theorem 3.3. *Let G be a coproximinal subspace of a metric linear space (X, d), then the following are equivalent:*
(i) R_G is one-valued and additive.
(ii) $R_G^{-1}(0)$ is a closed additive subgroup of X.

Proof. (i)$\Rightarrow$ (ii) It is known that $R_G^{-1}(0)$ is a closed set. Let $x, y \in R_G^{-1}(0)$, then $R_G(x) = \{0\}$ *and* $R_G(y) = \{0\}$. Since R_G is additive, we have $R_G(x + y) = R_G(x) + R_G(y) = 0$. This implies that $x + y \in R_G^{-1}(0)$. As $x \in R_G^{-1}(0)$, we have $d(0, g) \le d(x, g)$ for every $g \in G$. This gives $d(0, -g) \le d(-x, -g)$ for every $g \in G$ i.e. $-x \in R_G^{-1}(0)$. Hence $R_G^{-1}(0)$ is additive subgroup of X.
(ii)$\Rightarrow$ (i) Let $g_1, g_2 \in R_G(x)$. This gives $x - g_1, x - g_2 \in R_G^{-1}(0)$. Since $R_G^{-1}(0)$ is additive subgroup, we have $(x - g_1) - (x - g_2) \in R_G^{-1}(0)$ i.e. $g_2 - g_1 \in R_G^{-1}(0)$. This gives $g_2 - g_1 \in R_G^{-1}(0) \bigcap G = \{0\}$ and so $g_1 = g_2$. Hence R_G is one-valued.

Let $x, y \in X$ be such that $g_1 \in R_G(x)$ and $g_2 \in R_G(y)$. This gives $x - g_1$, $y - g_2 \in R_G^{-1}(0)$. Since $R_G^{-1}(0)$ is additive subgroup, we have $(x - g_1) + (y - g_2) \in R_G^{-1}(0)$ i.e. $(x+y)-(g_1+g_2) \in R_G^{-1}(0)$. Now, $R_G(x+y)-(g_1+g_2) = R_G(x+y-(g_1+g_2)) = 0$, as R_G is single-valued. Hence $R_G(x+y) = g_1+g_2 = R_G(x)+R_G(y)$. □

Before proving the next result, we prove the following lemma:

Lemma 3.4. *Let G be a linear subspace of a metric linear space (X, d) then the following statements are equivalent:*
(i) G is co-proximinal.
(ii) $X = G + R_G^{-1}(0)$.

Proof. (i)$\Rightarrow$ (ii) Let $x \in X$ be arbitrary and $g_0 \in R_G(x)$. Since $g_0 \in R_G(x)$, we have $x - g_0 \in R_G^{-1}(0)$. Then $x = g_0 + (x - g_0) \in G + R_G^{-1}(0)$ i.e. $X \subseteq G + R_G^{-1}(0)$ and consequently

$$X = G + R_G^{-1}(0).$$

(ii)$\Rightarrow$ (i) Let $x \in X$ be arbitrary, then $x = g_0 + y$ for some $g_0 \in G,\ y \in R_G^{-1}(0)$. Since $y \in R_G^{-1}(0),\ 0 \in R_G(y)$ i.e. $0 \in R_G(x - g_0)$ i.e. $x - g_0 \in R_G^{-1}(0)$ and so $g_0 \in R_G(x)$. Hence G is co-proximinal. □

Remarks 3.5. *1. For normed linear spaces the above lemma was proved in [4].*
2. For best approximation, analogous results were proved for normed linear spaces in [14] and for metric linear spaces in [15].

Using above lemma, we prove the following characterization of the co-Chebyshev subspaces:

Theorem 3.6. *For a closed linear subspace G of a metric linear space (X, d), the following are equivalent:*
(i) G is co-Chebyshev.
(ii) $X = G \bigoplus R_G^{-1}(0)$, where $\bigoplus$ means that the sum decomposition of each $x \in X$ is unique.
(iii) G is co-proximinal and $[R_G^{-1}(0) - R_G^{-1}(0)] \bigcap G = \{0\}$.
(iv) G is co-proximinal and the restriction map $w_G\ |_{R_G^{-1}(0)}$ is one to one.

