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Semigroups of Full Transformations with Restriction on the Fixed Set is Bijective

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Abstract : Let $T(X)$ be the full transformation semigroup of the set X and let $S(X,Y) = \{\alpha \in T(X) : Y\alpha \subseteq Y\}$ where Y is a nonempty subset of X. Then $S(X,Y)$ is a subsemigroup of $T(X)$. In this paper, for a fixed nonempty subset Y of X, let

$$PG_Y(X) = \{\alpha \in T(X) : \alpha|_Y \in G(Y)\}$$

where $G(Y)$ is the permutation group on Y. Then $PG_Y(X)$ is a subsemigroup of $S(X,Y)$. Some relationships between $PG_Y(X)$ it's subsemigroup and $S(X,Y)$ are considered. Moreover, it is shown that $PG_Y(X)$ is regular and characterizations of left regularity, right regularity, and completely regularity of elements of $PG_Y(X)$ are also described.

Keywords : transformation semigroup; regularity; left regularity; right regularity; completely regularity.
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1 Introduction

Let X be a nonempty set and let $T(X)$ denote the semigroup of the full transformations from X into itself under composition of mappings. This semigroup is an important object in semigroup theory, combinatorics, many-valued logic, etc. It is known that $T(X)$ is a regular semigroup, that is, for every $\alpha \in T(X), \alpha = \alpha\beta\alpha$

for some $\beta \in T(X)$. For a fixed nonempty subset Y of X, we denote

$$S(X,Y) = \{\alpha \in T(X) : Y\alpha \subseteq Y\}.$$

Then $S(X,Y)$ is a semigroup of full transformations on X which leave Y invariant. In 1966, Magill [1] introduced and studied this semigroups. Later, many classical notions of this semigroup have been investigated, see [2], [3] and [4].

In [3], Nenthein, Youngkhong and Kemprasit showed that $S(X,Y)$ is regular if and only if $X = Y$ or Y contains exactly one element.

In 1994, Umar [5] constructed a subsemigroup of $T(X)$ as follows:

$$F_Y(X) = \{\alpha \in T(X) : C(\alpha)\alpha \subseteq Y = Y\alpha \text{ and } \alpha|_Y \text{ is injective}\}$$

where

$$C(\alpha) = \bigcup\{y\alpha^{-1} : y \in X\alpha \text{ and } |y\alpha^{-1}| \geq 2\}.$$

$F_Y(X)$ is called an *Umar semigroup.* He proved that $F_Y(X)$ is a regular semigroup and considered the Green's relations on this semigroup. It is clear that $F_Y(X)$ is a subsemigroup of $S(X,Y)$.

Later, Sanwong and Sommanee [6] investigated regularity and Green's relations on a subsemigroup of $S(X,Y)$ which defined by

$$T(X,Y) = \{\alpha \in T(X) : X\alpha \subseteq Y\}.$$

Recently, a subsemigroup of $S(X,Y)$ defined by $F(X,Y) = \{\alpha \in T(X,Y) : X\alpha \subseteq Y\alpha\}$ was studied by Sanwong [7].

It is the aim of the paper to introduce a new subsemigroup of $S(X,Y)$ which is defined as follows:

$$PG_Y(X) = \{\alpha \in T(X) : \alpha|_Y \in G(Y)\}$$

where $G(Y)$ is the permutation group on a nonempty subset Y of X. Some algebraic properties of $PG_Y(X)$ are studied. For examples, $PG_Y(X)$ is a regular semigroup, $T(X)$ can be embeddable into $PG_Y(Z)$ for some set Z and relationships between $PG_Y(X)$, it's subsemigroup and $S(X,Y)$ are given. In the last section, left regularity, right regularity, and completely regularity of elements of $PG_Y(X)$ are determined.

Throughout of the paper, the symbol $\pi(\alpha)$ will denote the partition of X induced by $\alpha \in T(X)$ namely,

$$\pi(\alpha) = \{y\alpha^{-1} : y \in X\alpha\}.$$

The set X can be finite or infinite. The cardinality of a set A is denoted by $|A|$.

2 Preliminaries

Let X be an arbitrary set and Y a nonempty subset of X. Define a subset of $T(X)$ as follows:

$$PG_Y(X) = \{\alpha \in T(X) : \alpha|_Y \in G(Y)\}$$

where $G(Y)$ is the permutation group on Y. Note that id_X, the identity mapping on X, belongs to $PG_Y(X)$.