Proof. (i)$\Rightarrow$ (ii) Since G is co-proximinal, we have $X = G + R_G^{-1}(0)$. Now it is sufficient to show that this sum decomposition is unique. Let $x \in X$ be such that $x = g_1 + y_1 = g_2 + y_2$, where $g_1, g_2 \in G$ *and* $y_1, y_2 \in R_G^{-1}(0)$. This gives $g_1 - g_2 = y_2 - y_1$. Now, $y_1 \in R_G^{-1}(0)$ implies that $g_1 \in R_G(y_1 + g_1) = R_G(x)$. Similarly, $g_2 \in R_G(y_2+g_2) = R_G(x)$ i.e. $g_1, g_2 \in R_G(x)$. Since G is co-Chebyshev, we have $g_1 = g_2$. Hence $y_1 = y_2$ and so the representation is unique. Therefore, $X = G \bigoplus R_G^{-1}(0)$.
(ii)$\Rightarrow$(iii) Since $X = G \bigoplus R_G^{-1}(0)$, G is co–proximianl. Suppose $0 \neq y \in [R_G^{-1}(0) - R_G^{-1}(0)] \bigcap G$. Then $y = y_1 - y_2,\ y_1, y_2 \in R_G^{-1}(0)$ and $y_1 \neq y_2$. So, $0 \in R_G(y_1)$ and $0 \in R_G(y_2)$. Now, $y_1, y_2 \in R_G^{-1}(0),\ y_1 - y_2 \in G\backslash\{0\}$ and $y_1 = 0 + y_1 = (y_1 - y_2) + y_2$ i.e. $y_1 \in X$ has two distinct representations, a contradiction. Hence $[R_G^{-1}(0) - R_G^{-1}(0)] \bigcap G = \{0\}$.
(iii)$\Rightarrow$(iv) Suppose $w_G\ |_{R_G^{-1}(0)}$ is not one to one i.e. there exist $y_1, y_2 \in R_G^{-1}(0),\ y_1 \neq y_2$ such that $w_G(y_1) = w_G(y_2)$ i.e. $y_1 + G = y_2 + G$. This implies that $y_1 - y_2 \in G,\ y_1 - y_2 \neq 0$. Then $0 \neq y_1 - y_2 \in [R_G^{-1}(0) - R_G^{-1}(0)] \bigcap G$, a contradiction. Hence $w_G\ |_{R_G^{-1}(0)}$ is one to one.
(iv)$\Rightarrow$(i) Suppose $x \in X$ has two distinct best coapproximation in G, say g_1 and g_2 i.e. $g_1, g_2 \in R_G(x)$. This gives $x - g_1, x - g_2 \in R_G^{-1}(0)$. Since $g_1 \neq g_2,\ x - g_1 \neq x - g_2$ and $w_G(x - g_1) = x + G = w_G(x - g_2)$, a contradiction. Hence G is co-Chebyshev. □

Remarks 3.7. *1. For normed linear spaces the equivalence of (i) and (ii) was proved in [4].*
2. For best approximation analogous results were proved for normed linear spaces in [14] and for metric linear spaces in [15].

Using above theorem, we have the following:

Theorem 3.8. *Let G be a coproximinal subspace of a metric linear space (X, d). If $R_G^{-1}(0)$ is an additive subgroup, then $R_G^{-1}(0)$ is a proximinal subset of X.*

Proof. Since $R_G^{-1}(0)$ is additive subgroup, using Theorem 3.3 we have R_G is one valued and so G is co-Chebyshev in X. Therefore by Theorem 3.6, $X = G \bigoplus R_G^{-1}(0)$. Let $x \in X \backslash R_G^{-1}(0)$ be arbitrary then $x = g_1 + g_2$ where $g_1 \in G$ and $g_2 \in R_G^{-1}(0)$. Consider $d(x, g_2) = d(x - g_2, 0) = d(g_1, 0) = d(g_1, R_G^{-1}(0))$ i.e. $d(x, g_2) = d(x - g_2, R_G^{-1}(0)) = d(x, g_2 + R_G^{-1}(0)) = d(x, R_G^{-1}(0))$ (as $R_G^{-1}(0)$ is additive subgroup). Hence $R_G^{-1}(0)$ is proximinal in X. □

Taking $R_G^{-1}(0)$ to be a closed linear subspace, we have the following result in normed linear spaces.

Theorem 3.9. *Let G be a coproximinal subspace of a normed linear space X, then the following statements are equivalent:*
(i) R_G is one-valued and linear.
(ii) $R_G^{-1}(0)$ is a closed linear subspace of X.
(iii) $R_G^{-1}(0)$ is a convex subset of X.

Proof. (i)⇒ (ii) It is known that $R_G^{-1}(0)$ is a closed set. Let $x, y \in R_G^{-1}(0)$ and α, β be scalars. Then $R_G(x) = \{0\}$ *and* $R_G(y) = \{0\}$. Since R_G is linear, we have $R_G(\alpha x + \beta y) = \alpha R_G(x) + \beta R_G(y) = 0$. This implies that $\alpha x + \beta y \in R_G^{-1}(0)$. Hence $R_G^{-1}(0)$ is a subspace of X.
(ii)⇒(iii) is obvious.
(iii)⇒ (i) Let $g_1, g_2 \in R_G(x)$. This gives $x - g_1, x - g_2 \in R_G^{-1}(0)$. Since $x - g_2 \in R_G^{-1}(0)$, we have $g_2 - x \in R_G^{-1}(0)$. Since $R_G^{-1}(0)$ is a convex set, we have $\frac{1}{2}(x - g_1) + \frac{1}{2}(g_2 - x) \in R_G^{-1}(0)$ i.e. $\frac{1}{2}(g_2 - g_1) \in R_G^{-1}(0)$. This gives $\frac{1}{2}(g_2 - g_1) \in R_G^{-1}(0) \bigcap G = \{0\}$ and so $g_1 = g_2$. Hence R_G is one-valued.

Let $x, y \in X$ be such that $g_1 \in R_G(x)$ and $g_2 \in R_G(y)$. This gives $x - g_1,\ y - g_2 \in R_G^{-1}(0)$. Since $R_G^{-1}(0)$ is a convex set, we have $\frac{1}{2}(x - g_1) + \frac{1}{2}(y - g_2) \in R_G^{-1}(0)$ i.e. $\frac{1}{2}(x+y) - \frac{1}{2}(g_1+g_2) \in R_G^{-1}(0)$ i.e. $\|g\| \leq \|\frac{1}{2}(x+y) - \frac{1}{2}(g_1+g_2) - g\|$ *for every* $g \in G$ i.e. $(x+y) - (g_1+g_2) \in R_G^{-1}(0)$. Now, $R_G(x+y) - (g_1+g_2) = R_G(x+y-(g_1+g_2)) = 0$ i.e. $R_G(x+y) = g_1 + g_2 = R_G(x) + R_G(x)$. Since $g_1 \in R_G(x)$, we have $\|g_1 - g\| \leq \|x - g\|$ for every $g \in G$ i.e. $\|\alpha g_1 - \alpha g\| \leq \|\alpha x - \alpha g\|$ for every $g \in G$ and for every scalar α. This implies that $\alpha g_1 \in R_G(\alpha x)$ i.e. $\alpha R_G(x) \subseteq R_G(\alpha x)$. Let $g_1 \in R_G(\alpha x)$ then

$$\|g_1 - g\| \leq \|\alpha x - g\| \text{ } for \text{ } every \text{ } g \in G$$

$$\Rightarrow \|\frac{g_1}{\alpha} - \frac{g}{\alpha}\| \leq \|x - \frac{g}{\alpha}\| \text{ } for \text{ } every \text{ } g \in G$$

i.e. $\frac{g_1}{\alpha} \in R_G(x)$ and so $g_1 \in \alpha R_G(x)$. This gives $R_G(\alpha x) \subseteq \alpha R_G(x)$. Consequently $R_G(\alpha x) = \alpha R_G(x)$ and hence R_G is linear. □