Remark 2.1. *We note that $PG_Y(X) = G(X)$ if $Y = X$. For arbitrary singleton subset Y of X, we obtain that $S(X,Y) = PG_Y(X)$. Moreover, if $|X| = 2$, then we have $S(X,Y) = PG_Y(X) = F_Y(X)$.*

Theorem 2.2. *$PG_Y(X)$ is a regular semigroup.*

Proof. To prove that $PG_Y(X)$ is a subsemigroup of $T(X)$, let $\alpha, \beta \in PG_Y(X)$. Then we have $\alpha|_Y, \beta|_Y \in G(Y)$ whence $\alpha\beta|_Y \in T(Y)$. It is easy to verify that $\alpha\beta|_Y \in G(Y)$. To show $PG_Y(X)$ is regular, let $\alpha \in PG_Y(X)$. We obtain via $Y\alpha = Y$ that $X\alpha = Y \cup (X\alpha \backslash Y)$. For each $x \in Y$, there exists a unique $x' \in Y$ such that $x'\alpha = x$ since $\alpha|_Y \in G(Y)$. For $x \in X\alpha \backslash Y$, we choose $x' \in x\alpha^{-1}$. Define $\beta : X \to X$ by

$$x\beta = \begin{cases} x', & \text{if } x \in X\alpha, \\ x, & \text{otherwise.} \end{cases}$$

Obviously, $\beta|_Y : Y \to Y$ is bijective, that is $\beta \in PG_Y(X)$. Let $x \in X$. Then $x\alpha\beta\alpha = (x\alpha)'\alpha = x\alpha$. This means that $\alpha = \alpha\beta\alpha$ whence $PG_Y(X)$ is a regular semigroup. □

From the definition of $PG_Y(X)$ and Theorem 2.2, we conclude that $F_Y(X)$ is a subsemigroup of $PG_Y(X)$ and $PG_Y(X)$ is a subsemigroup of $S(X,Y)$. Next, the conditions under which the semigroups coincide are given.

Theorem 2.3. *$F_Y(X) = PG_Y(X)$ if and only if $|X \backslash Y| \leq 1$.*

Proof. Assume that $|X \backslash Y| \geq 2$. There exist $a, b \in X \backslash Y$ such that $a \neq b$. We define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} a, & \text{if } x = b, \\ x, & \text{otherwise.} \end{cases}$$

We obtain that $\alpha|_Y$ is the identity mapping on Y, that is $\alpha \in PG_Y(X)$. Since $a\alpha^{-1} = \{a, b\}$, we have $a \in C(\alpha)\alpha$ whence $C(\alpha)\alpha \nsubseteq Y$. This implies that $\alpha \notin F_Y(X)$ and then $F_Y(X) \neq PG_Y(X)$.

Conversely, assume that $|X \backslash Y| \leq 1$. It is enough to show that $PG_Y(X) \subseteq F_Y(X)$. Let $\alpha \in PG_Y(X)$ and $x \in C(\alpha)$. Then we get $\alpha|_Y \in G(Y)$ and $x \in y\alpha^{-1}$ for some $y \in X$ and $|y\alpha^{-1}| \geq 2$. To verify that $y = x\alpha \in Y$, suppose that $y \notin Y$. Since $Y\alpha = Y$, we have $x \notin Y$. By the assumption, we conclude that $x = y$. This means that $y\alpha^{-1} = \{y\}$ which is a contradiction. Hence $y \in Y$ and so $C(\alpha)\alpha \subseteq Y$. It is clear that $\alpha|_Y$ is injective. Therefore $\alpha \in F_Y(X)$ whence $F_Y(X) = PG_Y(X)$. □

Theorem 2.4. *$PG_Y(X) = S(X,Y)$ if and only if $|Y| = 1$.*

Proof. Assume that $|Y| > 1$. Let $a, b \in Y$ be such that $a \neq b$ and define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} a, & \text{if } x = b, \\ x, & \text{otherwise.} \end{cases}$$

It is easy to verify that $Y\alpha \subseteq Y$ and $\alpha|_Y$ is not injective. Hence $\alpha \in S(X,Y)$ and $\alpha \notin PG_Y(X)$.