Remarks 3.10. *The corresponding results for the map P_G were proved in [12] (see also [13]).*

The following theorem characterizes the coproximinality of hyperplanes in normed linear spaces:

Theorem 3.11. *A closed hyperplane G in a normed linear space X is coproximinal if and only if there exist an element $z \in X\backslash\{0\}$ such that $0 \in R_G(z)$.*

Proof. Let G be coproximinal in X, $x \in X\backslash G$ and $g_0 \in R_G(x)$. Since $g_0 \in R_G(x)$, we have $x - g_0 \in R_G^{-1}(0)$ i.e. $0 \in R_G(x-g_0),\ x - g_0 \neq 0$.

Conversly, if $0 \in R_G(z)$ then $\|g\| \leq \|z - g\|$ for every $g \in G$ implies that $\|\alpha g\| \leq \|\alpha z - \alpha g\|$ for every $g \in G$ and scalar α. This gives $0 \in R_G(\alpha z)$. Let $x \in X\backslash G$. Since G is a closed hyperplane in X, we have $x - \alpha z \in G$ for some suitable $\alpha \neq 0$. Since $0 \in R_G(\alpha z)$, we have $\|g\| \leq \|\alpha z - g\|$ for every $g \in G$ i.e.

$$\|(x-\alpha z) - (x - \alpha z + g)\| \leq \|\alpha z - (x - \alpha z) + (x - \alpha z) - g\ \|$$

$$i.e.\ \|(x-\alpha z) - (x - \alpha z + g)\| \leq \|x - (x - \alpha z + g)\|\ for\ every\ g \in G$$

$$i.e.\ \|(x-\alpha z) - g'\| \leq \|x - g'\|\ for\ every\ g' \in G$$

i.e. $x - \alpha z \in R_G(x)$. Hence G is coproximinal in X. □

Remarks 3.12. *A similar result for best approximation was proved in [13].*

A set G in a metric space (X, d) is said to be very non-proximinal if G is closed and no element $x \in X\backslash G$ has an element of best approximation in G i.e. if $\overline{G} = G$ and $P_G(x) = \phi$ for every $x \in X\backslash G$. Concerning the very non-proximinality of G, we have

Theorem 3.13. *A linear subspace G of a metric linear space (X, d) is very non-proximinal if and only if there is no element $z \in X\backslash\{0\}$ such that $0 \in P_G(z)$.*

Proof. Suppose there exist $z \in X\backslash\{0\}$ such that $0 \in P_G(z)$, then $d(z, 0) = d(z, G)$ implies that $d(z + g, g) = d(z + g, G)$ i.e. $g \in P_G(z + g)$ and so G is not very non-proximinal, a contradiction.

Conversly, suppose that there exist $x \in X\backslash G$ such that $g_0 \in P_G(x)$ i.e. $d(x, g_0) = d(x, G)$ i.e. $d(x - g_0, 0) = d(x - g_0, G)$ i.e. $0 \in P_G(x - g_0),\ x - g_0 \neq 0$, a contradiction. Hence the result follows. □

Remarks 3.14. *The above theorem extends the corresponding result proved in [13] for normed linear spaces. Analogously, we now characterize the non-coproximinality of G.*

A set G in a metric space (X, d) is said to be very non-coproximinal if no element $x \in X\backslash G$ has an element of best coapproximation in G i.e. $R_G(x) = \phi$ for every $x \in X\backslash G$.

Concerning the very non-coproximinality of G, we have

Theorem 3.15. *A linear subspace G of a metric linear space (X, d) is very non-coproximinal if and only if there is no element $z \in X\backslash\{0\}$ such that $0 \in R_G(z)$.*

Proof. Suppose there exist $z \in X\backslash\{0\}$ such that $0 \in R_G(z)$, then $d(0,g) \leq d(z,g)$ for every $g \in G$. This implies that $d(g', g+g') \leq d(z+g', g+g')$ for every $g \in G$ i.e. $g' \in R_G(z+g')$. This implies that G is not very non-coproximinal, a contradiction.

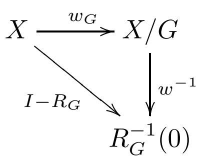
Conversly, assume that there exist no element $z \in X\backslash\{0\}$ such that $0 \in R_G(z)$. Suppose there exist $x \in X\backslash G$ such that $g_0 \in R_G(x)$, then $x - g_0 \in R_G^{-1}(0)$ and so $0 \in R_G(x-g_0),\ x - g_0 \neq 0$, a contradiction. ☐

4 Continuity Properties of Metric Projection, Metric Co-Projection and Related Maps

One of the main problem in the theory of approximation is the characterization of those Chebyshev (co-Chebyshev) subspaces for which P_G (R_G) is continuous. To discuss this problem, we start with the following:

Theorem 4.1. *For a co-Chebyshev subspace G of a metric linear space (X,d) the metric coprojection R_G is continuous if and only if the restriction $w = w_G\mid_{R_G^{-1}(0)}$ of the canonical mapping $w_G : X \to X/G$ to the set $R_G^{-1}(0)$ is a homeomorphism of $R_G^{-1}(0)$ onto X/G.*

Proof. We prove this result using the commuting diagram drawn below:



Assume w is a homeomorphism then so is w^{-1}. Since $(I - R_G) = w^{-1}w_G$, and w_G is continuous, $(I - R_G)$ is also continuous and so R_G is continuous.

Conversely, let R_G be continuous then so is $I - R_G$. Let U be an open set in $R_G^{-1}(0)$, then $(I - R_G)^{-1}(U)$ is open set in X and since w_G is an open mapping, $w_G[(I - R_G)^{-1}(U)]$ is open in X/G. Hence w^{-1} is continuous. Since w_G is continuous and onto(see [16]), and $w = w_G\mid_{R_G^{-1}(0)}$, we have w is continuous and onto. Moreover, it is one to one (Theorem 3.6). Hence w is a homeomorphism. ☐

Remarks 4.2. *An analogous result for best approximation map was proved in [17] for normed linear spaces.*

Concerning the continuity of R_G, we have the following:

Theorem 4.3. *For a co-Chebyshev subspace G of a metric linear space (X,d), the following statements are equivalent:*
(i) The metric coprojection R_G is continuous.
(ii) R_G is continuous at each point of $R_G^{-1}(0)$.
(iii) The direct sum decomposition $X = G \oplus R_G^{-1}(0)$ is topological (i.e. $\lim_{n\to\infty} x_n = x$ if and only if $\lim_{n\to\infty} R_G(x_n) = R_G(x)$ and $\lim_{n\to\infty}[x_n - R_G(x_n)] = x - R_G(x)$).
(iv) The functional $\phi_G(x) = d(R_G(x), 0),\ x \in X$ is continuous.

Proof. (i)⇒(ii) is obvious.
(ii)⇒(i) Suppose $x_n \to x$ but $R_G(x_n) \nrightarrow R_G(x)$, then $x_n - R_G(x) \to x - R_G(x) \in R_G^{-1}(0)$, but $R_G(x_n - R_G(x)) = R_G(x_n) - R_G(x) \nrightarrow 0$, a contradiction. Thus (i)⇔(ii).
(i)⇒(iii) Let $x_n \to x$. Since R_G is continuous, $R_G(x_n) \to R_G(x)$ and so $[x_n - R_G(x_n)] \to x - R_G(x)$. Conversly, let $R_G(x_n) \to R_G(x)$ and $[x_n - R_G(x_n)] \to x - R_G(x)$ then $x_n \to x$.
(iii)⇒(i) is obvious. Thus $(i) \Leftrightarrow (iii)$.
(i)⇒ (iv) is obvious.
(iv)⇒ (i) Let $x_n \to x \in R_G^{-1}(0)$, then $d(R_G(x_n), R_G(x)) = d(R_G(x_n), 0) \to d(R_G(x), 0) = 0$ implies that $R_G(x_n) \to R_G(x)$. This gives (iv)⇒ (ii). Since (ii)⇒(i), we have (iv)⇒ (i). Thus (i)⇔ (iv) and hence the proof is complete. □