The converse follows from Remark 2.1. ☐

Theorem 2.5. *If $PG_Y(X)$ is an inverse semigroup, then $|X \backslash Y| \leq 1$.*

Proof. Suppose that $|X\backslash Y| \geq 2$. Then there exist $a, b \in X\backslash Y$ such that $a \neq b$. Choose $c \in Y$ and define $\alpha, \beta : X \to X$ by

$$x\alpha = \begin{cases} c, & \text{if } x = b, \\ x, & \text{otherwise,} \end{cases}$$

and

$$x\beta = \begin{cases} a, & \text{if } x = b, \\ x, & \text{otherwise.} \end{cases}$$

We note that $\alpha|_Y$ and $\beta|_Y$ are the identity mapping. Thus $\alpha, \beta \in PG_Y(X)$. Moreover, we obtain that $\alpha = \alpha\beta\alpha, \beta = \beta\alpha\beta$ and $\alpha^2 = \alpha$. Hence $\alpha, \beta \in V(\alpha)$. Consequently, $PG_Y(X)$ is not an inverse semigroup. ☐

Corollary 2.6. *$PG_Y(X)$ is an inverse semigroup if and only if $|X| \leq 2$ or $Y = X$.*

Proof. Assume that $|X| > 2$ and $Y \neq X$. If $|X\backslash Y| > 1$, then we have $PG_Y(X)$ is not an inverse semigroup by Theorem 2.5. Suppose that $|X\backslash Y| = 1$. Let $a, b \in Y$ be such that $a \neq b$ and $X\backslash Y = \{c\}$. Define $\alpha, \beta : X \to X$ by

$$x\alpha = \begin{cases} a, & \text{if } x = c, \\ x, & \text{otherwise,} \end{cases}$$

and

$$x\beta = \begin{cases} b, & \text{if } x = c, \\ x, & \text{otherwise.} \end{cases}$$

Since $\alpha|_Y, \beta|_Y$ are the identity mappings, we deduce $\alpha, \beta \in PG_Y(X)$. Note that $\alpha = \alpha\beta\alpha, \beta = \beta\alpha\beta$ and $\alpha^2 = \alpha$ whence $\alpha, \beta \in V(\alpha)$. Therefore $PG_Y(X)$ is not an inverse semigroup.

From Remark 2.1, we obtain the converse. ☐

Let $P(X)$ be the partial transformation semigroup on X. We note that $T(X)$ is a subsemigroup of $P(X)$. The following theorem shows that each full transformations semigroup can be embedded into $PG_Y(X)$ for some set X.

Theorem 2.7. *Let $0 \notin X$. Then $P(X)$ is isomorphic to $PG_{\{0\}}(X \cup \{0\})$.*

Proof. Let $\alpha \in P(X)$. Then $\operatorname{dom}(\alpha) \subseteq X$. We let $\overline{\alpha} : X \cup \{0\} \to X \cup \{0\}$ be defined by

$$x\overline{\alpha} = \begin{cases} x\alpha, & \text{if } x \in \operatorname{dom}(\alpha), \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, $\overline{\alpha}|_{\{0\}} \in G(\{0\})$ whence $\overline{\alpha} \in PG_{\{0\}}(X \cup \{0\})$. We claim that $\overline{\alpha}\overline{\beta} = \overline{\alpha\beta}$ for each $\alpha, \beta \in P(X)$. Let $\alpha, \beta \in P(X)$ and $x \in X \cup \{0\}$.

Case 1. $x\overline{\alpha} \notin \operatorname{dom}(\beta)$. If $x \in \operatorname{dom}(\alpha)$ then $x\alpha \notin \operatorname{dom}(\beta)$ which implies $x \notin \operatorname{dom}(\alpha\beta)$. If $x \notin \operatorname{dom}(\alpha)$ then $x \notin \operatorname{dom}(\alpha\beta)$ since $\operatorname{dom}(\alpha\beta) \subseteq \operatorname{dom}(\alpha)$. Consequently, $x\overline{\alpha}\overline{\beta} = 0 = x\overline{\alpha\beta}$.

Case 2. $x\overline{\alpha} \in \operatorname{dom}(\beta)$. Then $x\alpha = x\overline{\alpha}$ and we conclude that $x \in \operatorname{dom}(\alpha\beta)$. Hence $x\overline{\alpha}\overline{\beta} = x\alpha\overline{\beta} = x\alpha\beta = x\overline{\alpha\beta}$.

These imply that $\overline{\alpha}\overline{\beta} = \overline{\alpha\beta}$ for all $\alpha, \beta \in P(X)$. It follows that the mapping $\varphi : P(X) \to PG_{\{0\}}(X \cup \{0\})$ defined by $\alpha\varphi = \overline{\alpha}$ is a homomorphism.

To verify injectivity of φ, let $\alpha, \beta \in P(X)$ be such that $\overline{\alpha} = \overline{\beta}$. Let $D = \{x \in X : x\overline{\alpha} \neq 0\}$. Obviously, $\operatorname{dom}(\alpha) = D = \operatorname{dom}(\beta)$. Moreover, we obtain that $x\alpha = x\beta$ for all $x \in D$ which implies $\alpha = \beta$. Finally, let $\beta \in PG_{\{0\}}(X \cup \{0\})$. Then define $\alpha \in P(X)$ by $x\alpha = x\beta$ for all $x \in \{x \in X : x\beta \neq 0\}$. Clearly, $\overline{\alpha} = \beta$. Hence $P(X)$ is isomorphic to $PG_{\{0\}}(X \cup \{0\})$. □

Immediately, we obtain the following corollary.

Corollary 2.8. *Let $0 \notin X$. Then $T(X)$ can be embedded into $PG_{\{0\}}(X \cup \{0\})$.*

3 Regularity

Recall that an element x in a semigroup S is called *left* [*right*] *regular* if $x = yx^2$ [$x = x^2y$] for some $y \in S$ and x is *completely regular* if $x = xyx$ and $xy = yx$ for some $y \in S$. In this section, the left regularity, right regularity, and completely regularity of elements in $PG_Y(X)$ are studied.

Theorem 3.1. *Let $\alpha \in PG_Y(X)$. Then α is a right regular element if and only if $\alpha|_{X\alpha}$ is injective.*

Proof. Assume that $\alpha = \alpha^2\beta$ for some $\beta \in PG_Y(X)$. Let $x, y \in X\alpha$ be such that $x\alpha = y\alpha$. Thus $x = x'\alpha$ and $y = y'\alpha$ for some $x', y' \in X$. It follows that

$$x = x'\alpha = x'\alpha^2\beta = x\alpha\beta = y\alpha\beta = y'\alpha^2\beta = y'\alpha = y.$$

Hence $\alpha|_{X\alpha}$ is injective.

Suppose that $\alpha|_{X\alpha}$ is injective. We will construct $\beta \in PG_Y(X)$ satisfying $\alpha = \alpha^2\beta$. Let $x \in X\alpha^2$. Then by the assumption, we have $x'\alpha = x$ for a unique $x' \in X\alpha$. Define $\beta : X \to X$ by

$$x\beta = \begin{cases} x', & \text{if } x \in X\alpha^2, \\ x, & \text{otherwise.} \end{cases}$$

We conclude via $\alpha|_Y \in G(Y)$ that $Y = Y\alpha = Y\alpha^2$. To verify $Y\beta = Y$, let $x \in Y$. Since $x \in Y = Y\alpha^2$ and by the uniqueness of x', we have $x' = y\alpha$ for some $y \in Y$ whence $x\beta \in Y$. On the other hand, let $y \in Y$. Since $Y = Y\alpha$, we conclude that $y\alpha \in Y\alpha^2$ and hence $(y\alpha)\beta = (y\alpha)' = y$. Thus $Y = Y\beta$. Let $x, y \in Y$ be such that $x\beta = y\beta$. Then $x, y \in Y = Y\alpha = Y\alpha^2$. By the uniqueness of x' and y', we obtain $\beta|_Y$ is injective. Hence $\beta \in PG_Y(X)$. Finally, let $x \in X$. Since $(x\alpha)\alpha = x\alpha^2$, we have $(x\alpha^2)' = x\alpha$. That is $x\alpha^2\beta = (x\alpha^2)' = x\alpha$. □

Theorem 3.2. *Let $\alpha \in PG_Y(X)$. Then α is a left regular element if and only if $X\alpha = X\alpha^2$.*

Proof. Assume that $\alpha = \beta\alpha^2$ for some $\beta \in PG_Y(X)$. Clearly, $X\alpha^2 \subseteq X\alpha$. Let $x \in X\alpha$. Then $x = x'\alpha$ for some $x' \in X$. Hence $x = x'\alpha = x'\beta\alpha^2 \in X\alpha^2$ which implies that $X\alpha = X\alpha^2$.

Suppose that $X\alpha = X\alpha^2$. We note from $\alpha|_Y \in G(Y)$ that for each $x \in Y$, there exists a unique $x' \in Y$ such that $x'\alpha = x$ whence $x'\alpha^2 = x\alpha$. Let $x \in X\backslash Y$. Then by the assumption, we choose $x' \in X$ such that $x'\alpha^2 = x\alpha$. We construct $\beta \in PG_Y(X)$ as follows: $x\beta = x'$ for each $x \in X$. To verify $Y = Y\beta$, let $x \in Y$. By the definition of x', we deduce $x\beta = x' \in Y$. Let $y \in Y$, then $y\alpha = x$ for some $x \in Y$ since $Y\alpha = Y$. By the uniqueness of x', we conclude that $x\beta = x' = y$ which implies that $Y = Y\beta$. Assume that $x\beta = y\beta$ for some $x, y \in Y$. Then $x' = y'$ which implies $x = x'\alpha = y'\alpha = y$. Therefore $\beta|_Y \in G(Y)$. Let $x \in X$. We conclude from the definition of x' that $x\beta\alpha^2 = x'\alpha^2 = x\alpha$. □