Remarks 4.4. *The corresponding result for the best approximation map is well known for normed linear spaces (see [13]) and for metric linear spaces (see [18]).*

The following result on the continuity of P_G (R_G) was proved in [19]([11]):

Theorem 4.5. *Let G be a Chebyshev (co-Chebyshev) subspace of a metric linear space (X, d) such that $P_G^{-1}(0)$ ($R_G^{-1}(0)$) is boundedly compact then P_G (R_G) is continuous.*

We now discuss the converse implications

Theorem 4.6. *Let G be a Chebyshev subspace of a metric linear space (X, d) such that every bounded sequence in X/G has a convergent subsequence. If the metric projection P_G onto G is continuous then $P_G^{-1}(0)$ is boundedly compact.*

Proof. Suppose $P_G^{-1}(0)$ is not boundedly compact. Let $\{f_n\}$ be a bounded sequence in $P_G^{-1}(0)$ which has no convergent subsequence. Then $\{f_n + G\}$ is a bounded sequence in X/G and by hypothesis, $\{f_n + G\}$ has a subsequence $\{f_{n_k} + G\} \to f + G$. Now, $f_n \in P_G^{-1}(0)$ i.e. $d(f_n, 0) = d(f_n, G)$ and $\{f_{n_k} + G\} \to f + G$ implies that there exist a sequence $\{g_{n_k}\}$ in G such that $\{f_{n_k} + g_{n_k}\} \to f$. Consider

$$\begin{aligned} d(P_G(f_{n_k} + g_{n_k}), P_G(f)) &= d(P_G(f_{n_k}) + g_{n_k}, P_G(f)) \\ &= d(g_{n_k}, P_G(f)) \\ &= d(f_{n_k} + g_{n_k}, f_{n_k} + P_G(f)) \\ &= d(f_{n_k} + g_{n_k} - f, f_{n_k} + P_G(f) - f) \\ &\leq d(f_{n_k} + g_{n_k} - f, 0) + d(f_{n_k} + P_G(f) - f, 0) \nrightarrow 0. \end{aligned}$$

Hence $d(P_G(f_{n_k} + g_{n_k}), P_G(f)) \nrightarrow 0$ a contradiction to the continuity of P_G. Hence our supposition is wrong and so $P_G^{-1}(0)$ is boundedly compact. □

Theorem 4.7. *Let G be a co-Chebyshev subspace of a metric linear space (X, d) such that every bounded sequence in X/G has a convergent subsequence. If the metric coprojection onto G is continuous then $R_G^{-1}(0)$ is boundedly compact.*

Proof. The proof runs on similar lines as that of Theorem 4.6. □

A mapping $u : X \to Y$ is said to be Lipschitzian if there exist a constant $k > 0$ such that $d(u(x), u(y)) \leq kd(x, y)$ for all $x, y \in X$.

Concerning the Lipschitzian property of the map R_G, we have

Theorem 4.8. *For a co-Chebyshev subspace G of a metric linear space (X, d), the metric coprojection R_G is Lipschitzian if and only if the mapping $w_G^{-1} : X/G \to R_G^{-1}(0)$ is a Lipschitzian homeomorphism of $R_G^{-1}(0)$ onto X/G.*

Proof. For any $x + G \in X/G$, we have $x - R_G(x) \in R_G^{-1}(0)$ and $w_G(x - R_G(x)) = x + G$.

Let w_G^{-1} be a Lipschitzian homeomorphism. Consider

$$\begin{aligned}
d(R_G(x), R_G(y)) &= d(R_G(x) - x + x, R_G(y) - y + y) \\
&= d(-w_G^{-1}(x + G) + x, -w_G^{-1}(y + G) + y) \\
&\leq d(w_G^{-1}(y + G), w_G^{-1}(x + G)) + d(x, y) \\
&\leq kd(y + G, x + G) + d(x, y) \\
&= k \inf_{g \in G} d(y - x, g) + d(x, y) \\
&\leq kd(x, y) + d(x, y) = (1 + k)d(x, y) = k'd(x, y)
\end{aligned}$$

i.e. R_G is Lipschitzian.