Theorem 3.3. *Let $\alpha \in PG_Y(X)$. Then α is a completely regular element if and only if $|P \cap X\alpha| = 1$ for all $P \in \pi(\alpha)$.*

Proof. Assume that $\alpha = \alpha\beta\alpha$ and $\alpha\beta = \beta\alpha$ for some $\beta \in PG_Y(X)$. Let $P \in \pi(\alpha)$. Then $P = x\alpha^{-1}$ for some $x \in X\alpha$. Choose $x' \in P$, we conclude that $x = x'\alpha = x'\alpha\beta\alpha = x\beta\alpha$ which implies $x\beta \in P$. Since $x\beta = x'\alpha\beta = x'\beta\alpha \in X\alpha$, we obtain $P \cap X\alpha \neq \emptyset$. To verify $|P \cap X\alpha| = 1$, suppose that $a, b \in P \cap X\alpha$. Then $a = a'\alpha, b = b'\alpha$ for some $a', b' \in X$ and $a\alpha = b\alpha$. It follows that

$$a = a'\alpha = a'\alpha\beta\alpha = a\beta\alpha = a\alpha\beta = b\alpha\beta = b\beta\alpha = b'\alpha\beta\alpha = b'\alpha = b.$$

Assume that for each $P \in \pi(\alpha)$, $|P \cap X\alpha| = 1$. Let $P \in \pi(\alpha)$. By assumption, we denote $x_P \in P \cap X\alpha$. Let $P' = x_P\alpha^{-1}$. Then $y\alpha = x_P$ for all $y \in P'$. In particular, $x_{P'}\alpha = x_P$. Define $\beta : X \to X$ by

$$x\beta = x_{P'} \text{ if } x \in P \text{ for some } P \in \pi(\alpha).$$

Since $\pi(\alpha)$ is a partition of X, β is well-defined. To show that $Y\beta = Y$, let $x \in Y$. Then $x \in P$ for some $P \in \pi(\alpha)$. We note from $Y = Y\alpha$ that $x \in P \cap X\alpha$ whence $x = x_P$. Since $Y = Y\alpha^2$, we have $x_P = y\alpha^2$ for some $y \in Y$. This means that $y\alpha \in x_P\alpha^{-1} = P'$. Thus $y\alpha \in P' \cap X\alpha$ which implies $y\alpha = x_{P'}$. It follows that $x\beta = x_{P'} = y\alpha \in Y\alpha = Y$. Hence $Y\beta \subseteq Y$. Let $y \in Y$. Then $y = y'\alpha$ for some $y' \in Y$ since $Y = Y\alpha$. From $\pi(\alpha)$ is a partition of X, we obtain that

$y'\alpha^2 \in P$ for some $P \in \pi(\alpha)$. Since $y'\alpha^2 \in P \cap X\alpha$, we have $y'\alpha^2 = x_P$. This implies that $y'\alpha \in x_P\alpha^{-1} = P'$ whence $y'\alpha \in P' \cap X\alpha$. Thus $y'\alpha = x_{P'}$. We conclude that $y'\alpha^2\beta = x_{P'} = y'\alpha = y$ then we get $Y\beta = Y$. Next, let $x, y \in Y$ be such that $x\beta = y\beta$. By the definition of β, we have that $x \in P$, $y \in Q$ for some $P, Q \in \pi(\alpha)$ and $x\beta = x_{P'}, y\beta = x_{Q'}$ where $x_{P'}\alpha = x_P$ and $x_{Q'}\alpha = x_Q$. Thus $x_P = x_{P'}\alpha = x\beta\alpha = y\beta\alpha = x_{Q'}\alpha = x_Q$ whence $P \cap Q \neq \emptyset$. Since $\pi(\alpha)$ is a partition of X, we have $P = Q$. We note that $x, y \in Y = Y\alpha$ which implies $x = x_P = y$. Hence $\beta|_Y$ is injective and so $\beta \in PG_Y(X)$.

To verify that $\alpha = \alpha\beta\alpha$ and $\alpha\beta = \beta\alpha$, let $x \in X$. We note that $x\alpha \in P$ for a unique $P \in \pi(\alpha)$. Then we obtain that $x\alpha = x_P$. Hence $x\alpha\beta\alpha = x_{P'}\alpha = x_P = x\alpha$. Since $x \in x_P\alpha^{-1} = P'$, we conclude that $x\beta\alpha = x_{P''}\alpha = x_{P'} = x\alpha\beta$. Therefore, α is a completely regular element. □

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