Conversly, suppose that R_G is Lipschitzian. Consider

$$\begin{aligned}
d(w_G^{-1}(x + G), w_G^{-1}(y + G)) &= d(x - R_G(x), y - R_G(y)) \\
&= d(x - R_G(x - g), y - R_G(y - g)) \ for\ all\ g \in G \\
&= d(x - R_G(x) + g, y - R_G(y - g)) \ for\ all\ g \in G \\
&\leq d(x + g, y) + d(R_G(x), R_G(y - g)) \ for\ all\ g \in G \\
&\leq d(x, y - g) + k_1 d(x, y - g) \ for\ all\ g \in G.
\end{aligned}$$

Therefore,

$$\begin{aligned}
d(w_G^{-1}(x + G), w_G^{-1}(y + G)) &\leq (1 + k_1) \inf_{g \in G} d(x, y - g) \\
&= (1 + k_1)d(x + G, y + G) \\
&= k''d(x + G, y + G)
\end{aligned}$$

i.e. w_G^{-1} is Lipschitzian.

Since R_G is Lipschitzian, it is continuois and so Theorem 4.1 implies that $w_G \mid_{R_G^{-1}(0)}$ is a homeomorphism and therefore $(w_G \mid_{R_G^{-1}(0)})^{-1}$ is also a homeomorphism.Hence w_G^{-1} is a Lipschitzian homeomorphism of $R_G^{-1}(0)$ onto X/G. □

Remarks 4.9. *For the map P_G, the corresponding result was proved for normed linear spaces in [17] and for metric linear spaces in [18].*

Let G be a set in a metric space (X, d). An element $g_0 \in G$ is said to be *strongly unique best approximation* (see [13]) *(strongly unique best coapproximation* [9]) of an element $x \in X$ if there exist a constant $r = r(x, G)$ with $0 < r \leq 1$ such that $d(x, g_0) + rd(g_0, g) \leq d(x, g)$ $(d(g_0, g) + rd(x, g_0) \leq d(x, g))$ for every $g \in G$. A subspace having strongly unique best approximation (strongly unique best coapproximation) for each $x \in X$ is called *strongly Chebyshev (strongly co-Chebyshev).*

Theorem 4.10. *For a strongly Chebyshev subspace G of a metric linear space (X, d), the metric projection P_G is pointwise Lipschitzian i.e. for each $x \in X$ there exist a constant $\lambda = \lambda(G, x)$ such that $d(P_G(x), P_G(y)) \leq \lambda d(x, y)(y \in X)$.*

Proof. If $r = r(G, x)$ as in the definition of strongly Chebyshev, then putting $g_0 = P_G(x)$ and $g = P_G(y)$, we obtain

$$\begin{aligned} rd(P_G(x), P_G(y)) &\leq d(x, P_G(y)) - d(x, P_G(x) \\ &\leq d(x, y) + d(y, P_G(y)) - d(x, P_G(x) \\ &\leq d(x, y) + d(y, P_G(x)) - d(x, P_G(x)) \\ &\leq d(x, y) + d(y, x) + d(x, P_G(x)) - d(x, P_G(x)) = 2d(x, y) \end{aligned}$$

This gives, $d(P_G(x), P_G(y)) \leq \frac{2}{r} d(x, y)$. So taking $\frac{2}{r} = \lambda$, we get the result. □

Remarks 4.11. *The converse of Theorem 4.10 is not true (see [13]): In $X = l^2$, the metric projection P_G is pointwise Lipschitzian, but it has no strongly Chebyshev subspace.*

Can we prove an analogous result for strongly co-Chebyshev subspaces?

Acknowledgements : The research work of the first author has been supported by U.G.C., India under Emeritus Fellowship and of the second author under Senior Research Fellowship.

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(Received 21 November 2014)
(Accepted 28 April 2015)